Symmetry **2019**, *xx*, 2-x; doi:10.3390/——

OPEN ACCESS

symmetry

ISSN 2073-8994

www.mdpi.com/journal/symmetry

Article

Hyperbolicity of First Order Quasi-Linear Equations

Vladimir Vasilyev [1], **Yuri Virchenko**[1,2]

[1] Department of Applied Mathematics and Computer Modeling, Belgorod State National Research University, Pobedy st. 85, Belgorod 308015, Russia

[2] Belgorod Shukhov State Technological University, Kostyukova st. 46, Belgorod, 308012 Russia

* Author to whom correspondence should be addressed; vladimir.b.vasilyev@gmail.com, tel.: +74722301300; fax: +74722301012.

Received: xx / Accepted: xx / Published: xx

Abstract: The theorem about equivalence of the strong hyperbolicity concept and the Friedrichs hyperbolicity one for partial quasi-linear differential equations of first order is proved. On the basis of this theorem, the necessary and sufficient conditions of hyperbolicity are found in terms of the matrix of corresponding linearized first order equations system.

Keywords: first order differential operators, quasi-linear equations system, Friedrichs' hyperbolicity, hyperbolic matrices

Classification: MSC 35F60

---

1. Introduction

Interest in hyperbolic quasi-linear equations has arisen firstly in mathematical physics and it continues to be maintained up to now due in the connection with the necessity of solving of gas dynamics problems (see, for example, [1]-[3]). At the same time, along with the search for exact solutions to the gas dynamics equations system and analogous equations of mathematical physics, a significant place in research is the study of the phenomenon which consists of the discontinuous solutions formation. Such solutions are called the weak Hugoniot ones (see, for example, [1], [4]-[6]). Later, there was interest in the appearance of dynamic mode in their solutions which is commonly called the chaotic one when nonlinearity present in such systems [7]. Finally, in connection with the problems of mathematical physics, in the formulation of which there is no a priori information about the general qualitative properties of solutions of the used quasi-linear systems of equations, there is an urgent need to determine the area of their hyperbolicity (see, [8]-[10]). This work is devoted just to solving this problem.

Consider the system quasi-linear evolution differential equations of first order. Let $\mathsf{u}(\boldsymbol{x},t) = \langle u_a(\boldsymbol{x},t)\,;\,a = 1 \div n\rangle\; :\; \mathbf{R}^m \mapsto \mathbf{R}^n$ be the collection of functions which depend on the coordinates $\boldsymbol{x} = \langle x_1,...,x_m\rangle \in \mathbf{R}^m$ and on the parameter $t \in \mathbf{R}$. Then this system has the form

$$\dot{u}_a(\boldsymbol{x},t) = \sum_{b=1}^{n}\sum_{k=1}^{m} A_k^{(a,b)}(\boldsymbol{x},t)\frac{\partial u_b(\boldsymbol{x},t)}{\partial x_k} + H_a(\mathsf{u},\boldsymbol{x},t)\,, \quad a = 1 \div n\,. \tag{1}$$

Later, throughout the work, we distinguish vectors from the definition space $\mathbf{R}^m$ of a vector function $\mathsf{u}(\boldsymbol{x},t)$ and the space $\mathbf{R}^n$ of the values of these functions. Therefore, we use different fonts to denote vectors from these spaces. Vectors from $\mathbf{R}^m$ are displayed in sanserif font, and vectors from $\mathbf{R}^n$ are displayed in bold.

Verification of hyperbolicity of the fixed system of quasi-linear equations of the first order in the case when the coefficients of the equation (1) are fixed numbers is carried out by a simple algorithm. On the contrary, if the coefficients depend on parameters which one can vary widely, and, moreover, if they are functionals, the study of the hyperbolicity of the system becomes dramatically more complicated. Solving such a problem can be a rather time-consuming procedure (see, for example, [9], [10]). Such a problem becomes especially difficult in the case when the system of equations has a large dimension n. The relevance of solving such a problem becomes very sharp due to the fact that the quasi-linear equations system with such differential operators may have a variable type, that is, at some their possible values, such equations may no longer have the property of "evolutionarity". It is always important if a system of first-order quasi-linear equations is intended for modeling physical processes. In particular, it is connected with its use for the describing the processes in continuous media when any physical dissipation mechanisms are neglected. This is due to the fact that for any solution $\mathsf{v}(\boldsymbol{x},t) \sim \mathsf{v}_0 \exp[i\omega t + (\boldsymbol{x},\boldsymbol{q})]$ of corresponding llnearized system of equations with the matrix $\mathsf{T}(\boldsymbol{q})$, the so-called "dispersion equation" must be fulfilled from a physical point of view, which connects the wave frequency ω with the wave vector $\boldsymbol{q}$. In such a situation all solutions branches $\omega_j(\boldsymbol{q})$, $j = 1 \div n$ of this algebraic equation (see, the defining equations (3), (4)) should be real when $\boldsymbol{q}$ is real. If a solution branch $\omega_k(\boldsymbol{q})$ has nonzero imaginary part for some vectors $\boldsymbol{q}$, then it leads to the existence of the complex-conjugate solution $\omega_j^*(\boldsymbol{q})$ due to the realness of the dispersion equation. In this case, the presence of the imaginary part leads not only to the existence of solutions $\mathsf{v}(\boldsymbol{x},t)$ of the system, tending asymptotically to some stationary evolutionary regime, but also to the mandatory existence of solutions $\mathsf{v}(\boldsymbol{x},t)$ having nothing physical sense, when there is unlimited increase in some areas of change of the spatial variable $\boldsymbol{x}$.

Thus, when selecting physically reasonable systems of equations, it is necessary to require the realness of all solutions $\omega_j(\boldsymbol{q})$, $j = 1 \div n$. In addition, a power-law increase of the solutions $\mathsf{v}(\boldsymbol{x},t)$ of the llnearized system with respect to t may be occurred under the conditions of the realness of all solutions $\omega_j(\boldsymbol{q})$, $j = 1 \div n$, but when the matrix $\mathsf{T}(\boldsymbol{q})$ of the system is not diagonalized. In this regard, when choosing a physically reasonable evolutionary system of quasi-linear equations, it is necessary to require diagonalizability of this matrix. As a result, we come to the conclusion that the hyperbolicity condition (see, Definition 1) of quasi-linear equations system of the first order, or the strong hyperbolicity, as such a property is also called, is a natural requirement imposed on them from the point of view of physics.

It should be emphasized that the study of the hyperbolicity conditions of systems of quasi-linear equations in the cited works [9], [10] was still limited to small-dimensional systems. At the same time, in general, the problem under study is connected with the so-called covariant differential equation systems (see, for example, [11]), the importance of which is due to their applicability in mathematical modeling in physics of complex condensed matter. The coefficients of such systems of equations are some arbitrary functions of the invariants of the group $\mathbf{O}_3$ transformations of the system of equations, and the dimension of such systems can vary from a minimum of 3 to about 20 (see, for example, [12], [13]). Therefore, the research presented in this paper is aimed to finding of necessary and sufficient features of the characteristic matrix $\mathsf{T}(\boldsymbol{q})$, which would guarantee strong hyperbolicity of a system of quasi-linear equations of the first order in general case.

2. Hyperbolic systems of first order equations

We find the connection between two concepts. First of them is the strong hyperbolicity of the quasi-linear equations system of first order. In future, we will call it simply as the hyperbolicity. Second concept is the Friedrichs hyperbolicity or it is so-called the t-hyperbolicity. But we will modified a little the definition of the last concept.

The linear system of equations (see, [1] -[3]) for the set $\mathsf{v}(\boldsymbol{x}, t)$ is connected with the system (1). This system is obtained by linearization of the system (1) at the point $\mathsf{u}(\boldsymbol{x}, t)$

$$\dot{v}_a(\boldsymbol{x}, t) = \sum_{b=1}^{n} \sum_{k=1}^{m} A_k^{(a,b)}(\mathsf{u}) \frac{\partial v_b(\boldsymbol{x}, t)}{\partial x_k} + \sum_{b=1}^{n} \frac{\partial H_a}{\partial u_b} v_b(\boldsymbol{x}, t) \,, \quad a = 1 \div n \tag{2}$$

where the set of functions $\mathsf{v}(\boldsymbol{x}, t) = \langle v_a(\boldsymbol{x}, t); a = 1 \div n \rangle$ corresponds to the fixed set $\mathsf{u} = \langle u_1, ..., u_n \rangle \in \mathbf{R}^n$ values of the set $\langle u_a(\boldsymbol{x}, t) \,; a = 1 \div n \rangle$ at the point $\boldsymbol{x}$ and at time moment t.

In order for this system of equations to be solvable at least locally with respect to t, for arbitrarily selected initial conditions that are "in general position", it is necessary and sufficient that it has the *hyperbolicity* property. This property assumes that the collection $\langle \mathsf{A}_k \,; k = 1 \div m \rangle$ of $n \times n$-matrix coefficients of the system (1), which are defined by matrix elements $A_k^{(a,b)} = \left(\mathsf{A}_k\right)_{a,b}$, $a, b = 1 \div n$, must have the following special property.

Definition 1.[1)] *The system (1) are called hyperbolic if, in the corresponding system (2), the $n \times n$-matrix $\mathsf{T}(\boldsymbol{q})$ with matrix elements*

$$T_{a,b}(\boldsymbol{q}) = \sum_{k=1}^{m} q_k \, A_k^{(a,b)}(\boldsymbol{x}, t) \tag{3}$$

is diagonalizable and it has only real eigenvalues $\omega^{(l)}$, $l = 1 \div n$ for any set $\boldsymbol{q} = \langle q_s; s = 1 \div m \rangle \in \mathbf{R}^m$, at any temporal point $t \in \mathbf{R}$ and for any set $\mathsf{u} \in \mathbf{R}^n$, $\boldsymbol{x} \in \mathbf{R}^m$.

Thus, the hyperbolicity of the system (1) consists in the realness of the roots $\omega^{(l)}$, $l = 1 \div n$ of the equation

$$\det(\omega - \mathsf{T}(\boldsymbol{q})) = 0 \tag{4}$$

1) The given definition corresponds to the so-called strong hyperbolicity.

with respect to ω and the presence of eigenvectors for each multiple solution. In future, we will call matrices $\mathsf{T}(\boldsymbol{q})$ satisfying this condition as the *hyperbolic* ones.

Due to the difficulty of establishing the fact of hyperbolicity of systems of quasi-linear equations in the cases pointed out in the introduction, they resort to checking the availability the property that is, generally speaking, the stronger one than hyperbolicity (see Theorem 1, *Sufficiency*), namely, it is the so-called t-hyperbolicity (the Friedrichs hyperbolicity) (see, for example, [4], [14]). We give the following definition of the t-hyperbolicity of the system (1), which is somewhat modernized in comparison with [1].

This will permit us to establish the equivalence of the upgraded definition with Definition 1 that opens the way for research on the hyperbolicity of systems (1) by a simpler method than if the original definition of hyperbolicity is used.

Definition 2. *The system* (1) *is called t-hyperbolic if the matrix* $\mathsf{T}(\boldsymbol{q})$ *is diagonalizable for any sets* $\boldsymbol{q} = \langle q_1, ..., q_n \rangle \in \mathbf{R}^m$ *and* $\mathsf{u} = \langle u_1, ..., u_n \rangle$ *and there exists such a symmetric positive* $n \times n$*-matrix* D *for which the matrix* $\mathsf{DT}(\boldsymbol{q})$ *is symmetric.*

3. Coincidence of hyperbolicity definitions

$n \times n$-matrix B is called diagonalizable if it has exactly n eigenvectors $\{\mathsf{e}_r; r = 1, \ldots, n\}$ or, in other words, these vectors form a basis in $\mathbf{R}^n$. In this section we will prove an important auxiliary theorem on diagonalizability of a matrix which can be attributed to the field of matrix analysis and which may represent an independent interest. In future, it will allow us to prove the equivalence of the definitions Definitions 1 and 2. Further, we follow the terminology of the monograph [15] when formulating and proving all statements.

Theorem 1. *In order for the real* $n \times n$*-matrix* B *to be diagonalizable and all its eigenvalues* $\langle \mu_1, ..., \mu_n \rangle$ *to be real, it is necessary and sufficient that there exists such a symmetric positive* $n \times n$*-matrix* D *for which the matrix* DB *is symmetric. In this case, the set of eigenvalues of the matrix* B *coincides with the set of eigenvalues of the matrix* DB.

Proof. *Necessity*. Let all eigenvalues of the matrix B be real. The proof of the existence of the matrix D consists of the following items **1-5**.

1. Let the real matrix B represents the implementation of the linear operator in $\mathbf{R}^n$ in the standard basis $\{\mathsf{e}_a^{(0)} \in \mathbf{R}^n; a = 1, \ldots, n\}$, $(\mathsf{e}_a^{(0)})_b = \delta_{ab}$, $a, b = 1, \ldots, n$. We will consider this operator in $\mathbf{C}^n$. Further, let $\mathsf{c} = \langle c_1, ..., c_n \rangle \in \mathbf{R}^n$ be an eigenvector in $\mathbf{R}^n$ corresponding to the real eigenvalue μ of the matrix B, that is $\mathsf{B}\,\mathsf{c} = \mu\mathsf{c}$. This eigenvector can always be chosen so that all its components are real.

In fact, under the specified condition, at least one of the real vectors $\mathrm{Re}\,\mathsf{c} = \langle \mathrm{Re}\,c_1, ..., \mathrm{Re}\,c_n \rangle$ or $\mathrm{Im}\,\mathsf{c} = \langle \mathrm{Im}\,c_1, ..., \mathrm{Im}\,c_n \rangle$ is not equal to zero. Assuming, for certainty, that such is the first of them and calculating the real part of both parts of the equality $\mathsf{Bc} = \mu\mathsf{c}$, we find that $\mathrm{Re}\mathsf{c}$ is the eigenvector of matrix B corresponding to the same eigenvalue μ. Therefore, since μ is an arbitrarily chosen eigenvalue of the matrix B, which, by supposition, has a complete set of real eigenvalues $\langle \mu_1, ..., \mu_n \rangle$ with corresponding eigenvectors $\{\mathsf{e}_1, ..., \mathsf{e}_n\}$, then all of them can be selected with real components.

2. For any basis $\{\mathsf{e}_1, ..., \mathsf{e}_n\}$ in the space $\mathbf{R}^n$, there is such a real nonsingular $n \times n$-matrix V, for which the set of vectors $\{\mathsf{V}\,\mathsf{e}_1, ..., \mathsf{V}\,\mathsf{e}_n\}$ is orthonormal. Indeed, we apply the Sonin-Schmidt orthogonalization process of the vector set $\{\mathsf{e}_1, ..., \mathsf{e}_n\}$. As a resuit, we get an orthonormal set $\{\mathsf{e}'_1, ..., \mathsf{e}'_n\}$ which is complete in the space $\mathbf{R}^n$.

This set is decomposed

$$\mathsf{e}'_a = \sum_{b=1}^{a} V_{ab}\mathsf{e}_b\,, \quad a = 1 \div n$$

on the basis of the original vector set $\{\mathsf{e}_1, ..., \mathsf{e}_n\}$, to which the process is applied where the coefficients of this expansion are real. The matrix V is defined by matrix elements $\big(\mathsf{V}\big)_{ab} = V_{ab}$. It has a non-zero determinant, since the set $\{\mathsf{e}'_1, ..., \mathsf{e}'_n\}$ is the basis in $\mathbf{R}^n$, and this matrix is triangular.

3. Since $\{\mathsf{e}_1, ..., \mathsf{e}_n\}$ are eigenvectors of the matrix B with eigenvalues $\langle\mu_1, ..., \mu_n\rangle$, $\mathsf{B}\,\mathsf{e}_r = \mu_r\mathsf{e}_r$,

$$\Big(\mathsf{V}\mathsf{B}\,\mathsf{V}^{-1}\Big)\mathsf{V}\mathsf{e}_r = \mathsf{V}\mathsf{B}\,\mathsf{e}_r = \mu_r\mathsf{V}\mathsf{e}_r\,, \quad r = 1 \div n\,.$$

Then vectors $\{\mathsf{V}\mathsf{e}_1, ..., \mathsf{V}\mathsf{e}_n\}$ are orthonormal and each of them is the eigenvector of the matrix $\mathsf{V}\mathsf{B}\,\mathsf{V}^{-1}$.

Thus, a consequence of items **1** and **2** is the fact that for any diagonalizable real $n \times n$-matrix B with real eigenvalues, there exists such a real nonsingular $n \times n$-matrix V, for which the matrix $\mathsf{V}\mathsf{B}\,\mathsf{V}^{-1}$ is diagonalizable and it has only real eigenvalues. All its eigenvectors form an orthonormal basis in $\mathbf{R}^n$.

4. Let us introduce $n \times n$-matrix $\mathsf{C} = \mathsf{V}\mathsf{B}\,\mathsf{V}^{-1}$. This matrix is symmetric, $\mathsf{c}^{\mathrm{T}} = \mathsf{C}$, since this property has any matrix with a complete orthonormal set of eigenvectors with real eigenvalues.

5. From the item **4**, it follows that the matrix $\mathsf{V}^{\mathrm{T}}\mathsf{C}\mathsf{V}$ is symmetric, since $\big(\mathsf{V}^{\mathrm{T}}\mathsf{C}\mathsf{V}\big)^{\mathrm{T}} = \mathsf{V}^{\mathrm{T}}\mathsf{C}^{\mathrm{T}}\mathsf{V} = \mathsf{V}^{\mathrm{T}}\mathsf{C}\mathsf{V}$. In addition, $\mathsf{D} = \mathsf{V}^{\mathrm{T}}\mathsf{V}$ is a positive symmetric matrix. Then

$$\mathsf{D}\mathsf{B} = (\mathsf{V}^{\mathrm{T}}\mathsf{V})\mathsf{B} = \mathsf{V}^{\mathrm{T}}\mathsf{C}\,\mathsf{V}$$

is also a symmetric matrix.

Sufficiency. Now, let it is known that there exists a real symmetric positive matrix D for a real $n \times n$-matrix B such that the matrix $\mathsf{D}\mathsf{B}$ is real and symmetric. The proof of the diagonalizability of the matrix B and the realness of its eigenvalues consists of the following items **6-10**.

6. Under given conditions, there exists an orthonormal set $\{\mathsf{e}_r; r = 1 \div n\}$ such that there are real positive eigenvalues $\mu_r > 0$ $r = 1 \div n$ of the matrix D for each its ejgenvector. At the same time, according to the item **1**, all eigenvectors of the matrix D corresponding to these eigenvalues can be chosen with real components in $\mathbf{C}^n$.

There is a unitary and, therefore, nonsingular matrix U, $\mathsf{U}\,\mathsf{U}^{+} = \mathbf{1}$ ($+$ denotes Hermitian conjugation), which diagonalizes the matrix D, that is $\mathsf{U}\,\mathsf{e}_r^{(0)} = \mathsf{e}_r$, $r = 1 \div n$ where $\{\mathsf{e}_r^{(0)} = \langle\delta_{r',r}; r' = 1 \div n\rangle; r = 1 \div n\}$ is the standard basis in $\mathbf{R}^n$ and $\mathsf{U}\,\mathsf{D}\,\mathsf{U}^{+} = \mathrm{diag}\langle\mu_r; r = 1 \div n\rangle$.

From the equalities $\mathsf{U}\,\mathsf{e}_r^{(0)} = \mathsf{e}_r$, taking into account the realness of the components of basis vectors $\{\mathsf{e}_r; r = 1 \div n\}$ and of vectors $\{\mathsf{e}_r^{(0)}; r = 1 \div n\}$, it follows $(\mathrm{Im}\,\mathsf{U})\,\mathsf{e}_r^{(0)} = 0$ for all $r = 1 \div n$, that is, for all vectors of the standard basis. Hence, the matrix $\mathrm{Im}\,\mathsf{U}$ is null. Thus, the matrix U is real and, therefore, $\mathsf{U}^{+} = \mathsf{U}^{\mathrm{T}}$. Then, the unitarity condition of the matrix U takes the form $\mathsf{U}^{\mathrm{T}}\mathsf{U} = \mathbf{1}$, that is, this matrix is orthogonal.

7. We define the real-valued matrix V by the equality $\mathsf{V}^{\mathrm{T}} = \mathsf{U}^{\mathrm{T}}\mathrm{diag}\langle\mu_r^{1/2}; r = 1, \ldots, n\rangle$. The matrix V is also non-singular since it is the product of non-singular real matrices. Then, from the equality $\mathsf{U}\,\mathsf{D}\,\mathsf{U}^{+} = \mathrm{diag}\langle\mu_r; r = 1, \ldots, n\rangle$ and the realness of the matrix U, it follows that

$$\mathsf{D} = \mathsf{U}^{\mathrm{T}}\,\mathrm{diag}\langle\mu_r; r = 1, \ldots, n\rangle\,\mathsf{U} = \mathsf{V}^{\mathrm{T}}\mathsf{V}\,.$$

8. Since the matrix V is non-singular, then we define $\mathsf{C} = \left(\mathsf{V}^{\mathrm{T}}\right)^{-1}\mathsf{D}\mathsf{B}\mathsf{V}^{-1}$. Since the matrix $\mathsf{D}\,\mathsf{B}$ is symmetric, the matrix C is also symmetric, $\mathsf{C}^{\mathrm{T}} = \left(\mathsf{V}^{-1}\right)^{\mathrm{T}}\left[\mathsf{D}\mathsf{B}\right]^{\mathrm{T}}\left[\left(\mathsf{V}^{\mathrm{T}}\right)^{-1}\right]^{\mathrm{T}} = \left(\mathsf{V}^{\mathrm{T}}\right)^{-1}\mathsf{D}\mathsf{B}\mathsf{V}^{-1} = \mathsf{C}$. Besides, it takes place

$$\mathsf{D}\,\mathsf{B} = \mathsf{V}^{\mathrm{T}}\mathsf{C}\,\mathsf{V} = \mathsf{V}^{\mathrm{T}}\mathsf{V}\,\mathsf{B}\,.$$

Consequently, $\mathsf{B} = \mathsf{V}^{-1}\mathsf{C}\,\mathsf{V}$.

9. All solutions of the characteristic equation $\det(\mathsf{C}-\mu) = 0$ are real since the matrix C is symmetric. Then, since

$$\det(\mathsf{B}-\mu) = \det\mathsf{V}\cdot\det(\mathsf{B}-\mu)\cdot\det\mathsf{V}^{-1} = \det(\mathsf{V}\,\mathsf{B}\,\mathsf{V}^{-1}-\mu) = \det(\mathsf{C}-\mu)\,,$$

all these solutions are eigenvalues of the matrix B. Thus, all eigenvalues of the matrix B are real.

10. Let e'_r, $r = 1,\ldots,n$ be the collection of eigenvectors of the symmetric matrix C and μ_r, $r = 1,\ldots,n$ be the collection of eigenvalues corresponding to it, $\mathsf{C}\mathsf{e}'_r = \mu_r\mathsf{e}'_r$. Then, substituting the expression $\mathsf{C} = \mathsf{V}\,\mathsf{B}\,\mathsf{V}^{-1}$ in $\mathsf{V}\,\mathsf{B}\,\mathsf{V}^{-1}\mathsf{e}'_r = \mu_r\mathsf{e}'_r$, we find that

$$\mathsf{B}\left(\mathsf{V}^{-1}\mathsf{e}'_r\right) = \mu_r\mathsf{V}^{-1}\,\mathsf{e}'_r\,,\quad r = 1,\ldots,n\,.$$

Therefore, $\mathsf{V}^{-1}\mathsf{e}'_r$, $r = 1,\ldots,n$ is the collection of eigenvectors of the matrix B. It means that the matrix B has strictly n eigenvectors, that it is diagonalizable.

The statement of the theorem one may reformulate as follows. △

Corollary 1. *Matrix* B *is diagonalizable and its eigenvalues are real if and only if it is representable as the product of two symmetric matrices and one of them is strictly positive.*

Proof. The statement follows from equalities $\mathsf{D}\mathsf{B} = \mathsf{V}^{\mathrm{T}}\mathsf{C}\,\mathsf{V}$, $\mathsf{B} = \mathsf{D}^{-1}\left(\mathsf{V}^{\mathrm{T}}\mathsf{C}\,\mathsf{V}\right)$. △

Remark. *By obvious way, the matrix* D *is defined up to arbitrary positive multiplier. However, even if we take into account this kind of ambiguity of its choice, the class of possible matrices* D*, the existence of which is stated in the theorem, significantly depends on the type of matrix* B*. It takes place even in the case when it is non-degenerate.*

The statement about the coincidence of the concepts of hyperbolicity and t-hyperbolicity is the consequence of Theorem 1.

Theorem 2. *In order for the system* (1) *of quasi-linear equations to be the hyperbolic one, it is necessary and sufficient that it should be* t*-hyperbolic.*

Proof. Based on the definition of hyperbolicity of first-order quasi-linear equations systems, the proof follows by applying the statement of the Theorem 1 to the matrix $\mathsf{T}(\boldsymbol{q})$ with elements (3). △

Thus, the question about the hyperbolicity of the equations (1) system boils down to determining of the factorization possibility of the matrix $\mathsf{T}(\boldsymbol{q})$ in the form of $\mathsf{T}(\boldsymbol{q}) = \mathsf{F}(\boldsymbol{q})\mathsf{G}(\boldsymbol{q})$ where $\mathsf{F}(\boldsymbol{q})$ and $\mathsf{G}(\boldsymbol{q})$ are symmetric matrices, with $\mathsf{F}(\boldsymbol{q}) > 0$. It is essential that the matrix $\mathsf{T}(\boldsymbol{q})$ depends linearly on the vector $\boldsymbol{q}$.

The proved statement simplifies the proof of hyperbolicity property of a given system with the large number of quasi-linear equations when the set of matrices $\langle \mathsf{A}_k(\boldsymbol{x},t); k = 1,\ldots,m\rangle$ is fixed, reducing it to finding a suitable matrix D while the direct search for conditions under which the matrix $\mathsf{T}(\boldsymbol{q})$ is hyperbolic, seems much more laborious.

For effective application of such a method, it is necessary to specify some transparently verifiable sign of hyperbolicity of an arbitrarily selected $n \times n$-matrix $\mathsf{T}(\boldsymbol{q})$ depending linearly on the vector $\boldsymbol{q} \in \mathbf{R}^m$. The following sections of the work are devoted to this problem. We will focus on the somewhat more general algebraic problem, namely, we study the matrix hyperbolicity of the fixed matrix T of dimension n.

Corollary 2. *In order for the matrix* T *to be hyperbolic, it is necessary and sufficient that there exists a symmetric positive matrix* D *that satisfies the equation*

$$\mathsf{T}^{\mathrm{T}}\mathsf{D} = \mathsf{D}\mathsf{T}\,. \tag{5}$$

Proof. The equality (5) follows from the symmetry of matrices DT and D, that is $(\mathsf{DT})^{\mathrm{T}} = \mathsf{T}^{\mathrm{T}}\mathsf{D}^{\mathrm{T}} = \mathsf{T}^{\mathrm{T}}\mathsf{D}$. △

In future, we will call the matrix D as the *binder* one. The equation (5) is also convenient to represent in the following form. Introducing matrices $\mathsf{T}_+ = (\mathsf{T} + \mathsf{T}^{\mathrm{T}})/2$ $\mathsf{T}_- = (\mathsf{T} - \mathsf{T}^{\mathrm{T}})/2$, we obtain

Corollary 3. *In order for the matrix* T *to be hyperbolic, it is necessary and sufficient that there exists such a symmetric positive matrix* D *that satisfies the equation*

$$[\mathsf{T}_+, \mathsf{D}] = \{\mathsf{T}_-, \mathsf{D}\} \tag{6}$$

where $[\cdot,\cdot]$ *is the commutator of the matrix pair and* $\{\cdot,\cdot\}$ *is the anticommutator of them.*

Proof. Equation (6) is valid due to $\mathsf{T} = \mathsf{T}_+ + \mathsf{T}_-$, $\mathsf{T}^{\mathrm{T}} = \mathsf{T}_+ - \mathsf{T}_-$. △

Corollary 4. *If* $[\mathsf{T}_+, \mathsf{T}_-] = 0$*, then the symmetric solution of the equation* (6) *relative the matrix* D *does not exist.*

Proof. In the case of matrix commutation, due to the self-conjugacy of the matrices T_+ and $i\mathsf{T}_-$ in $\mathbf{C}^n$, they have an orthonormal basis of eigenvectors common to them $\{\mathsf{e}_r : r = 1, \ldots, n\}$. Denote the sets $\{\lambda_r^{(+)} : r = 1, \ldots, n\}$ and $\{i\lambda_r^{(-)} : r = 1, \ldots, n\}$ of eigenvalues corresponding to these matrices with real values $\lambda_r^{(\pm)} : r = 1, \ldots, n$. We calculate the matrix elements $(\mathsf{e}_a, \cdot \mathsf{e}_b)$ of both sides of the equation (6),

$$(\mathsf{e}_a, [\mathsf{T}_+, \mathsf{D}]\mathsf{e}_b) = (\lambda_a^{(+)} - \lambda_b^{(+)})(\mathsf{e}_a, \mathsf{D}\mathsf{e}_b)\,,$$

$$(\mathsf{e}_a, \{\mathsf{T}_-, \mathsf{D}\}\mathsf{e}_b) = i(\lambda_a^{(-)} + \lambda_a^{(-)})(\mathsf{e}_a, \mathsf{D}\}\mathsf{e}_b)\,.$$

Then, at $a = b$, the left-hand side of the equality

$$(\mathsf{e}_a, [\mathsf{T}_+, \mathsf{D}]\mathsf{e}_b) = (\mathsf{e}_a, \{\mathsf{T}_-, \mathsf{D}\}\mathsf{e}_b)$$

is equal to zero. But at $\mathsf{T}_- \neq 0$, at least for one of eigenvalues $i\lambda_r^{(-)} \neq 0$, $r = 1 \div n$ and for this number l it is valid $(\mathsf{e}_l, \mathsf{D}\mathsf{e}_l) = 0$ that is impossible for positive matrix D. Consequently, this equality is possible only in the trivial case $\mathsf{T}_- = 0$. △

4. Hyperbolicity of diagonizable matrices T

Let R be a $n \times n$-matrix is hyperbolic in the sense of the definition of this term given in sec.2, that is whose all eigenvalues are real and they are not multiples. Next, let Q be an arbitrary $n \times n$-matrix. Consider the one-parametric family of matrices $\mathsf{R} + \eta\mathsf{Q}$, $\eta \in \mathbf{R}$. Now, we will prove the following statement.

Let us now prove the statements (Theorems 3-5) which show the fundamental possibility of analytical establishing of the strong hyperbolicity presence by studying the dependence of the matrix $\mathsf{T}(\boldsymbol{q})$ on the equations system parameters using the strong hyperbolicity of some reference system.

Theorem 3. *All eigenvalues of each matrix* $\mathsf{R} + \eta\mathsf{Q}$ *at* $\eta \in (\rho_-, \rho_+)$ *are real and do not multiples where* $0 \in (\rho_-, \rho_+)$ *and* ρ_-, ρ_+ *are points on real axis which are nearest on the left and on the right to the point* $\eta = 0$*, correspondingly, in which the equation* $\det(z\mathsf{E} - \mathsf{R} - \eta\mathsf{Q}) = 0$ *relative to* $z \in \mathbf{R}$ *has multiple roots. These eigenvalues are analytic functions on* $\eta \in \mathbf{R}$ *in the domain of complex plane, containing the interval* (ρ_-, ρ_+).

Proof. Let $P(z, \eta) = \sum_{r=0}^{n} a_r(\eta) z^{n-r}$ be a polynomial on $z \in \mathbf{R}$ with the degree n and with coefficients $a_r(\eta)$, $r = 0, \ldots, n$ which depend on the parameter η. The $P(z, 0)$ has n real which are not multiple at $|\eta| < \varepsilon$ when $\varepsilon > 0$ is sufficiently small. If coefficients of polynomials $P(z, \eta)$ depends on η by analytic way, its roots are also analytic functions on η. They are built by analytic continuation from the circle $\{\eta : |\eta| < \varepsilon\}$ with sufficiently small $\varepsilon > 0$.

We apply this statement to the polynomial

$$P(z, \eta) = \det(z\mathsf{E} - \mathsf{R} - \eta\mathsf{Q}) = z^n + \sum_{r=0}^{n-1} a_r(\eta) z^{n-1-r} ,$$

which has coefficients $a_r(\eta)$ depending on $\eta \in \mathbf{R}$ in a polynomial way (see, [16]) and its roots are eigenvalues of the matrix $\mathsf{R} + \eta\mathsf{Q}$. $\triangle$

From Theorem 3 we find the following sufficient criterium of the hyperbolicity of the matrix T.

Theorem 4. *Let the symmetric matrix* $[\mathsf{T} + \mathsf{T}^{\mathrm{T}}]$ *have no multiple eigenvalues and* $1 \in (\rho_-, \rho_+)$ *where* $\rho_\pm$ *are boundary points of the interval in which the parameter* η *specified by Theorem 3 is changed (*$\rho_\pm$ *may be infinite). The matrix* $(1 + \eta)\mathsf{T} + (1 - \eta)\mathsf{T}^{\mathrm{T}}$ *have no multiple roots in this interval and, therefore, the matrix* T *is hyperbolic.*

Proof. The statement follows directly from Theorem 3 at $\mathsf{R} = \mathsf{T} + \mathsf{T}^{\mathrm{T}}$ and $\mathsf{Q} = \mathsf{T} - \mathsf{T}^{\mathrm{T}}$. $\triangle$

The points $\rho_\pm$ are defined on the basis of the coefficients $a_r(\eta)$, $r = 0, \ldots, n-1$ by means of applying of the Bésout theorem for the polynomial $P(z; \eta)$. To do this one should apply the Euclid algorithm to this polynomial and its derivative $P_1(z; \eta) \equiv P'(z; \eta)$ on z (see, for example, [16]). The polynomial $P_n(\eta)$ of zero degree on z which is obtained as the remainder when this algorithm applying, is equal to zero in those points η where $P(z; \eta)$ has a multiple root. Thus, it is valid

Theorem 5. *If* $1 \in (\rho_-, \rho_+)$ *where the points* $\rho_\pm$ *are nearest roots of the equation* $P_n(\eta) = 0$*, correspondingly on the left and on the right of the point* $\eta = 0$*, and* $P_n(\eta)$ *is a result of the application of the Euclid algorithm to polynomials*

$$P(z; \eta) = \det(z\mathsf{E} - (1 + \eta)\mathsf{T} - (1 - \eta)\mathsf{T}^{\mathrm{T}})$$

and $P'(z;\eta)$, then the matrix T *is hyperbolic.*

In principle, by calculating the polynomial $P_n(\eta)$, we may find the necessary condition of the hyperbolicity of the matrix T on the basis of the statement of Theorem 5. However, its implementation faces a rather routine analysis if the matrix order n is not a small number. In this regard, we proceed to a deeper study of the hyperbolicity of the advance given matrix T using Theorem 3.

5. The case $\mathrm{n} = 2$

Let $n = 2$. This case is special at the future study. Let $\mathsf{S}^{(j)}$, $j = 1, 2, 3$ be the standard Pauli 2×2-matrices,

$$\mathsf{S}^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{S}^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathsf{S}^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They form the basis together with the unit matrix E of the dimension 2 in the linear space of all complex 2×2-matrices. We represent the real matrix T of the dimension 2 in the form of the expansion according to this basis

$$\mathsf{T} = t_0\mathsf{E} + \mathsf{T}', \quad \mathsf{T}' = t_1\mathsf{S}^{(1)} + it_2\mathsf{S}^{(2)} + t_3\mathsf{S}^{(3)} = \begin{pmatrix} t_3 & t_1 + t_2 \\ t_1 - t_2 & -t_3 \end{pmatrix} \tag{7}$$

where the coefficients t_j, $j = 0, 1, 2, 3$ are real. Eigenvalues $\lambda_\pm$ of the matrix T are defined by roots of quadratic trinomial $\lambda^2 - \lambda \mathrm{Sp}\mathsf{T} + \det \mathsf{T}$ relative λ. Since matrices $\mathsf{S}^{(j)}$, $j = 1, 2, 3$ have no traces, then $\mathrm{Sp}\mathsf{T} = 2t_0$. In this case, $\det \mathsf{T} = t_0^2 - t_1^2 + t_2^2 - t_3^2$. Consequently, eigenvalues are real if and only if $0 \le (\mathrm{Sp}\mathsf{T})^2/4 - \det \mathsf{T} = t_1^2 + t_3^2 - t_2^2$. On the basis of the analysis carried out, boundary points of the interval $[\rho_-, \rho_+]$ of the hyperbolicity violation are defined by the condition $t_1^2 + t_3^2 = t_2^2$.

Consider the question of the non-diagonalizability of the matrix T. This can only be in the case if the equality is realized and there exists the eigenvalue t_0. On the other hand, it is easy verified that the matrix T' (see, (7)) is nilpotent $\mathsf{T}'^2 = 0$ in such conditions that is, the T is really not diagonalizable.

We show that obtained condition of the matrix T hyperbolicity is consistent with the conclusion of Theorem 2. To do this, we introduce together with the decomposition (7) the analogous expansion $\mathsf{D} = d_0\mathsf{E} + d_1\mathsf{S}^{(1)} + d_3\mathsf{S}^{(3)}$ where $d_2 = 0$ due to the symmetry of the matrix D. Since matrices T_+ and T_- are represented in the form $\mathsf{T}_+ = t_0\mathsf{E} + t_1\mathsf{S}^{(1)} + t_3\mathsf{S}^{(3)}$, $\mathsf{T}_- = t_2\mathsf{S}^{(2)}$ in the case under consideration, then, using the famous commutation relations of the Pauli matrices

$$\{\mathsf{S}^{(j)}, \mathsf{S}^{(k)}\} = 2\delta_{jk}\mathsf{E}, \quad [\mathsf{S}^{(j)}, \mathsf{S}^{(k)}] = 2i\varepsilon_{jkl}\mathsf{S}^{(l)}, \quad j, k = 1, 2, 3$$

where ε_{jkl} is the Levi-Civita symbol, we find

$$\{\mathsf{T}_-, \mathsf{D}\} = \{it_2\mathsf{S}^{(2)}, d_0\mathsf{E} + d_1\mathsf{S}^{(1)} + d_3\mathsf{S}^{(3)}\} = 2it_2d_0\mathsf{S}^{(2)},$$

$$[\mathsf{T}_+, \mathsf{D}] = [t_0\mathsf{E} + t_1\mathsf{S}^{(1)} + t_3\mathsf{S}^{(3)}, d_0\mathsf{E} + d_1\mathsf{S}^{(1)} + d_3\mathsf{S}^{(3)}] =$$

$$= t_1d_3[\mathsf{S}^{(1)}, \mathsf{S}^{(3)}] + t_3d_1[\mathsf{S}^{(3)}, \mathsf{S}^{(1)}] = 2i(t_3d_1 - t_1d_3)\mathsf{S}^{(2)}.$$

Substituting obtained expression in (6), we find the condition of hyperbolicity of the matrix T in the form of the following restriction $t_2d_0 = t_3d_1 - t_1d_3$ of coefficients d_0, d_1, d_3. Then, in the case when the roots of the equation (4) are multiples, it should be fulfilled $\pm d_0\sqrt{t_1^2 + t_3^2} = t_3d_1 - t_1d_3$. Consequently, introducing angles φ and ψ such that $\cos\varphi = t_1\big(t_1^2+t_3^2\big)^{-1/2}$, $\sin\varphi = t_3\big(t_1^2+t_3^2\big)^{-1/2}$, $\cos\psi = d_1\big(d_1^2+d_3^2\big)^{-1/2}$, $\sin\psi = d_3\big(d_1^2+d_3^2\big)^{-1/2}$, we obtain the condition for coefficients d_0, d_1, d_3 in the form $d_0^2 = (d_1^2+d_3^2)\cos^2(\varphi-\psi)$. If roots are multiples, then there is such a choice of angle $\varphi = -\psi$ for any angle ψ when it is fulfilled $d_0^2 = (d_1^2+d_3^2)$, that is $\det \mathsf{D} = 0$ and, therefore, the matrix D does not positive.

6. Investigation of the equation for the matrix D

Consider the equation (5) in general case. If this equation has degenerate equation D with $\det \mathsf{D} = 0$, then the kernel of the matrix D is not empty. If the vector $\mathsf{g} \in \mathrm{Ker}\,\mathsf{D}$ and, therefore, $\mathsf{Dg} = 0$, then it is follows $\mathsf{DTg} = \mathsf{T}^{\mathrm{T}}\mathsf{Dg} = 0$ from the equation (5), that is, due to arbitrariness of the vector $\mathsf{g} \in \mathrm{Ker}\mathsf{D}$ choice, the matrix kernel is invariant relative the transformation of matrix T. For this reason, further, we exclude from our consideration degenerated solutions of the equation (5).

Since the matrix T has, at least, one eigenvalue and since all its eigenvalues coincides with eigenvalues of the matrix T^{+}, then the system of uniform equations relative matrix elements $(\mathsf{D})_{ab} \equiv D_{ab}$, which follows from the matrix equation $\mathsf{T}^{\mathrm{T}}\mathsf{D} - \mathsf{DT} = 0$, has, at least, one solution. We allow that there is the antisymmetric solution $\mathsf{D}^{\mathrm{T}} = -\mathsf{D}$. Then, it follows from (5) that the equality $(\mathsf{TD})^{\mathrm{T}} = \mathsf{T}^{\mathrm{T}}\mathsf{D}+\mathsf{D}^{\mathrm{T}}\mathsf{T} = 0$ is fulfilled for such a solution. Consequently, TD and, due to nondegeneracy of the matrix D, we have $\mathsf{T} = 0$. We exclude this trivial case from our consideration. Then, the equation (5) always has the solution in the form of symmetric matrix. Indeed, let D be a solution of nontrivial matrix T. If it is not a symmetric matrix, then, applying the transposition operation to both parts of the equation, we obtain that its solution is also the matrix D^{T}. Since $\mathsf{D}^{\mathrm{T}} \neq -\mathsf{D}$, the matrix $\mathsf{D}+\mathsf{D}^{\mathrm{T}}$ is the symmetric solution of the equation (5).

Thus, the main problem is to find the conditions for the existence of a solution D of the equation (5), which is diagonalizable and all its eigenvalues are positive. In order to solve this problem, we will introduce into consideration such a set $\mathcal{T}_-$ of matrices in the space of all possible antisymmetric $n\times n$-matrices, consisting of all matrices T_- for which there is a symmetric positive solution D. The following statements, namely Theorem 6 and its related consequences, clarify the qualitative structure of the set of all hyperbolic matrices. It is valid the following

Theorem 6. *The set $\mathcal{T}_-$ of all possible matrices T_- is the centrally symmetric relative the zero matrix.*

Proof. We fix the matrix T_+ and the symmetric positive matrix D. Let $\mathsf{T}_- \in \mathcal{T}_-$. The latter matrix obeys the equation (6) at given T_+ and D. Then, due to its antisymmetry, the matrix $-\mathsf{T}_- = \mathsf{T}_-^{\mathrm{T}}$ also obeys the equation (5), and, consequently, to the equation (6). Due to choosing arbitrariness of the matrix T_-, the set $\mathcal{T}_-$ is centrally symmetric. △

Now, we will prove an auxiliary statement which has a technical sense.

Theorem 7. *Let the matrix T_- be satisfy the equation (6) together with symmetric positive matrix D which has a set of eigenvalues $\zeta_k > 0$ with eigenvectors e_k, $k = 1,\dots,n$. Then, matrix elements $T^{(\pm)}_{ab} = (\mathsf{T}_\pm)_{ab}$, $a,b = 1 \div n$ of matrices $\mathsf{T}_\pm$ which are calculated in the orthonormal basis e_k, $k = 1,\dots,n$ satisfy the equation*

$$T^{(-)}_{ab} = \frac{\zeta_b - \zeta_a}{\zeta_b + \zeta_a}\,T^{(+)}_{ab}\,. \tag{8}$$

Proof. Let us calculate the matrix elements of linear operators in both sides of the equation (6), using the scalar production $(\cdot,\cdot)$ of vectors in $\mathbf{R}^n$. Since $(\mathsf{e}_a, \mathsf{D}\mathsf{e}_b) = \zeta_a\delta_{ab}$, $(\mathsf{e}_a, T_\pm \mathsf{e}_b) = T^{(\pm)}_{ab}$, $a,b = 1,\ldots,n$, then

$$(\mathsf{e}_a, \{\mathsf{T}_-, \mathsf{D}\}\mathsf{e}_b) = T^{(-)}_{ab}(\zeta_a+\zeta_b)\,, \quad (\mathsf{e}_a, [\mathsf{T}_+, \mathsf{D}]\mathsf{e}_b) = T^{(+)}_{ab}(\zeta_b-\zeta_a).$$

On the basis of (6) we conclude that it should be fulfilled the equality of these expressions. From where it follows (8), due to $\zeta_a+\zeta_b>0$. △

Corollary 4. *For the fixed symmetric matrix T_+, each matrix $\mathsf{T}_- \in \mathcal{T}_-$ is uniquely defined by matrix elements $T^{(-)}_{ab}$, which are calculated on the basis of the equality (8) by means of the orthogonal $n\times n$-matrix W that translates the standard basis $\{\mathsf{e}^{(0)}_a = \delta_{ab}\,; b=1,\ldots,n\}$ in $\mathbf{R}^n$ into the basis $\{\mathsf{e}^{(0)}_a \ : a = ,\ldots,n\}$, and by means of the collection of eigenvalues $\langle\zeta_a>0\,; a=1,\ldots,n\rangle$.*

Proof. It follows directly from (8). △

Corollary 5. *The dimension* $\dim\mathcal{T}_-$ *does not exceed the number of nonzero matrix elements $T^{(+)}_{ab}=0$ at $a>b$.*

Proof. If $T^{(+)}_{ab}=0$ in the formula (8), then it is valid $T^{(-)}_{ab}=0$ for these values a and b. The dimension of the space of matrices $T^{(-)}$ in (8) is determined by an arbitrary choice of nonzero matrix elements for $a>b$, since this matrix is antisymmetric and $T^{(-)}_{aa}=0$. The number of such elements $T^{(-)}_{ab}$ does not exceed the numbers of the corresponding elements of the matrix $T^{(+)}_{ab}$.

Then, the validity of the theorem statement follows from the fact that for every solution of D of the equation (5) and, consequently, of the equation (6) and for any orthogonal matrix W (the orthogonality of matrices W are connected with their realness), the matrix $\mathsf{D}_W = \mathsf{W}\mathsf{D}\mathsf{W}^{\mathrm{T}}$ is the solution of the equation $\mathsf{W}\mathsf{T}_+\mathsf{W}^{\mathrm{T}}\mathsf{D}_W = \mathsf{D}_W\mathsf{W}\mathsf{T}\mathsf{W}^{\mathrm{T}}$. △

Corollary 6. *The set $\mathcal{T}_-$ is compact $\|\mathsf{T}_-\| \le \|\mathsf{T}_+\|$.*

Proof. Since $|(\zeta_b-\zeta_a)/(\zeta_b+\zeta_a)| \le 1$ at $\zeta_l>0, l=1,\ldots,n$, then

$$\|\mathsf{T}_-\| = \max\{|T^{(-)}_{ab}|\,; a,b=1,\ldots,n\} \le \|\mathsf{T}_+\|\,. \quad \triangle$$

Corollary 7. *The set $\mathcal{T}_-$ is connected.*

Proof. Changing parameters $\zeta_l(s)$ by means of a continuous dependence on the parameter $s\in[0,1]$ such that $\zeta_l(0)=\zeta_l$, $\zeta(1)=0$, $l=1,\ldots,n$ and $\mathrm{sgn}(\zeta_b(s)-\zeta_a(s)) = \mathrm{sgn}(\zeta_b-\zeta_a)$, $a,b=1,\ldots,n$, we find that, for any solution $\mathsf{T}^- \in \mathcal{T}_-$, there exists always such a continuous curve located in the subset $\mathcal{T}_-$ of the matrix space which connects the matrix T_- with zero matrix. △

Let $T_{ab} = (\mathsf{e}_a, \mathsf{T}\mathsf{e}_b)$ be matrix elements of the matrix T in the basis of eigenvectors of the matrix D. Then, it follows from the equation (5) that

$$\zeta_a T_{ab} = \zeta_b T_{ba}\,, \quad a,b=1,\ldots,n\,. \tag{9}$$

We find from this equality that original equation for the matrix D has the solution if and only if nonzero values of productions $T_{a,j_1}T_{j_1,j_2}\ldots T_{j_{s-1},j_s}T_{j_s,b}$ should be unchanged for any pair $\{a,b\}$ and for any sequence of different numbers $\langle j_1,j_2,\ldots,j_s\rangle$, $s\le n-2$. At $n=2$, since there is only one equality in (9) at $a\ne b$, then such restrictions of matrix elements T_{ab} do not arise. Therefore, we studied this case separately in previous section. But, for example, at $n = 3$ there are already three independent equalities $\zeta_1 T_{12} = \zeta_2 T_{21}$, $\zeta_2 T_{23} = \zeta_3 T_{32}$ and $\zeta_1 T_{13} = \zeta_3 T_{31}$. In this case, we have

$$\zeta_1 T_{12} T_{23} T_{31} = \zeta_2 T_{21} T_{23} T_{13} = \zeta_3 T_{32} T_{21} T_{13} = \zeta_1 T_{13} T_{32} T_{21}$$

and, consequently, due to $\zeta_1 \neq 0$, it should be fulfilled $T_{12}T_{23}T_{31} = T_{13}T_{32}T_{21}$. Namely, the presence of the entire set of specified rather fierce conditions that the matrix elements T_{ab} must obey in the basis $\{\mathsf{e}_r : r = 1, \ldots, n\}$, determines the possibility of solvability of the equations (5) and (6).

Let us prove, now, the statement, which gives a sufficient condition of the hyperbolicity of the matrix T and significantly simplifies its establishment. Thus, it gives a sufficient indication of the hyperbolicity of the system of quasi-linear equations. It essentially simplifies the hyperbolicity analysis of the matrix T and, therefore, the hyperbolicity analysis of quasi-linear equations system. We will show that the equation (6) has a symmetric positive solution D which is sufficiently close to a positive matrix $s\mathsf{E}$ for any matrix T_- in the case of non-degenerate spectrum of the matrix T_+. It is obvious that this result is agreed with conclusions of previous section in the two-dimensional case.

Theorem 8. *Let $\{\lambda_l : l = 1, \ldots, n\}$ be the collection of eigenvalues of the matrix T_+, which are nonequal in pairs to each other. Let, further, $\delta = \min\{|\lambda_j - \lambda_k| : j \neq k; j, k = 1, \ldots, n\}$. Then, at $\delta > 2n\varepsilon\|\mathsf{T}_-\| > 0$, there exists symmetric positive matrix D, satisfying the equation*

$$[\mathsf{T}_+, \mathsf{D}] = \varepsilon\{\mathsf{T}_-, \mathsf{D}\}\,. \tag{10}$$

Proof. We represent the solution of the equation in the form of series

$$\mathsf{D} = \sum_{l=0}^{\infty} \varepsilon^l \mathsf{D}^{(l)} \tag{11}$$

where $[\mathsf{T}_+, \mathsf{D}^{(0)}] = 0$. Further, we choose the matrix $\mathsf{D}^{(0)}$ in the form $\mathsf{D}^{(0)} = s\mathsf{E}$ where $s > 0$ is sufficiently large value such that the matrix D is positive and $[\mathsf{T}_+, \mathsf{D}^{(l)}] = \{\mathsf{T}_-, \mathsf{D}^{(l-1)}\}$, $l \in \mathbf{N}$. Let $\{\mathsf{e}_l : l = 1, \ldots, n\}$ be the orthonormal basis of eigenvectors of the matrix T_+ with eigenvalues $\{\lambda_l : l = 1, \ldots, n\}$. Write down the equality of matrix elements in the basis $\{\mathsf{e}_l : l = 1, \ldots, n\}$, which is followed from the last recurrent relation

$$(\mathsf{e}_j, [\mathsf{T}_+, \mathsf{D}^{(l)}]\mathsf{e}_k) = (\lambda_j - \lambda_k)(\mathsf{e}_j, \mathsf{D}^{(l)}\mathsf{e}_k) = (\mathsf{e}_j, \{\mathsf{T}_-, \mathsf{D}^{(l-1)}\}\mathsf{e}_k)\,.$$

Since the matrix $\{\mathsf{T}_-, \mathsf{D}^{(l-1)}\}$ is antisymmetric when $\mathsf{D}^{(l-1)}$ is symmetric matrix, then right-hand side of the last equality is equal to zero at $j = k$. Without contradicting this equality, we suppose $(\mathsf{e}_k, \mathsf{D}^{(l)}\mathsf{e}_k) = 0$, $k = 1, \ldots, n$. Consequently, considering it as the equation relative nondiagonal matrix elements of the matrix, we find that it is $\mathsf{D}^{(l)}$, this equation is solvable and it takes place the recurrent communication

$$(\mathsf{e}_j, \mathsf{D}^{(l)}\mathsf{e}_k) = (\lambda_j - \lambda_k)^{-1} \times \\ \times \sum_{m=1}^{n} \Big[(\mathsf{e}_j, \mathsf{T}_-\mathsf{e}_m)(\mathsf{e}_m, \mathsf{D}^{(l-1)}\mathsf{e}_k) + (\mathsf{e}_j, \mathsf{D}^{(l-1)}\mathsf{e}_m)(\mathsf{e}_m, \mathsf{T}_-\mathsf{e}_k)\Big]\,.$$

From where, it follows the estimate

$$|(\mathsf{e}_j, \mathsf{D}^{(l)}\mathsf{e}_k)| \le \|\mathsf{D}^{(l)}\| \le \frac{2n}{\delta}\|\mathsf{T}_-\|\|\mathsf{D}^{(l-1)}\|\,.$$

Consequently the series is converged according to the matrix norm $\| \cdot \|$ at $2n\varepsilon\|\mathsf{T}_-\|/\delta < 1$ and the following estimate is valid

$$\|\mathsf{D}\| \leq \|\mathsf{D}^{(0)}\| \sum_{l=0}^{\infty} \varepsilon^l \left(\frac{2n\|\mathsf{T}_-\|}{\delta}\right)^l = \frac{s}{1 - 2n\varepsilon\|\mathsf{T}_-\|/\delta}. \quad \triangle$$

7. Conclusion

We have carried out the investigation devoted to the problem of strong hyperbolicity of quasi-linear equations systems of the first order. As a result, a modified formulation of the strong hyperbolicity concept is proposed and the equivalence of this formulation and the so-called hyperbolicity according Friedrichs is proved. The necessary and sufficient condition is formulated that this equations system is hyperbolic. It is done in terms of the matrix $\mathsf{T}(\boldsymbol{q})$ which defines evolution of a linearized (tangent) system of differential equations of the first order with constant coefficients corresponding to the original system. This condition is given in Corollary 3. We have named matrices $\mathsf{T}(\boldsymbol{q})$ satisfying this condition as the hyperbolic ones.

We conducted a qualitative study of the set of matrices $\mathsf{T}_-(\boldsymbol{q})$ for which the specified necessary and sufficient condition holds for a given matrix $\mathsf{T}_+(\boldsymbol{q})$, that is, when there is a solution to the equation (6). Moreover, we have found an effectively verifiable necessary condition for the class of matrices $\mathsf{T}_-(\boldsymbol{q})$, in order for such a solution to actually exist.

However, it is necessary further study of the problem despite the fact that the obtained results allow, in some cases (see, for example, [12], [13]), to prove the hyperbolicity of the quasi-linear equations systems of mathematical physics. Namely, it is necessary to obtain such quantitative results that will allow to accurately estimate both from above and from below the location of the hyperbolicity region boundary for each pre-defined matrix $\mathsf{T}(\boldsymbol{q})$. It should be noted that the problem closely related to the one studied in the paper is also of interest. Namely, it is important to find the criterion that the system (1) is elliptical. Apparently, in this case, the matrix $\mathsf{T}(\boldsymbol{q})$ is diagonalizable and it has purely imaginary eigenvalues. Such matrices are naturally to call as the elliptical ones.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Rozhdestvenskii, B.L., Yanenko N.N. *Systems of Quasilinear Equations and Their Applications to Gas Dynamics*. AMS, Providence, RI, 1983.
2 Lax, P.D. *Hyberbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*. Society for industrial and applied mathematics, Philadelphia, 1973.
3 Jeffrey, A. *Quasilinear Hyperbolic System and Waves*. Pitman, London, 1976.
4 Majda, A. The existence of multi-dimensional shock fronts. *Memoirs of the American Mathematical Society*. Providence, RI, 43, 281, 1983.
5 Rykov, V.V., Filimonov, A.M. Hyperbolic systems with multiple characteristic and applications. *UBS*. **2020**, *85*,72-86.

6 Rozhdestvenskii, B.L. Discontinuous solutions of hyperbolic systems of quasi-linear equations. *Russian Math. Surveys*, **1960**, **15(6)**, 55–111.

7 Palis, J., Takens, F. *Hyperbolicity and sensitive-chaotic dynamics at homoclinic bifurcations*. Cambridge University Press, 1993.

8 Araujo, V., Viana, M. Hyperbolic dynamical systems. arXiv:0804.3192v1 [math.DS] 20 Apr 2008.

9 Kulikovskli, A.G., Slobodkina, F.A. Equilibrium of arbitrary steady flows at the transonic points. *Applied mathematics and Mechanics*. **1968**, *31*, 593-602.

10 Kulikovskli A.G., Pogorelov N.V., Semenov A.Yu. *Mathematical Aspects of Numerical Solution of Hyperbolic Systems*. Chapman and Hall/CRC, New York, 2001.

11 Virchenko Yu.P. Subbotin A.V. The class of evolutionary ferrodynamic equations. Mathematical methods in Applied Sciences.– 2021.– 44.– 15.— P.11913-11922.

12 Virchenko, Yu.P., Novosel'tseva, A.E. Class of hyperbolic first order quasi-linear covariant equations of divergent type. In *Itogi nauki. Modern mathmatics. Thematic reviews.*; Gamreklidze R.V., Eds.; 2020, Moscow, pp.19–30 (in Russian).

13 Virchenko, Yu.P., Pleskanev, A.A. Hyperbolic spherically symmetric equation of the first order of divergent type for wector field. *Belgorod State University Scientific Bulletin. Mathematics & Physics.* **2019**, *51;2*, 280-286 (in Russian).

14 Friedrichs K. Nonlinear hyperbolic differential equations for functions of two independent variables. *Amer. J. Math.* **1948**, *70*, 555589.

15 Gantmacher, F.R. *The Theory of Matrices*, Chelsea, NY, 1959.

16 Roger, A.H., Charles, R.J. *Topics in matrix analysis*. Cambridge University Press, Cambridge, 1991.