ISOTROPIC COVARIANT TENSORS IN $\mathbb{R}^n$

Yu. P. Virchenko[1,*]

[1] *Belgorod State University, Pobedy ul. 18, Belgorod, 308000 Russia;*
Belgorod State Technological University named after V.G. Shukhov, Belgorod, 308000 Russia

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Abstract—Linear spaces $\mathfrak{L}_r^{(n)}$ of invariant tensors in $\mathbb{R}^n$, $n \geq 2$ with the rank $r \in \mathbb{N}$, $r \geq 2$ which are covariant under transformations of the $\mathbb{O}_n$-group are studied. Statements about the dimension $d(r,n)$ of $\mathfrak{L}_r^{(n)}$ and basis tensors in them are proved. It is shown that multiplicative δ-tensors are linear dependent at sufficiently large value of difference $r - 2n$ and in this case $\dim \mathfrak{L}_r^{(n)} < (r-1)!!$ is valid when r is even. The linear independence theorems are proved when $2n \geq r$ and $r = 6$, $n = 2$. It is found the upper estimate $\bar{d}(r,n)$ of the dimension $d(r,n)$ at $r > 2n$.

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1. INTRODUCTION

The concept of invariant tensors (isotropic tensors, at other terminology) in spaces $\mathbb{R}^n$ has been proposed at last century (see [1]-[3]) at the era of stormy development of tensor analysis connected with applications in physics and, in particular, with the A.Einstein thesis of the covariant fundamental equations describing physical fields. This concept turned out to be very important as from theoretical viewpoint (see, for example, [4]-[6]) in tensor algebra as from viewpoint of applications in mechanics and physics (see, for example,[7], [8]).

Invariant tensors of the fixed rank $r \in \mathbb{N}$, $r \geq 2$ and the definite type which are real- or complex-valued, which have the constancy property of their components at action of $\mathbb{O}_n$-group transformations in each space $\mathbb{R}^n$, $n \in \mathbb{N}$ form the linear space over the correspondent algebraic field. The proposed work is devoted to investigation of linear spaces of covariant tensors over real field and, further, we will use the designation $\mathfrak{L}_r^{(n)}$, $r \in \mathbb{N}$ for these spaces. The basic result of invariant tensors theory is the statement that each such a space is either trivial when r is odd or the linear space which is built on the basis of all multiplicative tensors of fixed rank r generated by the universe tensor $\boldsymbol{\delta}$ of second rank when r is even.

However, from our opinion, a sufficient attention is not devoted to the question concerns the linear independence of the pointed out tensor set and, therefore, to the determination of the dimension $d(r,n) \equiv \dim\, \mathfrak{L}_r^{(n)}$ (see, for example, the rather recent review [9], and also [10], [11]). This question arises naturally when the basis building in the space of form-invariant tensor-functions [6], [12], and also in connection with the calculation of the integral rational basis of invariants (see, for example, [4]). Answers on the pointed out questions were turned out essential in problems connected with the construction of covariant evolution equations (see, for example, [13]). Our work is devoted to the proof that the dimension $\dim \mathfrak{L}_r^{(n)} \equiv d(r,n)$ is equal $(r-1)!!$ at the even value r, $2n \geq r$. We give some necessary definitions and describe general properties of invariant tensors in the spaces $\mathbb{R}^n$, $n \geq 2$ and we propose the rather transparent proof of the coincidence $\mathfrak{L}_r^{(n)}$ with correspondent linear shell of δ-multiplicative tensors. We restrict our consideration only covariant tensors over real field since the transfer of obtained results for more general case must not caused some difficulties. Main results of the work are given in statements of Theorems 5 and 9-11.

* E-mail: virch@bsu.edu.ru

2. INVARIANT TENSORS

Let be $I_n = \{1, 2, ..., n\}$. We will call the mapping $\mathsf{A} : I_n^r \mapsto \mathbb{R}$, $n \in \mathbb{N}_+$, $r \in \mathbb{N}_+$ an *algebraic symbol* [1)] with the dimensionality n and the rank r. The value of the symbol A correspondent to ordered set $\langle j_1, ..., j_r \rangle \in I_n^r$ which we will call the symbol component and will write down in the form $A_{j_1,...,j_r}$. The components j_k, $k = 1 \div r$ of the ordered set are named indexes of the symbol A. They take its values in I_n. Natural linear operations are defined for symbols with fixed r and n. They are the addition of components and their multiplication by any real number $a \in \mathbb{R}$. Thus, the set of all symbols with fixed r and n forms the linear space by obvious way, which has the dimension n^r. The zero symbol in this space is $A_{j_1,...,j_r} = 0$, $\langle j_1, ..., j_r \rangle \in I_n^r$.

Let $\mathsf{A} = \{A_{j_1,...,j_r}; \langle j_1, ..., j_r \rangle \in I_n^r\}$ be the fixed symbol which has the dimension n and the rank r. Let each orthogonal matrix $\mathsf{U} \in \mathbb{O}_n$ with matrix elements $(\mathsf{U})_{j,k}$, $j, k \in I_n$ be connected with the symbol $\mathsf{A}' = \{A'_{j_1,...,j_r}; \langle j_1, ..., j_r \rangle \in I_n^r\} \in \mathfrak{A}$ in a one - to - one way according to the formula

$$A'_{j_1,...,j_r} = U_{j_1,k_1} ... U_{j_r,k_r} A_{k_1,...,k_r} \,, \qquad U_{j,k} = (\mathsf{U})_{j,k} \,. \tag{1}$$

Therefore, if $\mathsf{A}' = \mathsf{A} = \{A_{j_1,...,j_r}; \langle j_1, ..., j_r \rangle \in I_n^r\}$, then it corresponds to the unit matrix $\mathsf{E} \in \mathbb{O}_n$. The set $\mathfrak{A}$ of all such symbols generated by all matrices $\mathsf{U} \in \mathbb{O}_n$ and by the symbol A is called the covariant tensor in the space $\mathbb{R}^n$ or the tensor representation of the group $\mathbb{O}_n$ which have the rank $r \in \mathbb{N}$. Here and after in the text, we use the tensor agreement. The availability of pair-repeated indexes in a formula signs the summation on each such a pair in the limits $1 \div n$. In the formula (1) and further in the text, we do not use the rule of tensor algebra (see, for example, [5]) that the summation on pair-repeated indexes is possible only in the case when such a pair consists of a covariant index and the contravariant one. It is done because we consider only covariant tensors in the work.

Each symbol $\mathsf{A}' = \{A'_{j_1,...,j_r}; \langle j_1, ..., j_r \rangle \in I_n^r\}$, corresponding to the matrix U, which is contained in $\mathfrak{A}$, is called the coordinate representation of the tensor $\mathfrak{U}$. The fixed symbol $A'_{j_1,...,j_r}$ of $\mathfrak{A}$ with the value set $\langle j_1, ..., j_r \rangle$ of indexes is called the realization of the tensor $\mathfrak{A}$ in fixed basis defined by the matrix U. Further, we will say only about covariant tensors and, therefore, the word "covariant" is omitted.

Obviously, that linear operations are defined by hereditary way for tensors of fixed rank r in the space $\mathbb{R}^n$. Therefore, all set of such tensors compiles the linear space. The zero tensor is the null of this space such that $A_{j_1,...,j_r} = 0$, $\langle j_1, ..., j_r \rangle \in I_n^r$ in any orthonormalized basis.

Definition 1. *The tensor $\mathfrak{A}$ of the rank $r \in \mathbb{N}$ is called invariant in the space $\mathbb{R}^n$, if its coordinate representation $A_{j_1,...,j_r}$, $\langle j_1, ..., j_r \rangle \in I_n^r$ is not changed when any matrix $\mathsf{U} \in \mathbb{O}_n$ acts according to the formula* (1)*, i.e. for any such a matrix takes place* [2)]

$$U_{j_1,k_1} ... U_{j_r,k_r} A_{k_1,...,k_r} = A_{j_1,...,j_r} \,, \quad \langle j_1, ..., j_r \rangle \in I_n^r \,. \tag{2}$$

It is obvious that the zero tensor is invariant at any $r \in \mathbb{N}$. The result of application linear operations to invariant tensors is the invariant tensor. Consequently, it follows from the definition that the set of invariant tensors of fixed rank r in $\mathbb{R}^n$ forms the linear space. We denote it by $\mathfrak{L}_r^{(n)}$.

In future, we suppose that the invariant tensors in spaces $\mathfrak{L}_r^{(n)}$ are not zero when the statements concerning them are formulated and proved. Besides, since symbols of invariant tensors are not changed, when the set of symbols defining the invariant tensor $\mathfrak{A}$ consists of one element A, then, it has a sense to operate only symbols when invariant tensors studying. Therefore, not pointing out everytime, we will not further to distinguish invariant tensors and their symbols which are their realizations in fixed basis.

Theorem 1. *The rank of any nonzero invariant tensor is an even number.*

Proof. Let us suppose that $\mathsf{U} = -\mathsf{E}$ in (2), i.e. $U_j^k = \delta_{j,k} = \{1, j = k; 0, j \neq k\}$is the Kronecker symbol. Then $U_{j_1}^{k_1} ... U_{j_r}^{k_r} A_{k_1,...,k_r} = (-1)^r A_{j_1,...,j_r} \neq A_{j_1,...,j_r}$, $\langle j_1, ..., j_r \rangle \in I_n^r$ when r is odd. □

[1)] We use this term instead of the term *algebraic object* that is used in monographs [7], [8].

[2)] We does not consider so-called relatively invariant tensors.

Theorem 2. *The space $\mathfrak{L}_2^{(n)}$ is one-dimensional and each invariant tensor* A *of second rank has the form* $\mathsf{A} = c\,\mathsf{E}$*,* $c \in \mathbb{R}$.

Proof. We fix the dimension $n \geq 2$ of the space $\mathbb{R}^n$. Let us U is an orthogonal $n \times n$-matrix. The tensor A of second rank, according to its definition, has the form of the $n \times n$-matrix with elements $(\mathsf{A})_{j,k} = A_{j,k}$, $j, k = 1 \div n$ in coordinate representation. According to (1), the equality

$$U_{j,j'}U_{k,k'}A_{j',k'} = (\mathsf{UAU}^{\mathrm{T}})_{j,k} = A_{j,k}\,, \quad \langle j,k\rangle \in I_n^2$$

takes place for the matrix A due to the invariance of the tensor A. Consequently, the matrix A commutes with any orthogonal matrix U, $[\mathsf{U},\mathsf{A}] = 0$. From here, it follows that matrices A and U have the same set of orthonormalized eigenvectors which is complete in $\mathbb{C}^n$. The matrix $\mathsf{U} \in \mathbb{O}_n$ is arbitrary. Consequently, any vector in $\mathbb{C}^n$ is the eigenvector for the matrix A since there is such an orthogonal matrix for any complete set of orthonormalized vectors when all vectors of the set are eigenvectors. Such a situation is possible only in the case when $\mathsf{A} = c\,\mathsf{E}$, $c \in \mathbb{C}$. Due to the fact that the matrix A is real-valued, the multiplier c is real. □

From the invariance of tensor $\mathsf{E} = \{\delta_{jk}; \langle j,k\rangle \in I_n\}$ of second rank in $\mathbb{R}^n$, it follows the invariance of all tensors of any even rank belonging to tensor algebra with the operation of *tensor multiplication* (see, for example, [5]) and with one generating element E.

If the set $\mathfrak{c}$ of pairs $\{j,k\} \subset I_{2m}$ forms the disjunctive division $\bigcup_{\{j,k\}\in\mathfrak{c}}\{j,k\} = I_{2m}$, we call this set the paired division of I_{2m}. We designate the class of all its paired division $\mathfrak{c}$ by mens of $\mathfrak{C}_{2m}(I_n)$. It is obvious that the number of all paired division $\mathfrak{c}$ of the set I_{2m} is equal $(2m-1)!! \equiv (2m)!/2^m m!$. We introduce also the designation $\mathfrak{C}(I_n, \Sigma)$ for the class of all paired division $\mathfrak{c}$ of any set Σ consisting of an even number of elements.

Now, we will link each division $\mathfrak{c} \in \mathfrak{C}_{2m}(I_n)$ with the tensor $\mathsf{D}(\mathfrak{c})$ with components

$$D_{j_1,...,j_{2m}}(\mathfrak{c}) = \prod_{\{k,l\}\in\mathfrak{c}} \delta_{j_k,j_l}\,.$$

Theorem 3. *For any $n \geq 2$ and any paired division $\mathfrak{c} \in \mathfrak{C}_{2m}(I_n)$, $m \in \mathbb{N}$, the tensor $\mathsf{D}(\mathfrak{c})$ is invariant in $\mathbb{R}^n$, i.e. it belongs to the space $\mathfrak{L}_{2m}^{(n)}$.*

Proof. We will fix the value $n \geq 2$. Let U be an arbitrary matrix of $\mathbb{O}_n$. Using the disjunctivity of the division $\mathfrak{c} = \bigcup_{\{p,q\}\in\mathfrak{c}}\{p,q\} = I_{2m}$ and orthogonality of the matrix U, we will write down the transformation of components $D_{j_1,...,j_r}(\mathfrak{c})$ of the tensor $\mathsf{D}(\mathfrak{c})$ generated of this matrix,

$$U_{j_1,k_1}...U_{j_r,k_r}D_{k_1,...,k_r}(\mathfrak{c}) = U_{j_1,k_1}...U_{j_r,k_r}\prod_{\{p,q\}\in\mathfrak{c}}\delta_{k_p,k_q} = \prod_{\{p,q\}\in\mathfrak{c}} U_{j_p,k_p}\delta_{k_p,k_q}U_{j_q,k_q} =$$

$$= \prod_{\{p,q\}\in\mathfrak{c}} U_{j_p,k}U_{j_q,k} = \prod_{\{p,q\}\in\mathfrak{c}}\delta_{j_p,j_q} = D_{j_1,...,j_r}(\mathfrak{c})\,.$$

According to (2), this is meant the invariance of tensor $\mathsf{D}(\mathfrak{c})$. □

Let $\mathfrak{A} \in \mathfrak{L}_r^{(n)}$ be an arbitrary tensor for which the condition of invariance (2) is fulfilled for any orthogonal matrix U. In particular, it is fulfilled, if this matrix belongs to the subgroup $\mathbb{SO}_n$ of the group $\mathbb{O}_n$ (see, for example, [14]). In this case the equality $\det\mathsf{U} = 1$ takes place for the matrix U and, therefore, this matrix may be represented in the form $\mathsf{U} = \exp\big(\omega\mathsf{T}\big)$ where T is antisymmetric matrix and $\omega \in \mathbb{R}$.

Let us introduced the orthogonal matrices $\mathsf{U}^{(\alpha,\beta)} = \exp(\omega\mathsf{T}^{(\alpha,\beta)})$, $\{\alpha,\beta\} \subset I_n$, $\omega \in \mathbb{R}$ with matrix elements $U_{jk}^{(\alpha,\beta)} = \big(\mathsf{U}^{(\alpha,\beta)}\big)_{j,k}$, $\{j,k\} \subset I_n$ which belong to the subgroup $\mathbb{SO}_n$. Matrices $\mathsf{T}^{(\alpha,\beta)}$ are defined by such a way that $\big(\mathsf{T}^{(\alpha,\beta)}\big)_{j,k} = 0$ if $\{j,k\} \neq \{\alpha,\beta\}$, and their values at $\langle j,k\rangle = \langle\alpha,\beta\rangle \in I_2^2$

coincide with values of the antisymmetric symbol $\varepsilon_{j,k}$, $\varepsilon_{j,k} = -\varepsilon_{k,j}$, $\varepsilon_{j,k} = 1$, $k > j$ in two-dimensional space that is the linear shell of vectors $\mathbf{e}_j, \mathbf{e}_k$ in $\mathbb{R}^n$.

The matrices $\mathsf{T}^{(\alpha,\beta)}$ have eigenvalues $\{\pm i, 0\}$. The corresponding eigenvectors $\mathbf{v}^{(\pm)} = \langle x_1^{(\pm)}, ..., x_n^{(\pm)}\rangle$, $\mathbf{v}^{(0)} = \langle x_1^{(0)}, ..., x_n^{(0)}\rangle$ have components given by following formulas. If, for definiteness $\alpha > \beta$, then $x_\alpha^{(\pm)} = 1$, $x_\beta^{(\pm)} = \pm i$ and $x_j^{(\pm)} = 0$ at $j \neq \alpha, \beta$. Exactly the same, we suppose $x_\alpha^{(0)} = x_\beta^{(0)} = 0$ and $x_j^{(0)} = x_j \in \mathbb{C}$, $j \neq \alpha, \beta$ are arbitrary, but they are not equal to zero at the same time.

The proof of the fundamental theorem giving the description of all tensors in each space $\mathfrak{L}_r^{(n)}$ with r and $n \geq 2$, is based on the following statement.

Theorem 4. *For any $r = 2m$, $m \in \mathbb{N}$, the space $\mathfrak{L}_r^{(2)}$ coincides with the linear shell of all tensors* $\mathsf{D}(\mathfrak{c})$*, $\mathfrak{c} \in \mathfrak{C}_{2m}(I_n)$.*

Proof. Suppose $\mathsf{T} = \mathsf{T}^{(1,2)}$,

$$\mathsf{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{3}$$

We represent a component $A_{j_1,...,j_r}$, $j_s \in \{1, 2\}$, $s = 1 \div 2m$ of an arbitrary invariant tensor A in $\mathfrak{L}_{2m}^{(2)}$ in the form of following decomposition

$$A_{j_1,...,j_r} = \sum_{\langle\alpha_1,...,\alpha_r\rangle \in \{\pm\}^r} a_{\alpha_1,...,\alpha_r} \prod_{s=1}^{r} x_{j_s}^{(\alpha_s)} \tag{4}$$

with some coefficients $a_{\alpha_1,...,\alpha_r} \in \mathbb{C}$, $\alpha_s \in \{\pm\}$ which depend on $A_{j_1,...,j_r}$, $s = 1 \div r$, $r = 2m$, using an arbitrary pair of mutually orthogonal vectors $\mathbf{v}^{(\pm)} = \langle x_1^{(\pm)}, x_2^{(\pm)}\rangle$ in $\mathbb{C}^2$.

Further, let $\mathbf{v}^{(+)} = \langle 1, i\rangle$; $\mathbf{v}^{(-)} = \langle 1, -i\rangle$ be eigenvectors of the matrix T (3) with corresponding eigenvalues $+i$ and $-i$. The matrix $\mathsf{U} = \exp(\omega\mathsf{T}) \in \mathbb{SO}_2$ is orthogonal since $\mathsf{U}^T = \exp(\omega\mathsf{T}^T) = \exp(-\omega\mathsf{T})$ and, therefore, $\mathsf{U}\mathsf{U}^T = \mathsf{U}^T\mathsf{U} = \mathsf{E}$ and also $\det \exp(\omega\mathsf{T}) = \exp(\omega \mathrm{Sp}\mathsf{T}) = 1$. Then, for any such a matrix, we have

$$U_{j_1,k_1}...U_{j_r,k_r}A_{k_1,...,k_r} = \sum_{\langle\alpha_1,...,\alpha_r\rangle \in \{\pm\}^r} a_{\alpha_1,...,\alpha_r} U_{j_1,k_1}...U_{j_r,k_r} \prod_{s=1}^{r} x_{k_s}^{(\alpha_s)} =$$

$$= \sum_{\langle\alpha_1,...,\alpha_r\rangle \in \{\pm\}^r} a_{\alpha_1,...,\alpha_r} \exp(i\omega \sum_{s=1}^{r} \alpha_s) \prod_{s=1}^{r} x_{j_s}^{(\alpha_s)}$$

where $\sum_{s=1}^{r} \alpha_s = s_+ - s_-$, $s_+ = |\Sigma(\boldsymbol{\alpha})|$, $s_- = r - |\Sigma(\boldsymbol{\alpha})|$, $\Sigma(\boldsymbol{\alpha}) = \{j : \alpha_j = +\}$, $\boldsymbol{\alpha} = \langle\alpha_j \in \{\pm\}\, ; j = 1 \div r\rangle$.

Comparing with (4), we see that $A_{j_1,...,j_r}$ coincides with the obtained expression only in the case when each coefficient $a_{\alpha_1,...,\alpha_r}$ of the summand vanishes if the difference $(s_+ - s_-)$ does not equal to zero. Due to arbitrary value $\omega \in \mathbb{R}$, it takes place independently for all summands having the same value of the difference pointed out. This situation is connected with that $\mathbf{v}^{(+)} \perp \mathbf{v}^{(-)}$ and the products $\left(\prod_{s\in\Sigma(\boldsymbol{\alpha})} x_{j_s}^{(+)}\right)\left(\prod_{s\in I_r\setminus\Sigma(\boldsymbol{\alpha})} x_{j_s}^{(-)}\right)$ are independent from each other if multipliers are different.

Since the number set $\Sigma(\boldsymbol{\alpha}) \subset I_r$ corresponds by one-to-one way to each set $\boldsymbol{\alpha} = \langle\alpha_1, ..., \alpha_r\rangle$, the sum on sets $\langle\alpha_1, ..., \alpha_r\rangle$ may be changed on the sum on $\Sigma(\boldsymbol{\alpha})$

$$A_{j_1,...,j_r} = \sum_{\Sigma(\boldsymbol{\alpha})\subset I_r\, :\, |\Sigma(\boldsymbol{\alpha})| = r/2} a_{\alpha_1,...,\alpha_r} \prod_{s\in\Sigma(\boldsymbol{\alpha})} x_{j_s}^{(+)} x_{j'_s}^{(-)} \tag{5}$$

where j'_s are indexes from $I_r \setminus \Sigma(\boldsymbol{\alpha})$ which are written in the same order ascending as the indexes in $\Sigma(\boldsymbol{\alpha})$.

We notice that

$$x_j^{(+)} x_k^{(-)} = \begin{pmatrix} 1 \\ i \end{pmatrix}_j \begin{pmatrix} 1 \\ -i \end{pmatrix}_k = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}_{j,k} = (\mathsf{E} - i\mathsf{T})_{jk} = \delta_{jk} - i\varepsilon_{jk}$$

takes place for any ordered pair $\langle j, k\rangle \in I_n^2$. Then it follows from (5) that

$$A_{j_1,\dots,j_r} = \sum_{\Sigma(\boldsymbol{\alpha})\subset I_r\,:\,|\Sigma(\boldsymbol{\alpha})|=r/2} a_{\alpha_1,\dots,\alpha_r} \prod_{s\in\Sigma(\boldsymbol{\alpha})} (\mathsf{E} - i\mathsf{T})_{j_s,j_s'} =$$

$$= \sum_{\Sigma(\boldsymbol{\alpha})\subset I_r\,:\,|\Sigma(\boldsymbol{\alpha})|=r/2} a_{\alpha_1,\dots,\alpha_r} \prod_{s\in\Sigma(\boldsymbol{\alpha})} (\delta_{j_s,j_s'} - \varepsilon_{j_s,j_s'})\,. \tag{6}$$

Consequently, the symbol A with components $A_{j_1,\dots,j_r}$ belongs to the complex linear shell of symbols with components

$$\Big(\prod_{\{j,k\}\in\mathfrak{c}'\subset\mathfrak{c}} \delta_{j,k}\Big)\Big(\prod_{\{j,k\}\in\mathfrak{c}\setminus\mathfrak{c}'} \varepsilon_{j,k}\Big)\,, \quad \mathfrak{c}\in\mathfrak{C}_r(I_n)\,. \tag{7}$$

Besides, the imaginary part of each its component consists of real linear shell of components in which second product contains add number multipliers.

We use the famous identity [7]

$$\varepsilon_{jk}\varepsilon_{lm} = \det\begin{pmatrix} \delta_{jl} & \delta_{jm} \\ \delta_{kl} & \delta_{km} \end{pmatrix}\,, \quad \langle j,k,l,m\rangle \in I_2^4$$

for transformation of the expression (6). Then second products in (7) belongs to real linear shell of symbols

$$\sum_{\{l,m\}\in\mathfrak{c}\setminus\mathfrak{c}'} \varepsilon_{l,m} \prod_{\{j,k\}\in(\mathfrak{c}\setminus\mathfrak{c}')\setminus\{\{l,m\}\}} \delta_{j,k}\,.$$

Since it is fulfilled $U_{j,j'}U_{k,k'}\varepsilon_{j',k'} = -\varepsilon_{j,k}$ in the case when the orthogonal matrix of $\mathbb{O}_2$ is equal $\mathsf{U} = \mathrm{diag}\{1,-1\}$, the coefficients $a_{l_1,\dots,l_r}$ should be such that the summand is absent if it is the product of an odd number of symbols $\boldsymbol{\varepsilon}$. Finally, we find from (6) that $A_{j_1,\dots,j_r}$ belongs to real linear shell of symbols $\mathsf{D}(\mathfrak{c})$, $\mathfrak{c}\in\mathfrak{C}_r(I_n)$ at $n=2$. □

3. THE $\mathfrak{L}_r^{(n)}$ SPACES

Now, we will prove that the more general statement is valid. The space $\mathfrak{L}_r^{(n)}$ should be coincide with linear shell $\mathfrak{L}[D(\mathfrak{c}); \mathfrak{c}\in\mathfrak{C}_r(I_n)]$ at all values $n\in\mathbb{N}$, $n\ge 2$. We notice that this fact is proved for $n=3$ in frames of the theory of invariants (see [14]).

Theorem 5. *For any $r=2m$, $m\in\mathbb{N}$, the space $\mathfrak{L}_r^{(n)}$, $n=2,3,\dots$ coincides with the linear shell of all tensors $\mathsf{D}(\mathfrak{c})$, $\mathfrak{c}\in\mathfrak{C}_{2m}(I_n)$.*

Proof. Let A be an arbitrary tensor of the even rank $r\in\mathbb{N}$ in the space $\mathfrak{L}_r^{(n)}$. We prove that it belongs to the linear shell $\mathfrak{L}[D(\mathfrak{c}); \mathfrak{c}\in\mathfrak{C}_r(I_n)]$, $r=2m$, i.e. it has the form

$$\mathsf{A} = \sum_{\mathfrak{c}\in\mathfrak{C}_r(I_n)} a(\mathfrak{c})\mathsf{D}(\mathfrak{c})\,, \tag{8}$$

$a(\mathfrak{c})\in\mathbb{R}$, $\mathfrak{c}\in\mathfrak{C}_r(I_n)$. The proof consists of following items **1** - **5**.

1. For any orthogonal matrix $\mathsf{U} \in \mathbb{O}_n$, one may find the orthogonal basis of its eigenvectors $\mathbf{x}^{(\alpha)}$, $\alpha = 1 \div n$ such that $\mathsf{U}\mathbf{x}^{(\alpha)} = e^{\omega_\alpha}\mathbf{x}^{(\alpha)}$, $\mathbf{x}^{(\alpha)} = \langle x_j^{(\alpha)}; j = 1 \div n\rangle$, $\omega_\alpha \in \mathbb{R}$, $\alpha = 1 \div n$. Then we represent the symbol A of the rank $r \in \mathbb{N}$ in the form of the decomposition

$$A_{j_1,...,j_r} = \sum_{\langle \alpha_1,...,\alpha_r\rangle \in I_n^r} a_{\alpha_1,...,\alpha_r} x_{j_1}^{(\alpha_1)} ... x_{j_r}^{(\alpha_r)} .$$

Since A is the invariant tensor, the following equality takes place

$$A_{j_1,...,j_r} = U_{j_1,k_1}...U_{j_r,k_r} A_{k_1,...,k_r} = \sum_{\langle \alpha_1,...,\alpha_r\rangle \in I_n^r} a_{\alpha_1,...,\alpha_r} \big(U_{j_1,k_1} x_{j_1}^{(\alpha_1)}\big)...\big(U_{j_r,k_r} x_{j_r}^{(\alpha_r)}\big) =$$

$$= \sum_{\langle \alpha_1,...,\alpha_r\rangle \in I_n^r} a_{\alpha_1,...,\alpha_r} \exp(\omega_{\alpha_1} + ... + \omega_{\alpha_r})\, x_{j_1}^{(\alpha_1)} ... x_{j_r}^{(\alpha_r)} \qquad (9)$$

due to arbitrariness of the matrix U.

Due to orthogonality of the matrix U, for each vector $\mathbf{x}^{(\alpha)}$, there is the unique eigenvector $\mathbf{x}^{(\bar\alpha)}$ such that $\mathsf{U}\mathbf{x}^{(\bar\alpha)} = e^{-\omega_\alpha}\mathbf{x}^{(\bar\alpha)}$ and also $\mathbf{x}^{(\bar{\bar\alpha})} = \mathbf{x}^{(\alpha)}$, $\alpha = 1 \div n$ takes place. Since the matrix U is arbitrary, so the set $\langle \omega_\alpha; \alpha = 1 \div n\rangle$ may be arbitrary. Consequently, for the validity of the equality (9), it is necessary and sufficient that the sum $\sum_{\langle \alpha_1,...,\alpha_r\rangle \in I_n^r}$ contains only such summands with ordered sets $\langle \alpha_1, ..., \alpha_r\rangle \in I_n^r$ for which the correspondent set $\langle \omega_\alpha; \alpha = 1 \div n\rangle$ is divided on pairs mutually opposite values. We denote this condition of the summing by the sign $*$ over the sum. Then

$$A_{j_1,...,j_r} = \sum^{*}_{\langle \alpha_1,...,\alpha_r\rangle \in I_n^r} a_{\alpha_1,...,\alpha_r} x_{j_1}^{(\alpha_1)} ... x_{j_r}^{(\alpha_r)} =$$

$$= \sum_{\alpha=1}^{n}\sum_{k=1}^{r-1} x_{j_r}^{(\alpha)} x_{j_k}^{(\bar\alpha)} \sum^{*}_{\langle \alpha_1,...\not\alpha_k,...,\alpha_{r-1}\rangle \in I_n^{r-2}} a_{\alpha_1,...,\alpha_r} \prod_{l \in I_{r-1}\setminus\{k\}} x_{j_l}^{(\alpha_l)} = \sum_{\alpha=1}^{n}\sum_{k=1}^{r-1} x_{j_r}^{(\alpha)} x_{j_k}^{(\bar\alpha)} A^{(\alpha)}_{j_1,...\not j_k,...,j_{r-1}} , \qquad (10)$$

where $\alpha \equiv \alpha_r$, $\bar\alpha = \alpha_k$ and the tensors

$$A^{(\alpha)}_{j_1,...\not j_k,...,j_{r-1}} = \sum^{*}_{\langle \alpha_1,...,\not\alpha_k,...,\alpha_{r-1}\rangle \in I_n^{r-2}} a_{\alpha_1,...,\alpha_r} \prod_{l \in I_{r-1}\setminus\{k\}} x_{j_l}^{(\alpha_l)} , \quad \alpha = 1 \div n$$

of the rank $r-2$ are introduced. Due to the condition of the summing $\overset{*}{\sum}$, it is fulfilled $\sum_{j \in I_{r-1}\setminus\{k\}} \omega_{\alpha_j} = 0$ and, therefore, these tensors are invariant.

2. Let us find the eigenvectors $\mathbf{x}^{(\alpha)} = \mathbf{x}^{(\beta,\gamma;\alpha)}$, $\alpha = 1 \div n$ in the case when $\mathsf{U} = \mathsf{U}^{(\beta,\gamma)}$, $\{\beta, \gamma\} \subset I_n$. Since $\mathsf{U}^{(\beta,\gamma)} = \exp(\omega T^{(\beta,\gamma)})$, these vectors are defined as eigenvectors of the above-introduced matrix $\mathsf{T}^{(\beta,\gamma)}$ such that

$$T_{ij}^{(\beta,\gamma)} = \begin{cases} -1\,, & i = \max\{\beta,\gamma\}, j = \min\{\beta,\gamma\}\,; \\ 1\,, & i = \min\{\beta,\gamma\}, j = \max\{\beta,\gamma\}\,; \\ 0\,, & \text{in other cases}\,. \end{cases}$$

Then, eigenvectors $\mathbf{x}^{(\beta,\gamma;\alpha)}$, $\alpha = 1 \div n$ are defined by components

$$x_j^{(\beta,\gamma;\alpha\pm)} = \begin{cases} \pm i\,, & j = \beta\,; \\ 1, & j = \gamma\,; \\ 0\,, & j \notin \{\beta,\gamma\}\,, \end{cases} \qquad \{\beta,\gamma\} \subset I_n\,,$$

$\alpha_+ = \max\{\beta,\gamma\}$ и $\alpha_- = \min\{\beta,\gamma\}$, $x_j^{(\beta,\gamma;\alpha)} = \{0, j \in \{\beta,\gamma\}; 1, j \notin \{\beta,\gamma\}$ and $j = \alpha; 0, j \neq \alpha\}$ for $\alpha \neq \alpha_\pm$.

3. For eigenvectors pointed out in the item 2, we take into account that the number $\bar{\alpha}$ at $\alpha = \alpha_+$ is equal α_- and $\alpha = \bar{\alpha}$ for numbers $\alpha \neq \alpha_\pm$. Then,

$$x_j^{(\beta,\gamma,\alpha_+)} x_{j'}^{(\beta,\gamma,\alpha_-)} = \delta_{j,j'}^{(\beta,\gamma)} = \begin{cases} 0\,, & j, j' \notin \{\beta,\gamma\}\,; \\ \delta_{j,j'}\,, & j, j' \in \{\beta,\gamma\} \end{cases} \tag{11}$$

where $x_j^{(\beta,\gamma,\alpha)} x_{j'}^{(\beta,\gamma,\alpha)} = \delta_{j,j'} - \delta_{j,j'}^{(\beta,\gamma)}$, $\alpha \notin \{\alpha_+,\alpha_-\}$. We substitute these expressions in (10), using (11)

$$A_{j_1,\ldots,j_r} = \sum_{\alpha=1}^{n} \sum_{k=1}^{r-1} x_{j_r}^{(\beta,\gamma,\alpha)} x_{j_k}^{(\beta,\gamma,\bar{\alpha})} A^{(\alpha)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} =$$

$$= \sum_{k=1}^{r-1} \Big(\delta_{j_r,j_k}^{(\beta,\gamma)} (A^{(\alpha_+)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} + A^{(\alpha_-)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}}) + \sum_{\alpha\notin\{\alpha_+,\alpha_-\}} (\delta_{j,j'} - \delta_{j,j'}^{(\beta,\gamma)}) A^{(\alpha)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} \Big) =$$

$$= \sum_{k=1}^{r-1} \Big(\delta_{j_r,j_k}^{(\beta,\gamma)} B^{(\beta,\gamma)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} + \delta_{j_r,j_k} C^{(\beta,\gamma)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} \Big) \tag{12}$$

where

$$B^{(\beta,\gamma)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} = A^{(\alpha_+)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} + A^{(\alpha_-)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} - \sum_{\alpha\notin\{\alpha_+,\alpha_-\}} A^{(\alpha)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}}\,,$$

$$C^{(\beta,\gamma)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} = \sum_{\alpha\notin\{\alpha_+,\alpha_-\}} A^{(\alpha)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}}$$

are tensors of the rank $r-2$ with values of indexes in $I_r \setminus \{r,k\}$. (Here, we take into account also that the set of tensors $A^{(\alpha)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}}$, $\alpha = 1 \div n$ depends on choice of the pair $\{\beta,\gamma\}$.)

4. Since the pair $\{\beta,\gamma\}$ is arbitrary, choosing any other pair $\{\beta',\gamma'\}$ which does not coincide with $\{\beta,\gamma\}$, we find that it should be fulfilled the equality

$$\sum_{k=1}^{r-1} \Big(\delta_{j_r,j_k}^{(\beta,\gamma)} B^{(\beta,\gamma)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} + \delta_{j_r,j_k} C^{(\beta,\gamma)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} \Big) =$$

$$= \sum_{k=1}^{r-1} \Big(\delta_{j_r,j_k}^{(\beta',\gamma')} B^{(\beta',\gamma')}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} + \delta_{j_r,j_k} C^{(\beta',\gamma')}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} \Big)\,.$$

The values of indexes j_r, j_k in this equality are arbitrary and tensors $C^{(\beta,\gamma)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}}$, $C^{(\beta',\gamma')}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}}$, $B^{(\beta,\gamma)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}}$, $B^{(\beta',\gamma')}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}}$, do not depend on them. Then, we put firstly that $j_r \notin \{\beta,\gamma\} \cup \{\beta',\gamma'\}$. In this case $\delta_{j_r,j_k}^{(\beta,\gamma)} = \delta_{j_r,j_k}^{(\beta',\gamma')} = 0$ and there are one summand at $j_k = j_r$ in both parts of the equality,

$$C^{(\beta,\gamma)}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}} = C^{(\beta',\gamma')}_{j_1,\ldots,\not{j_k},\ldots,j_{r-1}}\,. \tag{13}$$

Thus, the tensors $C^{(\beta,\gamma)}_{j_1,\dots \not{j}_k,\dots,j_{r-1}} \equiv C_{j_1,\dots \not{j}_k,\dots,j_{r-1}}$ do not depend on choice of the pair $\{\beta,\gamma\}$. In a result, we obtain the equality

$$\sum_{k=1}^{r-1} \delta^{(\beta,\gamma)}_{j_r,j_k} B^{(\beta,\gamma)}_{j_1,\dots \not{j}_k,\dots,j_{r-1}} = \sum_{k=1}^{r-1} \delta^{(\beta',\gamma')}_{j_r,j_k} B^{(\beta',\gamma')}_{j_1,\dots \not{j}_k,\dots,j_{r-1}} .$$

Let $j_r \in \{\beta,\gamma\} \setminus \{\beta',\gamma'\}$ in this equality. Therefore, its right-hand side vanishes. Since the pair $\{\beta',\gamma'\}$ may be chosen arbitrary, this equality takes place as for $j_r = \beta$ as for $j_r = \gamma$. Thus, for any value j_r it is fulfilled the equality

$$\sum_{k=1}^{r-1} \delta^{(\beta,\gamma)}_{j_r,j_k} B^{(\beta,\gamma)}_{j_1,\dots \not{j}_k,\dots,j_{r-1}} = 0 . \tag{14}$$

Substituting the equalities (13) and (14) in (12), we find that the invariant tensor A is represented by components

$$A_{j_1,\dots,j_r} = \sum_{k=1}^{r-1} \delta_{j_r,j_k} C_{j_1,\dots \not{j}_k,\dots,j_{r-1}} . \tag{15}$$

5. Now, we prove that the invariant tensor A has the form (8) using the induction on even values r. For $r = 2$ this statement follows from 2. The step of induction is built on the basis of (15). The following equality takes place

$$A_{j_1,\dots,j_r} = \sum_{k=1}^{r-1} \delta_{j_r,j_k} \prod_{l=1, l\neq k}^{r-1} U_{j_l,k_l} C_{j_1,\dots \not{j}_k,\dots,j_{r-1}}$$

due to invariance of the tensor A. Having fixed the value $j_r = j_k$ in this equality, we find that for any $k \in I_n$

$$\prod_{l=1, l\neq k}^{r-1} U_{j_l,k_l} C_{l_1,\dots \not{l}_k,\dots,l_{r-1}} = C_{j_1,\dots \not{j}_k,\dots,j_{r-1}}$$

is valid, i.e. the tensor of the rank $r-2$ is invariant. Therefore, due to induction supposition, for this tensor, the representation (8) is valid in the form of the following linear superposition of multiplicative tensors compiled of the tensor $\boldsymbol{\delta}$

$$C_{j_1,\dots \not{j}_k,\dots,j_{r-1}} = \sum_{\mathfrak{c} \in \mathfrak{C}(I_n, I_r \setminus \{k,r\})} a(\mathfrak{c}) \prod_{\{l,m\} \in \mathfrak{c}} \delta_{j_l,j_m} .$$

Substituting this representation in (15), we find that (8) takes place. □

4. DIMENSION OF THE SPACE $\mathfrak{L}^{(n)}_r$

Natural question about the dimension $d(r,n) = \dim\, \mathfrak{L}^{(n)}_r$ arises. In particular, this question concerns determination of the case when the set of invariant tensors $\mathsf{D}(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}_r(I_n)$ is linear independent. We give some simplest statements about the value $d(r,n)$.

From 5, it follows that for any $r = 2m$, $m \in \mathbb{N}$ the inequality $\dim \mathfrak{L}^{(n)}_r \equiv d(n,r) \leq (r-1)!!$ takes place. The following statement sets the case when this inequality transforms into equality.

Theorem 6. *If r is an even number, $r \leq 2n$, the symbols $\mathsf{D}(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}_r$ are linear independent.*

Proof. Let us take the set $\{j_1, \dots, j_{r/2}\}$ of first $r/2$ indexes of components $A_{j_1,\dots,j_r}$ of symbols $\mathsf{D}(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}_r(I_n)$. Let $\mathfrak{c} = \{j_1, j_{k_1}\}, \dots, \{j_{r/2}, j_{k_{r/2}}\}$ of $\mathfrak{C}_r(I_n)$ be the fixed pair-division. Suppose that the symbols

$\mathsf{D}(\mathfrak{c}')$, $\mathfrak{c}' \in \mathfrak{C}_r(I_n)$ are linear independent. Therefore, there is such a nonzero set $a(\mathfrak{c}')$, $\mathfrak{c}' \in \mathfrak{C}_r(I_n)$, for which the equality

$$\sum_{\mathfrak{c}' \in \mathfrak{C}_r(I_n)} a(\mathfrak{c}')\mathsf{D}(\mathfrak{c}') = 0 \tag{16}$$

is fulfilled. We put $j_1 = j_{k_1} = 1, ..., j_{r/2} = j_{k_{r/2}} = r/2$ in this equality. Therefore, in the sum of left-hand side of the equality only one summand with $\mathfrak{c}' = \mathfrak{c}$ is present. From here, we obtain that $a(\mathfrak{c}) = 0$. Due to arbitrariness of pair-division $\mathfrak{c} \in \mathfrak{C}_r(I_n)$, all coefficients in (16) are equal to zero. □

Consequence 1. $d(r, n) = (r-1)!!$ *at* $r \le 2n$.

The equality $d(r, n) = (r-1)!!$ cannot fulfilled in general case at $r > 2n$, i.e. for tensors of sufficiently large rank when the space dimension is fix. Indeed, the number of all numerical equations for coefficients $a(\mathfrak{c})$ in the equation (16) which define linear independent set between symbols $\mathsf{D}(\mathfrak{c}) \in \mathfrak{L}_r^{(n)}$, $\mathfrak{c} \in \mathfrak{C}_r(I_n)$ at $r > 2n$ does not exceed n^r. But the number of summands in the sum (16), i/e/ the number of all coefficients is equal $(r-1)!!$.

Since $n^r/(r-1)!! \to 0$ at $r \to \infty$ for any $n \ge 2$, so for all sufficiently large values r, the inequality $d(r, n) < (r-1)!!$ should be valid at the fixed value n. This inequality takes place for such values r which unlikely will represent an interest when tensor algebra is applied practically. Despite this, from the point of development of the general theory, it is important to determine nontrivial values of the function $d(r, n)$ and to extract the natural set of basis tensors. In particular, it is needed to determine those cases when the equality $d(r, n) = (r-1)!!$ still has a place at $r > 2n$. In this regard, the following statement is very important.

Theorem 7. *If the equality* $d(r, n') = (r-1)!!$ *takes place for two spaces* $\mathfrak{L}_r^{(n)}$ *and* $\mathfrak{L}_r^{(n')}$ *at* $n' < n \le r/2$, *then the equality* $d(n, r) = (r-1)!!$ *is also fulfilled.*

Proof. Let us consider the equation system (16) for coefficients $a(\mathfrak{c})$ in the space $\mathfrak{L}_r^{(n)}$. We will assume that the indexes $j_1, ..., j_{n'}$ take the values $1 \div n'$ and we put that the indexes $j_{n'+1}, ..., j_d$ are equal $n'+1$. Therefore, homogeneous linear system of $(r-1)!!$ equations for the same number of unknowns coincides with the system of equations for coefficients $a(\mathfrak{c})$ in the space $\mathfrak{L}_r^{(n')}$. According to the theorem condition, this system has the unique solution $a(\mathfrak{c}) = 0$. □

Consequence 2. *In all spaces* $\mathbb{R}^n$, $n \ge 2$, *the set of three invariant tensors of 4th rank* $\mathsf{D}(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}_4(I_n)$ *в* $\mathbb{R}^n$ *with components* $D_{jklm}(\mathfrak{c}_1) = \delta_{jk}\delta_{lm}$, $D_{jklm}(\mathfrak{c}_2) = \delta_{jl}\delta_{km}$, $D_{jklm}(\mathfrak{c}_3) = \delta_{jm}\delta_{kl}$ *is linear independent.*

Proof. The statement follows from 6 at $n = 2$. Having applied 7, we make sure that proving statement is correct at any $n \ge 2$. □

Theorem 8. *If the set of tensors* $\mathsf{D}(\mathfrak{c})$ *of the rank* r *in* $\mathbb{R}^n$, $\mathfrak{c} \in \mathfrak{C}_r(I_n)$ *forms the basis in* $\mathfrak{L}_r^{(n)}$, *then the set of tensors* $\mathsf{D}(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}_r(I_{n+1})$ *of the rank* r *in the space* $\mathbb{R}^{n+1}$ *also forms the basis.*

Proof. We consider the equation

$$\sum_{\mathfrak{c} \in \mathfrak{C}_r(I_{n+1})} a(\mathfrak{c})\mathsf{D}(\mathfrak{c}) = 0 \tag{17}$$

for coefficients $a(\mathfrak{c})$ where $\mathsf{D}(\mathfrak{c})$ are tensors of the rank r in $\mathbb{R}^{n+1}$, $\mathfrak{c} \in \mathfrak{C}_r(I_{n+1})$.

Let the values of indexes in sequences $\langle j_1, ..., j_r \rangle$ of all components $D_{j_1,...,j_r}(\mathfrak{c})$ do not equal $n+1$. Therefore, (17) is written in the form of the equation (16) which has the unique zero solution according to the theorem condition. □

The solution of problem connected with the dimensionality of the system of the invariant tensors $\mathsf{D}(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}_r(I_n)$ is based on determination those numbers r and n when the equation system (16) has a nontrivial solution. For this, it is necessary and sufficient to study nontrivial solvability the homogeneous system $\bar{\mathfrak{S}}$ of numerical equations for coefficients $a(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}_r(I_n)$ at $r/2 > n$. We will call this system the *reduced* one. It is obtained from (16) on the basis of the construction described below.

Let us build the homogeneous equation system $\mathfrak{S}_0$ with the rectangular $n^r \times (r-1)!!$-matrix of coefficients. Each equation is obtained from (16) by assignment to ordered set $\langle j_1, ..., j_r \rangle$ of indexes some concrete values from I_n. In a result, each linear homogeneous equation for the set of coefficients $a(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}_r(I_n)$ corresponds each numerical set $\langle j_1, ..., j_r \rangle$. We restrict ourselves only such sets $\langle j_1, ..., j_r \rangle$ where each numerical value from I_n is met even number of times. In opposite case, all components of tensors $D_{j_1,...,j_r}$ are equal to zero and, therefore, such an equation is absent in the system.

Further, we notice that changing of component values in any set $\langle j_1, ..., j_r \rangle$ without changing of their order does not change the equation in $\mathfrak{S}$ corresponding to it, because coefficients $D_{j_1,...,j_r}$ of the equation are stayed the same. Therefore, we extract all equations from the system $\mathfrak{S}$ which are obtained by permutation of values in corresponding set of indexes from a fixed representative of all such equations. In a result, we obtain the system of homogeneous linear equations $\mathfrak{S}$ with a rectangular matrix S of coefficients. In the case when $\operatorname{rank}\mathsf{S} < (r-1)!!$, the set of tensors $\mathsf{D}(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}_r(I_n)$ is linear dependent.

Now, we extract the system $\bar{\mathfrak{S}}$ from the system $\mathfrak{S}$. This system is characterized by the rectangular matrix $\bar{\mathsf{S}}$. Each nonempty subset of values in sequences $\langle j_1, ..., j_r \rangle \in I_n$ which are defined equations of the system $\mathfrak{S}$ generates the disjunctive division $\{\mathfrak{s}_1, ..., \mathfrak{s}_s\}$, $s \le r/2$ of the set I_r. Components of this division are not necessarily some pairs. But each component $\mathfrak{s}_i$ of such a division contains an even number of elements.

Let us consider any equation of the system $\mathfrak{S}$ which corresponds to anything division $\{\mathfrak{s}_1, ..., \mathfrak{s}_s\}$ of pointed out type. In order the summand $a(\mathfrak{c})$ of this equation of the type (16) to have nonzero coefficient, i.e. in order the symbol $D_{j_1,...,j_r}(\mathfrak{c})$ to be equal 1, it is necessary and sufficient that each pair $\mathfrak{p} \in \mathfrak{c}$ should be contained in one of sets $\mathfrak{s}_i$, $i = 1 \div s$. Consequently, the equation under consideration has the form

$$\sum_{\substack{\mathfrak{c}\in\mathfrak{C}_r(I_n):\\ \mathfrak{p}\in\mathfrak{c},\exists(i\in I_s:\mathfrak{p}\subset\mathfrak{s}_i)}} a(\mathfrak{c}) = 0\,. \tag{18}$$

Let $\{\mathfrak{s}'_1, ..., \mathfrak{s}'_{s'}\}$, $\{\mathfrak{s}_1, ..., \mathfrak{s}_s\}$ are two divisions of pointed out type. We put that second division is a subdivision of the first one, i.e. for each set $\mathfrak{s}'_{i'}$, $i' = 1 \div s'$, $s' < s$, there are such a family of sets $\mathfrak{s}_{i_1}, ..., \mathfrak{s}_{i_l}$, $l_{i'} \le s$ in first division that

$$\mathfrak{s}'_{i'} = \bigcup_{k=1}^{l_{i'}} \mathfrak{s}_{i_k}$$

takes place. Having written down the linear equation, corresponding to this division, in the form

$$\sum_{i=1}^{s'} \sum_{\substack{\mathfrak{c}\in\mathfrak{C}_r(I_n):\\ \mathfrak{p}\in\mathfrak{c},\exists(i\in I_s:\mathfrak{p}\subset\mathfrak{s}'_i)}} a(\mathfrak{c}) = 0\,, \tag{19}$$

we transform its left-hand side by the following way

$$\sum_{i'=1}^{s'}\sum_{k=1}^{l_{i'}} \sum_{\substack{\mathfrak{c}\in\mathfrak{C}_r(I_n):\mathfrak{p}\in\mathfrak{c},\\ \exists(i'\in I_{s'},k\in I_{l_{i'}}:\mathfrak{p}\subset\mathfrak{s}_{i_k})}} a(\mathfrak{c}) = \sum_{k=1}^{s} \sum_{\substack{\mathfrak{c}\in\mathfrak{C}_r(I_n):\mathfrak{p}\in\mathfrak{c},\\ \exists(k\in I_s:\mathfrak{p}\subset\mathfrak{s}_{i_k})}} a(\mathfrak{c})$$

where the sum is transformed according to such method of the accounting of summands such that $\{\mathfrak{s}_1, ..., \mathfrak{s}_s\}$ is the subdivision of $\mathfrak{s}'_{i'}$, $i' = 1 \div s'$. Since the internal sum coincides with the sum in left-hand side of the equation (18), the equation (19) is the consequence of (18).

We will call those divisions $\{\mathfrak{s}_1, ..., \mathfrak{s}_n\}$ the *maximal* ones. From above given arguments, it follows that all equations of the system $\mathfrak{S}$ which do not correspond to maximal divisions, we may extract from the system without changing of the rank of its matrix. We call the system $\bar{\mathfrak{S}}$ of linear homogeneous equations which is obtained as a result of application of the pointed out operation to $\mathfrak{S}$ the *reduced* one. Thus, we prove

Theorem 9. *If $n < r/2$, the dimension of space $\mathfrak{L}_r^{(n)}$ does not exceed the dimension $\bar{d}(r, n)$ of the reduced system $\bar{\mathfrak{S}}$ of linear homogeneous equations* (18)*, corresponding to maximal divisions with n components.*

Calculation of the upper estimate of rank $\bar{\mathsf{S}}$ is based on the following statement.

Theorem 10. *The dimension $\bar{d}(r, n)$ of reduced system of linear homogeneous equations which is considered as the function on $r = 2m, m \in \mathbb{N}$ and $n \in \mathbb{N}$, $2 \le n < r$, is defined by unique way on the basis of the solution of the difference equation*

$$\bar{d}(r, n+1) = (r-1)\bar{d}(r-2, n) + \mathrm{C}^3_{r-1}\bar{d}(r-4, n) + \mathrm{C}^5_{r-1}\bar{d}(r-6, n) + ...$$

$$+ \mathrm{C}^{2[(r-n)/2]-1}_{r-1}\bar{d}(2[(n+1)/2], n)\,,^{3)} \tag{20}$$

which satisfies the condition

$$\bar{d}(r, 2) = \mathrm{C}^2_{r-1} + \mathrm{C}^4_{r-1} + ... + \mathrm{C}^{r-2}_{r-1} = 2^{r-2} - 1\,. \tag{21}$$

Proof. According to the definition of the reduced system $\bar{\mathfrak{S}}$, the number $\bar{d}(r, 2)$ represents the cardinality of the set of such disjunctive divisions of the sequence $\langle 1, 2, ..., r\rangle$ on two nonempty subsequences $\langle j_{i_1}, ..., j_{i_{2s}}\rangle$, $\langle j_{i_{2s+1}}, ..., j_{i_r}\rangle$ with even numbers of components in them. The corresponding sequences $\langle j_{i_1}, ..., j_{i_{2s}}\rangle$, $\langle j_{i_{2s+1}}, ..., j_{i_r}\rangle$ do not transform to each other when changing of values of their components by $1 \leftrightarrow 2$. Since the value j_1 one may choose by any way, so, having put $j_1 = 1$, we find that this number is equal to the number of nonempty sets in I_r with even number of elements. The elements of these sets are the marks of indexes with the value 2. Therefore, this number is equal to the sum in the formula (21).

Reasoning by analogous way in general case, we obtain the formula (20) at $2(n+1) < r$. Let us represent any division, corresponding to the equation of reduced system, in the form of the subset $\mathfrak{s} \subset I_r$ with even number of elements, which represents the marks of indexes being equal $n+1$, and its complement $I_r \setminus \mathfrak{s}$. The last is the union of sets which compile the division of the set $I_r \setminus \mathfrak{s}$ on n component with even numbers of elements. Each component represents the marks of indexes equaled, correspondingly, to 1, 2, ... n. Therefore, having put $j_1 = n+1$ and accounting all possibilities of the choice of the set $\mathfrak{s} \ni 1$, we obtain (20). Here, we take into account that only such summands may be present in the sum (20) for which the cardinality in the set $I_r \setminus \mathfrak{s}$ is not less $2n$ in order that the division of the set I_r has $n+1$ components. Besides, we take into account the identity $2[(r-n)/2] = r - 2[(n+1)/2]$. □

Consequence 3. *The inequality $\bar{d}(r, 2) < (r-1)!!$ takes place at $r > 6$.*

Proof. The equality $\bar{d}(6, 2) = 5!! = 15$ is valid at $r = 6$. The step of induction from r to $r+2$ is built by the following way

$$\bar{d}(r+2, 2) = 2^r - 1 = 4\cdot(2^{r-2} - 1) + 3 \le 4(r-1)!! + 3 < (r+1)!!$$

at values $r \ge 6$ due to $4(r-1)!! + 3 < (r-1)!!(r+1)$ at $r > 4$. □

Remark. In view of the statement of Consequence 3, the set of tensors $\mathsf{D}(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}_8(I_2)$ are linear independent. For $n = 3$, the linear dependence of tensors is realized certainly, starting out the value $r = 14$, since $\bar{d}(12, 3) = 13695 > 11!! = 10395$, but $\bar{d}(14, 3) = 65793 < 13!! = 135135$.

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[3]) Here, the square brakets denote the integer part of the number in it.

5. DIMENSION OF SPACES $\mathfrak{L}_r^{(2)}$

In connection with the investigation in previous sections, natural question arises. What is the value of the rank r, from which the dimension of space $\mathfrak{L}_r^{(n)}$ does not equal to $(r-1)!!$. Since, due to above remark, in the simplest case $n=2$ dimension $\mathfrak{L}_r^{(2)}$ is less $(r-1)!!$ at $r \geq 8$, so in this case the complete answer on this question gives the following

Theorem 11. *The set of tensors* $\mathsf{D}(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}_6(I_n)$ *of 6th rank in the space* $\mathbb{R}^2$ *is linear dependent and* $\dim\, \mathfrak{L}_6^{(n)} = 10$.

Proof. The proof consists of following items **1** − **5**.

1. It is necessary to find the rank of the reduced system which is obtained from the equation

$$\sum_{\mathfrak{c}' \in \mathfrak{C}_6(I_2)} a(\mathfrak{c}')\mathsf{D}_{j_1,\dots,j_6}(\mathfrak{c}') = 0 \tag{22}$$

for the set $a(\mathfrak{c}')$, $\mathfrak{c}' \in \mathfrak{C}_6(I_2)$. The corresponding reduced system of numerical equations is obtained by choice of the set of values of $\{1,2\}$ for sequences $\langle j_1, ..., j_6 \rangle$ such that each of them corresponds to the disjunctive division of the set I_6 on two sets $\mathfrak{p} \subset I_6$ and $I_6 \setminus \mathfrak{p}$, $|\mathfrak{p}| = 2$, $\mathfrak{p} \in \mathfrak{c}$, $\mathfrak{c} \in \mathfrak{C}_6(I_2)$. One may consider that $j_l = 2$, $l \in \mathfrak{p}$, $j_m = 1$, $m \in I_6 \setminus \mathfrak{p}$ takes place for such a pair. Thus, for all equations of the system, the one-to-one correspondence with the set of pairs $\mathfrak{p} \subset I_6$ are determined.

In the equation corresponding to the pair $\mathfrak{p}$, the sum in (22) contains only those summands for which $D_{j_1,\dots,j_6}(\mathfrak{c}') \neq 0$. Therefore, for each pair $\mathfrak{p} \in I_6$, the correspondent equation is written down in the form

$$\sum_{\mathfrak{c}' \in \mathfrak{C}_6(I_2)\,:\mathfrak{p} \in \mathfrak{c}'} a(\mathfrak{c}') = 0\,. \tag{23}$$

The number of elements in the set $I_6^{(2)}$ of all pairs $\mathfrak{p} \subset I_6$ is equal to the number of elements in the set $\mathfrak{C}_6(I_2)$, $|I_6^{(2)}| = 15$. We introduce the operator $\mathsf{Q} : \mathbb{R}^{15} \mapsto \mathbb{R}^{15}$ which is defined by the matrix $Q_{\mathfrak{p},\mathfrak{c}'} = \{1, \mathfrak{p} \in \mathfrak{c}'\,; 0, \mathfrak{p} \notin \mathfrak{c}'\}$ such that a correspondence with the basis vectors in $\mathbb{R}^{15}$ is defined for pairs $\mathfrak{p} \subset I_6$ and paired-divisions $\mathfrak{c} \in \mathfrak{C}_6(I_2)$. In terms of this operator, the equation (23) is written down in the form

$$(\mathsf{Q}\langle a(\mathfrak{c}); \mathfrak{c} \in \mathfrak{C}_6(I_2)\rangle)(\mathfrak{p}) = \sum_{\mathfrak{c}' \in \mathfrak{C}_6(I_2)\,:\,\mathfrak{p} \in \mathfrak{c}'} Q_{\mathfrak{p},\mathfrak{c}'} a(\mathfrak{c}') = 0\,.$$

2. We will calculate the matrix elements of the operator $\mathsf{Q}^T\mathsf{Q}$. According to the definition of Q, we have

$$\left(\mathsf{Q}^T\mathsf{Q}\right)_{\mathfrak{c}_1,\mathfrak{c}_2} = \sum_{\mathfrak{p}\,:\,\mathfrak{p} \in \mathfrak{c}_1, \mathfrak{p} \in \mathfrak{c}_2} Q_{\mathfrak{p},\mathfrak{c}_1}\, Q_{\mathfrak{p},\mathfrak{c}_2}\,.$$

Let be $\mathfrak{c}_1 = \mathfrak{c}_2 = \mathfrak{c}$. Therefore,

$$\left(\mathsf{Q}^T\mathsf{Q}\right)_{\mathfrak{c},\mathfrak{c}} = \sum_{\mathfrak{p}\,:\,\mathfrak{p} \in \mathfrak{c}} Q_{\mathfrak{p},\mathfrak{c}} = 3\,,$$

since the division $\mathfrak{c}$ consists of three pairs. If $\mathfrak{c} \neq \mathfrak{c}'$, but $\mathfrak{c} \cap \mathfrak{c}' \neq \emptyset$, there is only one nonzero summand in the sum, since $\mathfrak{c}$ and $\mathfrak{c}'$ are paired-divisions of I_6. If two elements are present in their intersection, so $\mathfrak{c} = \mathfrak{c}'$. If $\mathfrak{c} \cap \mathfrak{c}' = \emptyset$, all summands in the sum are equal to zero. Thus, the self-adjoint operator $\mathsf{Q}^T\mathsf{Q}$ are represented in the form $\mathsf{Q}^T\mathsf{Q} = 2\mathsf{E} + \mathsf{R}$, where the operator R is defined by the matrix such that $R_{\mathfrak{c},\mathfrak{c}'} = 1$ if $\mathfrak{c} \cap \mathfrak{c}' \neq \emptyset$ and $R_{\mathfrak{c},\mathfrak{c}'} = 0$ in opposite case.

3. We will calculate the matrix of operator R^2,

$$\left(\mathsf{R}^2\right)_{\mathfrak{c}_1,\mathfrak{c}_2} = \sum_{\mathfrak{c} \in \mathfrak{C}_6(I_2)} R_{\mathfrak{c}_1,\mathfrak{c}} R_{\mathfrak{c},\mathfrak{c}_2} = \left|\Sigma_{\mathfrak{c}_1,\mathfrak{c}_2}\right|$$

where $\Sigma_{\mathfrak{c}_1,\mathfrak{c}_2} = \big\{\langle \mathfrak{c}_1, \mathfrak{c}_2\rangle \,:\, c_j \in \mathfrak{C}_6(I_2), \exists(\mathfrak{c} \in \mathfrak{C}_6(I_2) \,:\, \mathfrak{c} \cap \mathfrak{c}_j \neq \emptyset), j = 1, 2\big\}$.

Let be $\mathfrak{c}_1 = \mathfrak{c}_2 = \mathfrak{c}'$. There is one summand with $\mathfrak{c} = \mathfrak{c}'$.

We denote $\mathfrak{N}(\mathfrak{c}) = \{j \,:\, j \in \mathfrak{p}, \mathfrak{p} \in \mathfrak{c}\}$ for any set $\mathfrak{c}$ of intersecting pairs. So, it is valid $\mathfrak{N}(\mathfrak{c}_1) = \mathfrak{N}(\mathfrak{c}_2) = I_6$. If the relation $\mathfrak{c} \cap \mathfrak{c}' \neq \emptyset$ at $\mathfrak{c} \neq \mathfrak{c}'$ is fulfilled for summands in the sum, this intersection consists of one pair. In the opposite case, it should be $\mathfrak{c} = \mathfrak{c}'$. Therefore, let be $\mathfrak{c} \cap \mathfrak{c}' = \{\mathfrak{p}\}$. This pair may be choose by three way in the division $\mathfrak{c}$. So, the nonintersecting divisions $\mathfrak{c}' \setminus \{\mathfrak{p}\}$, $\mathfrak{c} \setminus \{\mathfrak{p}\}$ consists of two pairs and the set $\mathfrak{N}(\mathfrak{c} \setminus \{\mathfrak{p}\}) = \mathfrak{N}(\mathfrak{c}' \setminus \{\mathfrak{p}\})$ consists of of four elements. On the basis of four elements, one may compile 3 nonintersecting divisions consisting of two pairs. Therefore, we have two possibilities of divisions $\mathfrak{c} \setminus \{\mathfrak{p}\}$, which is not coincided with $\mathfrak{c}' \setminus \{\mathfrak{p}\}$. Thus, taking into account all possibilities, we come to a conclusion that $\Sigma_{\mathfrak{c},\mathfrak{c}} = 1 + 2 \cdot 3 = 7$.

Let be $\mathfrak{c}_1 \neq \mathfrak{c}_2$. There are two cases: a) $\mathfrak{c}_1 \cap \mathfrak{c}_2 \neq \emptyset$ and b) $\mathfrak{c}_1 \cap \mathfrak{c}_2 = \emptyset$. In further, we will use the following representation of divisions in the form of pairs: $\mathfrak{c}_j = \{\mathfrak{p}_j, \mathfrak{q}_j, \mathfrak{r}_j\}$, $j \in \{1, 2\}$. Since the divisions $\mathfrak{c}_j$ may have only one common pair in the case a), we will consider two variants: a$_1$) $\mathfrak{p} \in \mathfrak{c}$ and a$_2$) $\mathfrak{p} \notin \mathfrak{c}$ having put $\mathfrak{c}_j = \{\mathfrak{p}, \mathfrak{q}_j, \mathfrak{r}_j\}$, $j \in \{1, 2\}$, $\{\mathfrak{q}_1, \mathfrak{r}_1\} \cap \{\mathfrak{q}_2, \mathfrak{r}_2\} = \emptyset$.

In the case a$_1$), there are only three possibilities of the compiling of pairs $\mathfrak{q}$, $\mathfrak{r}$ in $\mathfrak{c} \setminus \{\mathfrak{p}\}$ using elements of the set $\mathfrak{N}(\mathfrak{c} \setminus \{\mathfrak{p}\})$: 1) the pairs compile the division $\mathfrak{c}_1 \setminus \{\mathfrak{p}\}$ and, therefore, $\mathfrak{c} = \mathfrak{c}_1$, 2) these pairs compile the division $\mathfrak{c}_2 \setminus \{\mathfrak{p}\}$ and, therefore, $\mathfrak{c} = \mathfrak{c}_2$, 3) these pairs form the division not intersecting the set $\mathfrak{c}_1 \setminus \{\mathfrak{p}\} \cup \mathfrak{c}_2 \setminus \{\mathfrak{p}\}$. Consequently, $|\Sigma_{\mathfrak{c}_1,\mathfrak{c}_2}| = 3$.

In the case a$_2$) $(\mathfrak{c}_1 \setminus \{\mathfrak{p}\}) \cap \mathfrak{c} \neq \emptyset$ and $(\mathfrak{c}_2 \setminus \{\mathfrak{p}\}) \cap \mathfrak{c} \neq \emptyset$ should be fulfilled when $\mathfrak{p} \notin \mathfrak{c}$. These sets should be contain only one pair $\mathfrak{q}_1$ or $\mathfrak{q}_2$, correspondingly. In the opposite case, it is valid $\mathfrak{c}_1 = \mathfrak{c}$ or $\mathfrak{c}_2 = \mathfrak{c}$. Indeed, for example, $\mathfrak{c}_1 = \mathfrak{c}$ is realized, therefore, $\mathfrak{c} = \mathfrak{c}_2$, since $\mathfrak{c}$ contains always the pair $\mathfrak{p}$ which is common with $\mathfrak{c}_2$, but this is impossible in the case under consideration. Due to the same reason, the pairs $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are different. Moreover, $\mathfrak{q}_1 \cap \mathfrak{q}_2 = \emptyset$, since in the opposite case, the set $\mathfrak{N}(\mathfrak{c} \setminus \{\mathfrak{p}\}) \setminus (\mathfrak{q}_1 \cup \mathfrak{q}_2)$ contains one element, that is impossible. Consequently, the choice of the pair $\mathfrak{q}_1$ from $\mathfrak{N}(\mathfrak{c}_1 \setminus \{\mathfrak{p}\})$, which one may do by three way, determines completely the division $\mathfrak{c}$, that has the form $\{\mathfrak{p}, \mathfrak{q}_1, \mathfrak{q}_2\}$. Therefore, $|\Sigma_{\mathfrak{c}_1,\mathfrak{c}_2}| = 3$ is valid as in the first case.

We consider the case b) and we will build such divisions $\mathfrak{c}$ that $\mathfrak{c} \cap \mathfrak{c}_1 \neq \emptyset$, $\mathfrak{c} \cap \mathfrak{c}_2 \neq \emptyset$. These intersections of the divisions consist of one pair: $\mathfrak{p}_1 \in \mathfrak{c}_1$ and, correspondingly, $\mathfrak{p}_2 \in \mathfrak{c}_2$. Moreover, $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \emptyset$, since these pairs are contained in the same division $\mathfrak{c}$. The pairs $\mathfrak{p}_1$ and $\mathfrak{p}_2$ do not intersect with the pair $\mathfrak{q}$ which compiles the intersection of $\mathfrak{c}_1 \cap \mathfrak{c}_2$. Therefore, the choice of the pair $\mathfrak{p}_1$ of $\mathfrak{c}_1$ defines by unique way the pair $\mathfrak{p}_2$ in $\mathfrak{c}_2$ when divisions $\mathfrak{c}$ constructing at the condition of their disjunctivity, since $\mathfrak{p}_j \subset \mathfrak{N}(\mathfrak{c} \setminus \{\mathfrak{q}\})$, $j \in \{1, 2\}$. In turn, one may choose the pair $\mathfrak{p}_1$ by three ways in $\mathfrak{c}_1 \setminus \{\mathfrak{q}\}$.

In a result of our consideration of variants a) and b), we conclude that $\big(\mathsf{R}^2\big)_{\mathfrak{c}_1,\mathfrak{c}_2} = |\Sigma_{\mathfrak{c}_1,\mathfrak{c}_2}| = 3$ at $\mathfrak{c}_1 \neq \mathfrak{c}_2$.

4. Following transformations of the quadratic form $(\mathsf{Q}\mathsf{Q}^T\boldsymbol{\xi}, \boldsymbol{\xi})$ with vectors $\boldsymbol{\xi} = \langle \xi(\mathfrak{c}); \mathfrak{c} \in \mathfrak{C}_6(I_2)\rangle \in \mathbb{R}^{15}$ show that the operator R is non negative,

$$(\mathsf{Q}\mathsf{Q}^T\boldsymbol{\xi}, \boldsymbol{\xi}) = \sum_{\langle \mathfrak{c},\mathfrak{c}'\rangle \in \mathfrak{C}_6^2(I_2)} \xi(\mathfrak{c}) R_{\mathfrak{c},\mathfrak{c}'} \xi^*(\mathfrak{c}') = 3 \sum_{\mathfrak{c} \in \mathfrak{C}_6(I_2)} |\xi(\mathfrak{c})|^2 + \sum_{\substack{\langle \mathfrak{c},\mathfrak{c}'\rangle \in \mathfrak{C}_6^2(I_2)\,:\\ |\mathfrak{c} \cap \mathfrak{c}'| = 1}} \xi(\mathfrak{c}) \xi^*(\mathfrak{c}') =$$

$$= \sum_{\mathfrak{c} \in \mathfrak{C}_6(I_2)} \sum_{\mathfrak{p} \in \mathfrak{c}} |\xi(\mathfrak{c})|^2 + \sum_{\mathfrak{p} \subset I_6} \sum_{\substack{\langle \mathfrak{c},\mathfrak{c}'\rangle \in \mathfrak{C}_6^2(I_2)\,:\\ \mathfrak{c} \cap \mathfrak{c}' = \{\mathfrak{p}\}}} \xi(\mathfrak{c}) \xi^*(\mathfrak{c}') = \sum_{\mathfrak{p} \subset I_6} \Big[\sum_{\substack{\mathfrak{c} \in \mathfrak{C}_6(I_2)\,:\\ \mathfrak{p} \in \mathfrak{c}}} |\xi(\mathfrak{c})|^2 + \sum_{\substack{\langle \mathfrak{c},\mathfrak{c}'\rangle \in \mathfrak{C}_6^2(I_2)\,:\\ \mathfrak{c} \cap \mathfrak{c}' = \{\mathfrak{p}\}}} \xi(\mathfrak{c}) \xi^*(\mathfrak{c}') \Big] =$$

$$= \sum_{\mathfrak{p} \subset I_6} \sum_{\substack{\langle \mathfrak{c},\mathfrak{c}'\rangle \in \mathfrak{C}_6^2(I_2)\,:\\ \mathfrak{p} \in \mathfrak{c} \cap \mathfrak{c}'}} \xi(\mathfrak{c}) \xi^*(\mathfrak{c}') = \sum_{\mathfrak{p} \subset I_6} \Big| \sum_{\substack{\mathfrak{c} \in \mathfrak{C}_6(I_2)\,:\\ \mathfrak{p} \in \mathfrak{c}}} \xi(\mathfrak{c}) \Big|^2$$

where we take into account that $\mathfrak{c} = \mathfrak{c}'$ takes place when $|\mathfrak{c} \cap \mathfrak{c}'| > 1$.

5. It follows from the item **3** that $\mathsf{R}^2 = 4\mathsf{E} + 3\mathsf{P}$ where all matrix elements of the self-adjoint operator P are equal to 1. Therefore, the operator $\mathsf{P}/15$ is the one-dimensional orthogonal projector in $\mathbb{R}^{15}$. It has 14 eigenvalues equal to 0 and one eigenvalue equal to 1. Therefore, the eigenvalues of the operator$\mathsf{P}$ are equal to 0 or 15 with correspondent multiplicities. From here, we conclude that eigenvalues of the operator R^2 are equal to 4 or 49 with the same multiplicities. Correspondingly, the self-adjoint operator R may have the eigenvalues from the set $\{\pm 2, \pm 7\}$. Only one of values is realized and it is non-degenerated. Therefore, eigenvalues of the operator $\mathsf{Q}^T\mathsf{Q}$ belong to $\{0, 4, 9\}$ and 9 is non-degenerated value.

Now, we notice that $\mathrm{Sp}\,\mathsf{QQ}^T = 2\mathrm{Sp}\mathsf{E} + \mathrm{Sp}\mathsf{R} = 45$. Since the trace $\mathrm{Sp}\,\mathsf{QQ}^T$ is equal to the sum of all eigenvalues, the value 4 has the multiplicity 9 and the value 0 has the multiplicity 5. So, the rank of the matrix QQ^T and, consequently, the rank of the matrix Q is equal to 10. □

6. CONCLUSION

Apparently, the question about linear independence of the set of multiplicative invariant tensors in linear spaces of tensors which are invariant relative to a classical group does not studied earlier [4]. It arises in connection with the building of general covariant systems of evolutionary differential equations [13]. Unfortunately, there is one statement formulated in the pointed out work, turned out to be incorrect. In connection with results of the work under consideration, significant number of problems arises which may be represent the interest for mathematical physics.

As concerns the $\mathbb{O}_n$-group, the simplest question that is stayed without answer in our work consists of the determination rank $\bar{\mathsf{S}}$ in general case. The proof of this fact should be solved completely the question about the dimension of $\mathfrak{L}_r^{(n)}$. Besides, it is important to point out the constructive method of formation of the basis tensors in the space $\mathfrak{L}_r^{(n)}$ if $\dim \mathfrak{L}_r^{(n)} < (r-1)!!$. Further, in the development of the theory, it is important to solve some analogous questions for other Lie's groups, in which the tensor algebra connected with them plays important role in mathematical physics, namely, for groups $\mathbb{SO}_n$, for the Poincaré group, for groups $\mathbb{SU}(n)$.

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