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Frames of solutions and discrete analysis of pseudo-differential equations

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We construct discrete analogs of multidimensional singular integral operators and study their invertibility. Moreover, we give a comparison between continual and discrete case. We give the theory of periodic Riemann problem also, because it is needed for studying invertibility of so-called paired equations. For more general case of pseudo-differential operators, we construct the solvability theory for discrete pseudo-differential equations in discrete analogs of Sobolev–Slobodetskii spaces. Some comparison results for discrete and continuous solutions are given also in appropriate discrete normed spaces.

KEYWORDS
Calderon–Zygmund operator, discrete equation, discrete pseudo-differential operator, invertibility, periodic factorization, periodic Riemann problem, symbol

MSC CLASSIFICATION
42B20; 47G30; 65N22

1 | INTRODUCTION

We study discrete equations for obtaining good approximations for elliptic pseudo-differential equations and related boundary value problems. For this purpose, we introduce discrete functional spaces depending on a sampling parameter, define discrete operators acting in these functional spaces, study solvability of corresponding discrete equations with possible additional conditions, and finally compare discrete and continuous solutions in certain discrete spaces with a norm. Solutions of discrete equations we will call frames of solutions for an initial pseudo-differential equation.

The paper consists of two parts. The first one is related to special class of simplest discrete pseudo-differential operators of order 0, and these operators are called singular integrals or Calderon–Zygmund operators. These operators were systematically studied in papers of A. Calderon and A. Zygmund (see, for example, one of latter papers[1] and historical remarks in Mikhlin and Proessdorf.[2] For simplest Calderon–Zygmund kernel, such operators are "singular convolution type operators." First results for discrete convolution operators in multidimensional Euclidean space were described in Gohberg and Feldman[3] and then it was generalized in distinct directions by a lot of authors.[4–10] We try to use classical non-algebraic approach with the discrete Fourier transform and a comparison for symbols of discrete and continuous operators. We widely use ideas of multidimensional Fourier series theory, and roughly speaking, we treat the discrete fourier transform as a certain Fourier series.[11] Some results in this direction were announced in previous studies.[12–16]

Last years, more general constructions related to discrete pseudo-differential operators were discussed.[17–19] We think this problem is connected with a desire to transfer ideas and methods of pseudo-differential theory[20,21] to other mathematical structures which are different from usual Euclidean space. But we look on the problem from another point of view. There are certain discrete theories for boundary value problems for differential equations[22,23] which are constructed as analogs of corresponding continuous theories. Moreover, these theories give a comparison between discrete and continuous solutions. At the same time, there is theory of boundary value problems for elliptic pseudo-differential equations.[24] We would like to construct a discrete analog of this theory including comparison theorems for discrete and continuous solutions. Although the discrete theory is interesting itself, nevertheless, we think that this theory should transfer to the continuous one when the sampling parameter tends to zero. That's why we need a comparison result for both discrete and continuous solutions also. Some results in this direction were announced in other works.[25–28]

Our constructions are based on the special periodic Riemann boundary value problem[14] and related periodic factorization for an elliptic symbol.[26] These two concepts play a crucial role for solvability theory of model pseudo-differential equations in canonical domains. Estimates for discrete and continuous solutions are obtained with the help of discrete Fourier transform and properties of discrete functional spaces. Also we widely use the symbol properties of elliptic pseudo-differential operators.

2 | PART I: SINGULAR INTEGRALS

2.1 | Discrete singular integrals in $\mathbb{R}^m$

Let $\mathbb{Z}^m$ be an integer lattice in m-dimensional space $\mathbb{R}^m$, $K(x,y)$ be a variable Calderon–Zygmund kernel,[1] $K(x,0) \equiv 0, \forall x \in D$, and K_d^h, be a discrete operator

$$K_h^d u_d(\tilde{x}) = \sum_{\tilde{y} \in D \bigcap hZ^m} K(\tilde{x}, \tilde{x} - \tilde{y}) u_d(\tilde{y}) h^m,$$

on functions of discrete variable $u_d(\tilde{x})$, which are defined on the set $D \bigcap Z_h^m, D \subset R^m$.

This operator is generated by the following integral:

$$v.p. \int_D K(x, x-y) u(y) dy, x \in D.$$

Such operators were studied in classical papers of A.P. Calderon and A. Zygmund in the spaces of integrable functions,[1] and it was established L_p-boundedness for them.

Our main goal is to describe invertibility (or Fredholm property) conditions for such discrete operators for simple cases at least and to give a comparison between discrete and continuous cases.

2.1.1 | Discrete functional spaces and discrete Fourier transform

The discrete Fourier transform F_d for functions of discrete argument, which are defined on the lattice $h\mathbb{Z}^m$, is given by the formula

$$(F_d u)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^m} u(\tilde{x}) e^{i\tilde{x}\cdot\xi} h^m \equiv \tilde{u}(\xi), \ \xi \in \hbar\mathbb{T}^m,$$

where $\hbar = h^{-1}, \mathbb{T} = [-\pi, \pi]$.

Such Fourier transform has the same properties as the classical one,[11] and its inverse transform is given by the formula

$$(F_d^{-1} \tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \int_{\hbar T^m} \tilde{u}_d(\xi) e^{-i\tilde{x}\cdot\xi} d\xi.$$

2.1.2 | Symbols and solvability for discrete equations

We start from simplest Calderon–Zygmund operators of convolution type

$$v.p. \int_{\mathbb{R}^m} K(x-y) u(y) dy = \lim_{\substack{\varepsilon \to 0 \\ N \to +\infty}} \int_{\varepsilon < |x-y| < N} K(x-y) u(y) dy,$$

where the kernel $K(x)$ satisfies the following conditions:

1) $K(tx) = t^{-m}K(x), \forall x \neq 0, t > 0$;
2) $\int\limits_{S^{m-1}} K(\theta)d\theta = 0, S^{m-1}$ is unit sphere in $\mathbb{R}^m$;
3) $K(x)$ is differentiable on $\mathbb{R}^m \backslash \{0\}$.

Let us consider a discrete operator generated by the Calderon–Zygmund kernel $K(x)$ and defined on functions $u_h(\tilde{x})$, $\tilde{x} \in h\mathbb{Z}^m$, $h > 0$, and the corresponding equation

$$au_h(\tilde{x}) + \sum_{\tilde{y} \in h\mathbb{Z}^m_+} K(\tilde{x} - \tilde{y})u_h(\tilde{y})h^m = v_h(\tilde{x}),\ \tilde{x} \in h\mathbb{Z}^m_+. \tag{1}$$

a is certain constant, in the discrete half-space $h\mathbb{Z}^m_+ = \{\tilde{x} \in h\mathbb{Z}^m : \tilde{x_m} > 0\}$, $u_h, v_h \in L_2(h\mathbf{Z}^m_+)$, u_h is unknown function, and v_h is given one.

By definition, we put $K(0) = 0$, and for the operator

$$u_h(\tilde{x}) \mapsto au(\tilde{x}) + \sum_{\tilde{y} \in h\mathbb{Z}^m} K(\tilde{x} - \tilde{y})u_h(\tilde{y})h^m,\ \tilde{x} \in h\mathbb{Z}^m,$$

we introduce its symbol by the formula

$$\sigma_h(\xi) = a + \sum_{\tilde{x} \in h\mathbb{Z}^m} e^{-i\xi\tilde{x}}K(\tilde{x})h^m;$$

it is periodic function with basic cube period $[-\pi h^{-1};\ \pi h^{-1}]^m$.

The sum for $\sigma_h(\xi)$ is defined as a limit of partial sums over cubes Q_N

$$\lim_{N\to\infty} \sum_{\tilde{x} \in Q_N} e^{-i\xi\tilde{x}}K(\tilde{x})h^m,$$

$$Q_N = \left\{ \tilde{x} \in h\mathbb{Z}^m : |\tilde{x}| \leq N, |\tilde{x}| = \max_{1\leq k\leq m} |\tilde{x}_k| \right\}.$$

It is very similar to the classical symbol of Calderon–Zygmund operator,[2] which is defined as Fourier transform of the kernel $K(x)$ in principal value sense

$$\sigma(\xi) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \int\limits_{\varepsilon<|x|<N} K(x)e^{i\xi x}dx.$$

Key point of our study is a theorem asserting that images of σ and σ_h are the same.

We also introduce continuous equation in a half-space $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x', x_m), x_m > 0\}$

$$au(x) + \int\limits_{\mathbb{R}^m_+} K(x - y)u(y)dy = v(x), x \in \mathbb{R}^m_+, \tag{2}$$

and we'll prove that Equations (1) and (2) are uniquely solvable or unsolvable simultaneously for all $h > 0$ in corresponding spaces.

We will discuss some properties of symbols $\sigma(\xi)$ and $\sigma_h(\xi)$, which are needed for us.

Theorem 1. *The ranges of σ and σ_h are the same, and their values are constant for any ray from origin.*

Proof. First, $\lim\limits_{h\to 0} \sigma_h(\xi) = \sigma(\xi),\ \forall \xi \neq 0$.

Indeed, if we fix $\xi \neq 0$, then by definition of integral as a limit of integral sums.

Further,

$$\sigma_h(\xi) = \sigma_1(h\xi),\ \forall h > 0,\ \xi \in [-\pi h^{-1}, \pi h^{-1}]^m.$$

Indeed,

$$\sigma_h(\xi) = \sum_{\tilde{x}\in h\mathbb{Z}^m} K(\tilde{x})e^{-i\tilde{x}\cdot\xi}h^m =$$
$$= \sum_{\tilde{y}\in\mathbb{Z}^m} K(h\tilde{y})e^{-i\tilde{y}\cdot h\xi}h^m = \sum_{\tilde{y}\in\mathbb{Z}^m} K(\tilde{y})e^{-i\tilde{y}\cdot h\xi} = \sigma_1(h\xi).$$

The proof follows from previous assertions immediately, because if we fix ξ, then $\lim\limits_{h\to 0}\sigma_1(h\xi) = \sigma(\xi)$, and therefore, $\lim\limits_{h\to 0}\sigma_h(\xi) = \sigma(\xi)$. □

2.1.3 | Digital Calderon–Zygmund operators and invertibility

Definition 1. The operator K_h^d we call a discrete Calderon–Zygmud operator.

Corollary 1. *Operators $K : L_2(\mathbb{R}^m) \to L_2(\mathbb{R}^m)$ and $K_h^d : L_2(h\mathbb{Z}^m) \to L_2(h\mathbb{Z}^m)$ are both invertible or both not invertible for arbitrary $h > 0$.*

Proof. In fact, the invertibility of operators is determined by their symbol; roughly speaking, if the symbol is non-vanishing, then the operator is invertible; more precisely, if $\inf|\sigma(\xi)| > 0$, then the operator K is invertible. The same is valid for the operator K_h^d. According to Theorem 1, the images of σ and σ_h^d are the same we obtain the required assertion. □

2.2 | Discrete equations in $\mathbb{R}_+^m$

2.2.1 | Preliminaries: Continuous case

Now we consider more general operator and equation in whole space $\mathbb{R}^m$

$$(K_1P_+ + K_2P_-)U = V, \tag{3}$$

taking into account that K_1, K_2 are Calderon–Zygmund operators with symbols $\sigma_{K_1}(\xi), \sigma_{K_2}(\xi)$, and P_+, P_- are restriction operators on $\mathbb{R}_\pm^m = \{x = (x_1, \dots, x_m), \pm x_m > 0\}$. It is easily verified that Equation (2) is equivalent in a solvability sense to special case of Equation (3) when $K_2 \equiv I$, I is identity operator.

If we'll denote the Fourier transform by letter F and use the notations

$$FP_+ = Q_{\xi'}F,\ FP_- = P_{\xi'}F,$$

$$P_{\xi'} = 1/2(I + H_{\xi'}),\ Q_{\xi'} = 1/2(I - H_{\xi'}),$$

where $H_{\xi'}$ is Hilbert transform on variable ξ_m for fixed $\xi' = (\xi_1, \dots, \xi_{m-1})$:

$$(H_{\xi'}u)(\xi', \xi_m) \equiv \frac{1}{\pi i} v.p. \int_{-\infty}^{+\infty} \frac{u(\xi', \tau)}{\tau - \xi_m} d\tau,$$

then Equation (3) after applying the Fourier transform will be the following equation with the parameter ξ':

$$\frac{\sigma_{M_1}(\xi', \xi_m) + \sigma_{M_2}(\xi', \xi_m)}{2}\tilde{U}(\xi) +$$

$$+ \frac{\sigma_{M_1}(\xi', \xi_m) - \sigma_{M_2}(\xi', \xi_m)}{2\pi i} v.p. \int_{-\infty}^{+\infty} \frac{\tilde{U}(\xi', \eta)}{\eta - \xi_m} d\eta = \tilde{V}(\xi)$$

($\sim$ over a function denotes its Fourier transform).

This equation is closely related to boundary Riemann problem (with the parameter ξ') with coefficient[29,30]

$$G(\xi',\xi_m) = \sigma_{M_1}(\xi',\xi_m)\sigma_{M_2}^{-1}(\xi',\xi_m).$$

2.2.2 | Preliminaries: Discrete case

We consider here the discrete analog for the equation (3) in the space $L_2(h\mathbb{Z}^m)$, namely, the equation

$$(K_{1,h}^d P_+^d + K_{2,h}^d P_-^d)U_d = V_d, \tag{4}$$

assuming that $P_\pm^d$ in (4) are the restriction operators on $h\mathbb{Z}_\pm^m$, and $K_{1,h}^g, K_{2,h}^d$ are discrete Calderon–Zygmund operators generated by kernels $K_1(x)$, and $K_2(x)$; such operators are bounded in the spac $L_2(h\mathbb{Z}^m)$.

Further, our main goal is to derive the periodical analog of the Hilbert transform with respect to the variable ξ_m ($\xi \in \hbar\mathbb{T}^m$, ξ' is fixed) as the formula

$$(H_{\xi'}^{per}u)(\xi',\xi_m) = \frac{h}{2\pi i}\int_{-\pi\hbar}^{\pi\hbar} u(\xi',t)\cot\frac{h(t-\xi_m)}{2}dt. \tag{5}$$

Periodical analogs of projectors $P_{\xi'}, Q_{\xi'}$ will look as follows:

$$P_{\xi'}^{per} = 1/2(I + H_{\xi'}^{per}),\ Q_{\xi'}^{per} = 1/2(I - H_{\xi'}^{per}),$$

and the periodical analog of the singular integral equation above will be

$$\frac{\sigma_{1,h}(\xi',\xi_m)+\sigma_{2,h}(\xi',\xi_m)}{2}\tilde{U}(\xi) + \frac{\sigma_{1,h}(\xi',\xi_m)+\sigma_{2,h}(\xi',\xi_m)}{4\pi i}\times$$

$$\times v.p.\int_{-\pi\hbar}^{\pi\hbar} h\tilde{U}(\xi',\eta)\cot\frac{h(\eta-\xi_m)}{2}d\eta = \tilde{F}(\xi), \tag{6}$$

where $\sigma_{1,h}, \sigma_{2,h}$ are the symbols of discrete operators $K_{1,h}^d, K_{2,h}^d$. Of course, Equation (6) is related to corresponding Riemann boundary value problem, and the left piece of Part I will be devoted to studying solvability conditions for Equation (6) and statement of corresponding periodic Riemann boundary value problem. The solvability conditions for the discrete Equation (4) will be a final accord.

2.2.3 | Holomorphic functions in complex strips and periodic analog of Sokhotskii formulas

According to form of the operator (5), we start from one-dimensional case and $h = 1$. We will show that this operator is connected to a certain boundary value problem.

Let Π_+, Π_- be an upper and lower half-strip in a complex plane $\mathbb{C}$:

$$\Pi_\pm = \{z \in \mathbb{C} : z = t + is,\ t \in [-\pi;\pi], \pm s > 0\}.$$

By periodic Riemann problem, we mean the following: finding two periodic bounded functions $\Phi^\pm(z)$, which are holomorphic in $\Pi_\pm$, and their boundary values under $s \to 0\pm$ satisfy the following linear relation

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t),\ t \in [-\pi;\pi], \tag{7}$$

where $G(t),\ g(t)$ are given functions on the $[-\pi;\pi]$, $G(-\pi) = G(\pi)$, $g(-\pi) = g(\pi)$.

To solve this problem, we introduce the integral of type

$$\Phi(z) = \frac{i}{4\pi} \int_{-\pi}^{\pi} \varphi(x) \cot \frac{z-x}{2} dx, \ z \in \Pi_{\pm};$$

it is a periodic analog of the Cauchy type integral. Further, it is desirable to obtain the analog of Sokhotskii formulas.[29,30]

Lemma 1. *Let the function $\varphi(t)$ satisfies the Hőlder condition on the segment $[-\pi; \pi]$, $\varphi(-\pi) = \varphi(\pi)$. Then the function $(z = t + is)$ $\Phi(z)$ has boundary values $\Phi^{\pm}(t)$ under $s \to 0\pm$, which are given by the formulas:*

$$\Phi^{+}(\tau) = \frac{1}{4\pi i} \int_{-\pi}^{\pi} \varphi(t) \cot \frac{t-\tau}{2} dt + \frac{\varphi(\tau)}{2},$$

$$\Phi^{-}(\tau) = \frac{1}{4\pi i} \int_{-\pi}^{\pi} \varphi\left(t \cot \frac{t-\tau}{2} dt - \frac{\varphi(\tau)}{2};\right.$$

the integral is treated in principal value-sense.

Proof of Lemma 1 can be found in Vasil'ev A and Vasil'ev B.[14]

Corollary 2. *An arbitrary function from $\varphi \in L_2[-\pi, \pi]$ can be uniquely represented as a sum of boundary values of holomorphic functions*

$$\Phi^{\pm}(z) = \pm \frac{i}{4\pi} \int_{-\pi}^{\pi} \varphi(x) \cot \frac{z-x}{2} dx, \ z \in \Pi_{\pm}.$$

Proof. According to Lemma 1, this assertion is valid for smooth functions. Since such functions are dense in $L_2[-\pi, \pi]$ and the operator

$$\varphi(x) \mapsto \int_{-\pi}^{\pi} \varphi(t) \cot \frac{t-x}{2} dt$$

is a linear bounded operator $L_2[-\pi, \pi] \to L_2[-\pi, \pi]$,[11] then we obtain the required representation. A uniqueness follows from Liouville theorem. □

2.2.4 | Periodic analog of the Hilbert transform

Let's denote $\mathbb{Z}_+ = 0, 1, 2, \dots, \mathbb{Z}_- = \mathbb{Z} \backslash \mathbb{Z}_+$. The Fourier transform for function of discrete variable is the series

$$(F_d u_d)(\xi) = \sum_{k=-\infty}^{+\infty} u_d(k) e^{ik\xi}, \ \xi \in [-\pi, \pi].$$

Let's consider the one-dimensional discrete Fourier transform F_d for the indicator of $\mathbb{Z}_+$:

$$\chi_{\mathbb{Z}_+}(x) = \begin{cases} 1, \ x \in \mathbb{Z}_+ \\ 0, \ x \notin \mathbb{Z}_+ \end{cases}.$$

For summable functions, their product transforms to divided by 2π convolution of their Fourier images on the segment $[-\pi, \pi]$, but for our case $F_d(\chi_{\mathbb{Z}_+} \cdot u_d)$, one of functions $\chi_{\mathbb{Z}_+}$ is not summable. Thus, first, we introduce some regularizing multiplier and evaluate the following Fourier transform

$$F_d(e^{-\tau k} \cdot \chi_{\mathbb{Z}_+})(\xi) = \sum_{k \in \mathbb{Z}_+} e^{-\tau k} e^{ik\xi} = \sum_{k \in \mathbb{Z}_+} e^{-\tau k + ik\xi} =$$

$$= \sum_{k\in\mathbb{Z}_+} e^{ik(\xi+i\tau)} = \sum_{k\in\mathbb{Z}_+} e^{ikz}, \ \tau\to 0, \ z=\xi+i\tau, \ \tau>0.$$

The Fourier transform for the function $u_d(n)$ we'll denote $\tilde{u}(\xi)$, it is left to find the sum for e^{ikz},

$$\sum_{k\in\mathbb{Z}_+} e^{ikz} = 1 + e^{iz} + e^{2iz} + \ldots = \frac{1}{1-e^{iz}},$$

and after some transformations

$$\frac{1}{1-e^{iz}} = \frac{1}{2} - \frac{1}{2}\left(1 + \frac{2}{e^{iz}-1}\right) = \frac{1}{2} - \frac{1}{2}\frac{e^{iz}+1}{e^{iz}-1} = \frac{1}{2} - \frac{1}{2}\frac{e^{iz/2}+e^{-iz/2}}{e^{iz/2}-e^{-iz/2}}$$

we obtain

$$\sum_{k\in\mathbb{Z}_+} e^{ikz} = \frac{1}{2} - \frac{1}{2i}\cot\frac{z}{2}.$$

Therefore, according to the theorem on Fourier transform for a product of two functions, we have

$$F(\chi_{\mathbb{Z}_+}\cdot u_d)(\xi) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\tilde{u}(\xi)d\xi + \frac{i}{4\pi}\lim_{\tau\to 0+}\int_{-\pi}^{\pi}\tilde{u}(t)\cot\frac{z-t}{2}dt, \ z=\xi+i\tau.$$

According to Sokhotskii formulas from Lemma 1 (these are almost same for periodic kernel $\cot(x)$) (see also classical books[29,30]),

$$F(\chi_{\mathbb{Z}_+}\cdot u_d)(\xi) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\tilde{u}(\xi)d\xi + \frac{\hat{u}(\xi)}{2} - \frac{1}{4\pi i}v.p.\int_{-\pi}^{\pi}\tilde{u}(t)\cot\frac{\xi-t}{2}dt.$$

If we introduce the function $\chi_{\mathbb{Z}_-}(x)$ and consider the Fourier transform for the product $F(\chi_{\mathbf{Z}_-}\cdot u)$ with preliminary regularization, then we have

$$F(e^{-\tau k}\cdot\chi_{\mathbb{Z}_-}) = \sum_{-\infty}^{-1} e^{-\tau k}e^{ik\xi} = \sum_{-\infty}^{-1} e^{-\tau k+ik\xi} =$$
$$= \sum_{-\infty}^{-1} e^{ik(\xi+i\tau)} = \sum_{-\infty}^{-1} e^{ikz}, \ \tau\to 0, \ z=\xi+i\tau, \ \tau<0.$$

Further,

$$\sum_{-\infty}^{-1} e^{ikz} = -1 + 1 + e^{-iz} + e^{-2iz} + \ldots = -1 + \frac{1}{1-e^{-iz}} = -\frac{1}{2} + \frac{1}{2i}\cot\frac{z}{2}.$$

Then we obtain

$$F(\chi_{\mathbb{Z}_-}\cdot u_d) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\tilde{u}(\xi)d\xi - \lim_{\tau\to 0-}\frac{i}{4\pi}\int_{-\pi}^{\pi}\tilde{u}(t)\cot\frac{z-t}{2}dt, \ z=\xi+i\tau, \ \tau<0.$$

Applying Lemma 1 once again, we have

$$F(\chi_{\mathbb{Z}_-}\cdot u_d)(\xi) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\tilde{u}(\xi)d\xi + \frac{\tilde{u}(\xi)}{2} + \frac{1}{4\pi i}v.p.\int_{-\pi}^{\pi}\hat{u}(t)\cot\frac{\xi-t}{2}dt.$$

To verify, one can find the sum for $F(\chi_{\mathbf{Z}_+}\cdot u_d)$, $F(\chi_{\mathbf{Z}_-}\cdot u_d)$ and obtain

$$F(\chi_{\mathbb{Z}_+}\cdot u_d) + F(\chi_{\mathbb{Z}_-}\cdot u_d) = \tilde{u}(\xi).$$

Subtracting $F(\chi_{\mathbb{Z}_-}\cdot u_d)$ from $F(\chi_{\mathbb{Z}_+}\cdot u_d)$, we obtain

$$F(\chi_{\mathbf{Z}_+}\cdot u_d)-F(\chi_{\mathbf{Z}_-}\cdot u_d)=-\frac{1}{2\pi i}\int_{-\pi}^{\pi}\tilde{u}(t)\cot\frac{\xi-t}{2}dt,-$$

so that the operator

$$\tilde{u}(t)\mapsto\frac{i}{2\pi}\int_{-\pi}^{\pi}\tilde{u}(t)\cot\frac{\xi-t}{2}dt$$

in the space $L_2[-\pi,\pi]$ is unitary equivalent to the multiplier

$$sign\tilde{x}=\begin{cases}+1, & \text{if } \tilde{x}\in\mathbb{Z}_+;\\ -1, & \text{if } \tilde{x}\in\mathbb{Z}_-\end{cases}$$

in the space $L_2(\mathbb{Z})$. It permits to call this operator by periodic Hilbert transform.

Corollary 3. [*] *There are two projectors in the space $L_2[-\pi,\pi]$*

$$P=\frac{1}{2}(I+H^{per}),\ Q=\frac{1}{2}(I-H^{per})$$

such that

$$P+Q=I,\ PQ=QP-0.$$

Proof. If we denote by H^{per} the operator

$$(H^{per}u((\xi)=\frac{i}{2\pi}\int_{-\pi}^{\pi}\hat{u}(t)\cot\frac{\xi-t}{2}dt,$$

then we have

$$(H^{per})^2=I,$$

and the required properties follow from this fact immediately. □

It is very desirable to describe ranges of projector P and Q, but we will do it in the next section for h-lattice.

h-lattice

Here, we will consider the case $h\mathbb{Z}_\pm$ and will obtain analogs of above formulas for the h-lattice. It will be almost the same, but there are some nuances in calculations.

First, for $\tau>0$,

$$\sum_{\tilde{x}\in h\mathbb{Z}_+}e^{i\tilde{x}\xi}e^{-\tau\tilde{x}}h=h\sum_{\tilde{x}\in h\mathbb{Z}_+}e^{i\tilde{x}(\xi+i\tau)}=h\sum_{\frac{\tilde{x}}{h}\in\mathbb{Z}_+}e^{i\frac{\tilde{x}}{h}hz},\ z=\xi+i\tau.$$

Therefore,

$$\sum_{\tilde{x}\in h\mathbb{Z}_+}e^{i\tilde{x}\xi}e^{-\tau\tilde{x}}h=\frac{h}{1-e^{ihz}}=\frac{h}{2}-\frac{h}{2i}\cot\frac{hz}{2}.$$

[*] Really we have additional summands $\pm\frac{1}{4\pi}\int_{-\pi}^{\pi}\tilde{u}(\xi)d\xi$ for the projectors, but these summands correspond to values of u_d at zero point. Since we interested in positive or negative points, we assume that this value is zero without loss of generality. The same arguments we use further.

Analogously, for $\tau < 0$,

$$\sum_{\tilde{x}\in h\mathbb{Z}_-} e^{i\tilde{x}\xi} e^{-\tau\tilde{x}} h = -\frac{h}{2} + \frac{h}{2i}\cot\frac{hz}{2}.$$

According to above calculations to obtain the Fourier images of multipliers $\chi_{h\mathbb{Z}_\pm}(\tilde{x})$, we need to study boundary values of the integral

$$\Phi(z) = \frac{ih}{4\pi}\int_{-\hbar\pi}^{\hbar\pi} \varphi(x)\cot\frac{h(z-x)}{2}dx,\ z \in \hbar\Pi_\pm.$$

Using previous calculations, we change of variable $hx = y$ and denote $hz = \zeta \in \Pi_\pm, \Psi(\zeta) = \Phi\left(\frac{\zeta}{h}\right)$. Then we have $\zeta = h\xi + hi\tau$

$$\Psi(\zeta) = \frac{i}{4\pi}\int_{-\pi}^{\pi} \varphi\left(\frac{y}{h}\right)\cot\frac{\zeta - y}{2}dy,\ \zeta \in \Pi_\pm.$$

Then by Lemma 1, we conclude

$$\Psi^+(h\xi) = \frac{1}{4\pi i}\int_{-\pi}^{\pi} \varphi\left(\left(\frac{y}{h}\right)\right)\cot\frac{y - h\xi}{2}dy + \frac{\varphi(\xi)}{2}$$

$$\Psi^-(h\xi) = \frac{1}{4\pi i}\int_{-\pi}^{\pi} \varphi\left(\left(\frac{y}{h}\right)\right)\cot\frac{y - h\xi}{2}dy - \frac{\varphi(\xi)}{2}.$$

Now if we return to the variable x, then we obtain final formulas foe boundary values

$$\Phi^+(\xi) = \frac{h}{4\pi i}\int_{-\hbar\pi}^{\hbar\pi} \varphi(x)\cot\frac{h(x-\xi)}{2}dx + \frac{\varphi(\xi)}{2},$$

$$\Phi^-(\xi) = \frac{h}{4\pi i}\int_{-\hbar\pi}^{\hbar\pi} \varphi(x)\cot\frac{h(x-\xi)}{2}dx - \frac{\varphi(\xi)}{2}.$$

So we can introduce two projectors in the space $L_2(\hbar\mathbb{T})$. Let us denote

$$H_h^{per} = \frac{h}{4\pi i}\int_{-\hbar\pi}^{\hbar\pi} \varphi(x)\cot\frac{h(x-\xi)}{2}dx$$

and

$$P_h = \frac{1}{2}(I + H_h^{per}),\ Q_h = \frac{1}{2}(I - H_h^{per}).$$

Further, let us denote $\mathbb{A}_\pm$ subspaces of functions $u_\pm(\xi)$ from $L_2(\hbar\mathbb{T})$ which admit analytical continuation into $\Pi_\pm$ satisfying the condition

$$\sup_{\tau\in\mathbb{R}_\pm}\int_{-\hbar\pi}^{+\hbar\pi} |u_\pm(\xi + i\tau)|^2 d\xi < +\infty. \tag{8}$$

Lemma 2. *The space $L_2(\hbar\mathbb{T})$ can be uniquely represented as the following direct sum:*

$$L_2(\hbar\mathbb{T}) = \mathbb{A}_1 \oplus \mathbb{A}_2,$$

and for a function $u \in L_2(\hbar\mathbb{T})$, this representation looks as follows:

$$u(\xi) = (P_h u)(\xi) + (Q_h u)(\xi).$$

Proof. This proof is similar to proofs of Corollaries 2 and Corollary 3; only one additional parameter h is added. A reader can verify these facts himself. □

More fine moment is to describe images of projectors P_h and Q_h.

Let $D_d \subset h\mathbb{Z}$. By the space $L_2(D_d)$, we mean the space of functions from $L_2(h\mathbb{Z})$ which vanish outside of D_d with induced norm. Let us denote Fourier image of the space $L_2(D_d)$ by $L_2(\tilde{D}_d)$.

Lemma 3. *The following relations*

$$L_2(\tilde{h\mathbb{Z}}_+) = \mathbb{A}_1, \ L_2(\tilde{h\mathbb{Z}}_-) = \mathbb{A}_2$$

are valid.

Proof. We have proved earlier that an image of an arbitrary function from $L_2(h\mathbb{Z}_+)$ belongs to $\mathbb{A}_1$. It is left to prove that an arbitrary function from $\mathbb{A}_1$ has its own inverse image in the space $L_2(h\mathbb{Z}_+)$. Let's take the function $\tilde{u}_+ \in \mathbb{A}_1$ and consider its inverse Fourier image

$$u_d(\tilde{x}) = \frac{1}{2\pi}\int_{-\hbar\pi}^{\hbar\pi} \tilde{u}(\xi)e^{-i\tilde{x}\cdot\xi}d\xi.$$

Obviously, $u_d \in L_2(h\mathbb{Z})$, and we need to prove that $u_d(\tilde{x}) = 0$ for $\tilde{x} \in \mathbb{Z}_-$. We will introduce the function $e^{\tilde{x}\cdot\tau}u_d(\tilde{x},\tau)$,

$$u_d(\tilde{x},\tau) = \frac{1}{2\pi}\int_{-\hbar\pi}^{\hbar\pi} \tilde{u}(\xi + i\tau)e^{-i\tilde{x}\cdot\xi}d\xi,$$

and will prove that it does not depend on τ. Indeed, we have

$$\frac{\partial}{\partial\tau}(e^{\tilde{x}\cdot\tau}u_d(\tilde{x},\tau)) = \tilde{x}e^{\tilde{x}\cdot\tau}u_d(\tilde{x},\tau) + e^{\tilde{x}\cdot\tau}\frac{\partial u_d(\tilde{x},\tau)}{\partial\tau}.$$

Further,

$$\frac{\partial u_d(\tilde{x},\tau)}{\partial\tau} = \frac{1}{2\pi}\int_{-\hbar\pi}^{\hbar\pi} \frac{\partial\tilde{u}(\xi + i\tau)}{\partial\tau}e^{-i\tilde{x}\cdot\xi}d\xi.$$

Since $\tilde{u}_d \in \mathbb{A}_1$, we have Cauchy–Riemann equations according to them

$$\frac{\partial\tilde{u}(\xi + i\tau)}{\partial\tau} = i\frac{\partial\tilde{u}(\xi + i\tau)}{\partial\xi}.$$

Now we apply the Fourier transform properties or by integration by parts, we obtain

$$\frac{1}{2\pi}\int_{-\hbar\pi}^{\hbar\pi} \frac{\partial\tilde{u}(\xi + i\tau)}{\partial\tau}e^{-i\tilde{x}\cdot\xi}d\xi = \frac{i}{2\pi}\int_{-\hbar\pi}^{\hbar\pi} \frac{\partial\tilde{u}(\xi + i\tau)}{\partial\xi}e^{-i\tilde{x}\cdot\xi}d\xi =$$

$$\frac{i}{2\pi}\left(\tilde{u}(\xi + i\tau)e^{-i\tilde{x}\cdot\xi)}\Big|_{-\hbar\pi}^{\hbar\pi} + i\tilde{x}\int_{-\hbar\pi}^{\hbar\pi}\tilde{u}(\xi + i\tau)e^{-i\tilde{x}\cdot\xi}d\xi\right) = -\frac{\tilde{x}}{2\pi}\int_{-\hbar\pi}^{\hbar\pi}\tilde{u}(\xi + i\tau)e^{-i\tilde{x}\cdot\xi}d\xi.$$

Therefore, we have

$$\frac{\partial u_d(\tilde{x},\tau)}{\partial \tau} = -\frac{\tilde{x}}{2\pi}\int_{-\hbar\pi}^{\hbar\pi}\tilde{u}(\xi+i\tau)e^{-i\tilde{x}\cdot\xi}d\xi. = -\tilde{x}u_d(\tilde{x},\tau).$$

Thus, we have proved that

$$\frac{\partial}{\partial\tau}(e^{\tilde{x}\cdot\tau}u_d(\tilde{x},\tau)) = 0.$$

It means that there exist the function $v_d(\tilde{x})$ such that $e^{\tilde{x}\cdot\tau}u_d(\tilde{x},\tau) = v_d(\tilde{x})$ os $u_d(\tilde{x},\tau) = e^{-\tilde{x}\cdot\tau}v_d(\tilde{x})$.

Let us show that $u_d(\tilde{x},\tau) \in L_2(h\mathbb{Z}^m)$ with respect to the variable $\tilde{x}$. According to Parceval equality, we have

$$\sum_{\tilde{x}\in h\mathbb{Z}^m}|u_d(\tilde{x},\tau)|^2 h = \frac{1}{2\pi}\int_{-\hbar\pi}^{\hbar\pi}|\tilde{u}(\xi+i\tau)|^2 d\xi$$

for $\forall\tau\in\mathbb{R}_+$, and taking into account the formula (8), we obtain

$$\sum_{\tilde{x}\in h\mathbb{Z}^m}|u_d(\tilde{x},\tau)|^2 h \le C$$

with C non-depending on τ. Then

$$\sup_{\tau\in\mathbb{R}_+}\sum_{\tilde{x}\in h\mathbb{Z}^m}|v_d(\tilde{x})|^2 e^{-2\tilde{x}\cdot\tau}h \le C, \tag{9}$$

that is, the series (9) converges $\forall\tau\in\mathbb{R}_+$. But if $\tilde{x}<0$, then $e^{-2\tilde{x}\cdot\tau}\to+\infty$ under $\tau\to+\infty$, and a convergence for the series (9) is impossible. Hence, $v_d(\tilde{x}) = 0$ for $\tilde{x}<0$. Also, it follows from (9), $v_d\in L_2(h\mathbb{Z}_+)$, and

$$v_d(\tilde{x}) = u_d(\tilde{x},0) = (F_d^{-1}\tilde{u})(\tilde{x}).$$

It completes the proof. □

2.2.5 | Statement and solving periodic Riemann problem

Here, we will return to Section 2.2.3 and will show how the periodic Riemann boundary value problem is connected with certain integral equation which is generated by projectors P_h and Q_h (see classical books[29–31]). We consider the equation

$$(aP_h + bQ_h)u = v \tag{10}$$

in the space $L_2(\hbar\mathbb{T}^m)$, where a, b, v are given functions, $v\in L_2(\hbar\mathbb{T}^m)$ and a, b are smooth periodic functions defined in $\hbar\mathbb{T}^m$. Let us show that this equation is closely related to the periodic Riemann boundary value problem (7).

A solvability theory for periodic Riemann boundary value problem is constructed in Vasil'ev A and Vasil'ev V,[14] and here, we will use only one fragment of the theory, namely, necessary and sufficient condition for unique solvability of the boundary value problem. We give this property in terms of functions a and b like Gohberg and Krupnik.[31]

We will remind some connections between Equation (10) and periodic Riemann boundary value problem (7) for the segment $\hbar\mathbb{T}$.

Let us introduce the function

$$\Phi(z) = \frac{ih}{4\pi}\int_{-\hbar\pi}^{\hbar\pi}u(x)\cot\frac{h(z-x)}{2}dx,\ z\in\hbar\Pi_\pm,\ z=\xi+i\tau.$$

Then according to Section 2.2.4, we can write

$$(P_h u)(\xi) = \Phi^-(\xi),\ (Q_h u)(\xi) = \Phi^+(\xi),$$

so that

$$a(\xi)\Phi^-(\xi) + b(\xi)\Phi^+(\xi) = v(\xi),$$

or in other notation

$$\Phi^+(\xi) = -\,-\frac{a(\xi)}{b(\xi)}\Phi^-(\xi) + \frac{v(\xi)}{b(\xi)}$$

under the condition $b(\xi) \neq 0 \forall \xi \in \hbar\mathbb{T}$. If we put

$$G(\xi) = -\frac{a(\xi)}{b(\xi)},\ g(\xi) = fracv(\xi)b(\xi),$$

we obtain the periodic Riemann boundary value problem (7).

Let $G(\xi)$ be a periodic continuous non-vanishing function defined in $\hbar\mathbb{T}$.

Definition 2. Index of the function $G(\xi)$ is called the integer

$$æ = \frac{1}{2\pi}\int_{-\hbar\pi}^{+\hbar\pi} d\arg G(\xi).$$

Now we will rewrite the periodic Riemann boundary value problem for $\hbar\mathbb{T}$ and will prove its unique solvability under a special condition. Thus, we study the following problem: finding a pair of functions $\Phi^{\pm}(\xi) \in L_2(\hbar\mathbb{T})$ which admit holomorphic continuation into $\hbar\Pi_{\pm}$ and their boundary values on $\hbar\mathbb{T}$ should satisfy the lineal relation

$$\Phi^+(\xi) = G(\xi)\Phi^-(\xi) + g(\xi), \tag{11}$$

where G, g are given functions, $g(\xi) \in L_2(\hbar\mathbb{T})$, and $G(\xi)$ is continuous non-vanishing periodic function in $\hbar\mathbb{T}$.

Proposition 1. *Let $æ = 0$. Then the periodic Riemann boundary value problem (11) has a unique solution.*

Proof. If $æ = 0$, then $G(\xi)$ admits a special factorization

$$G(\xi) = \frac{G_+(\xi)}{G_-(\xi)} \tag{12}$$

such that the factors $G_{\pm}(\xi)$ admit bounded holomorphic continuation into $\hbar\Pi_{\pm}$, respectively. Indeed, the function $\ln G(\xi)$ is a univalent function, and according to Lemma 1 and Section 2.2.4, it can be represented by the formula

$$\ln G(\xi) = \Theta^+(\xi) - \Theta^-(\xi),$$

where

$$\Theta(z) = \frac{ih}{4\pi}\int_{-\hbar\pi}^{\hbar\pi} \ln G(x)\cot\frac{z-x}{2}dx,\ z = \xi + i\tau \in \hbar\Pi_{\pm},$$

Now we put $G_{\pm}(\xi) = \exp(\Theta^{\pm}(\xi))$, and the representation (14) is obtained.

Further, we rewrite the problem (11) in the form

$$\frac{\Phi^+(\xi)}{G_+(\xi)} - \frac{\Phi^-(\xi)}{G_-(\xi)} = \frac{g(\xi)}{G_+(\xi)},$$

from which we obtain the solution for the problem (11) as follows:

$$\Phi^{\pm}(\xi) = \pm\frac{g(\xi)}{2} + \frac{hG_+(\xi)}{4\pi i}v.p.\int_{-\hbar\pi}^{\hbar\pi}\frac{g(\xi)}{G_+(\xi)}\cot\frac{\xi - x}{2}dxi.$$

Uniqueness at each step follows from Liouville theorem. □

Remark 1. If æ ≠ 0, then there is no uniqueness theorem in the problem (13); one has either formula for a general solution or certain solvability conditions (see Vasil'ev A and Vasil'ev V[14]).

Corollary 4. *Equation (10) is uniquely solvable in the space $L_2(\hbar\mathbb{T})$ if and only if*

$$Ind\,\frac{a(\xi)}{b(\xi)} = 0.$$

Proof. We have shown above that Equation (10) is equivalent in a solvability sense to the problem (13). Using this fact and correspondence between a, b, and G, we obtain the required result. □

2.2.6 | Solvability conditions for discrete operators in a half-space

Before starting, we remind that a solvability of Equation (2) in the space $L_2(\hbar\mathbb{Z}^m_+)$ is equivalent to a solvability of Equation (10) and the space $L_2(\hbar\mathbb{Z}^m)$ with $b(\xi)\equiv 1$. Indeed, if we have two equations

$$(Au)(x) = f(x),\ x \in \mathbb{R}^m_+, \tag{13}$$

and

$$(AP_+U + IP_-U)(x) = F(x),\ x \in \mathbb{R}^m, \tag{14}$$

where $f(x)$ coincides with $F(x)$ in $\mathbb{R}^m_+$ then

$$u(x) = U(x),\ x \in \mathbb{R}^m_+.$$

Conversely, if $u(x)$ is solution of Equation (13), then

$$U(x) = \begin{cases} u(x), & \text{if } x \in \mathbb{R}^m_+; \\ (Au)(x), & \text{if } \tilde{x} \in \mathbb{R}^m_- \end{cases}$$

is solution of the equation (14) with

$$F(x) = \begin{cases} f(x), & \text{if } \tilde{x} \in \mathbb{R}^m_+; \\ (Au)(x), & \text{if } \tilde{x} \in \mathbb{R}^m_-. \end{cases}$$

Here, we suppose, additionally, that the symbol $\sigma(\xi', \xi_m)$ satisfies the condition

$$\sigma(0, \dots, 0, -1) = \sigma(0, \dots, 0, +1). \tag{15}$$

Theorem 2. *Equations (1) and (2) are uniquely solvable or unsolvable simultaneously for all $h > 0$.*

Proof. If we fix ξ' in the cube $[-\pi, \pi]^m$, then under varying ξ_m on $[-\pi, \pi]$, the argument of $\sigma_1(\xi)$ will vary along the curve on cubical surface of $[-\pi, \pi]^m$, which unites the points $(0, \dots, 0, -\pi)$ and $(0, \dots, ., \pi)$ (for the case $m\geq 3$, all such curves are homotopic, and for the case $m = 2$, there are two curves left and right one).

This varying corresponds to the varying of the argument of function $\sigma(\xi)$ along the curve from point A_1 to point A_2 on the unit sphere. Further, if we consider the symbol $\sigma_h(\xi)$ now on the cube $[-h^{-1}\pi, h^{-1}\pi]^m$, then $h\xi_m$ will be varied on $[-\pi, \pi]$ also under fixed $h\xi'$. In other words, the argument of $\sigma_h(\xi)$ for fixed ξ' (we consider small $h > 0$) will be varied along curve on cubical surface of $[-h^{-1}\pi, h^{-1}\pi]^m$, which unites the points $(0, \dots, 0, -h^{-1}\pi)$ and $(0, \dots, 0, h^{-1}\pi)$. It corresponds to varying argument of function $\sigma(\xi)$ from point B_1 to point B_2 on unit sphere. Obviously, under decreasing h, the sequence $A_1, B_1, \dots$ will be convergent to south pole of the unit sphere $(0, \dots, 0, -1)$, and the sequence $A_2, B_2, \dots$ to north pole $(0, \dots, 0, +1)$. Thus, because the variation of argument of $\sigma_h(\xi)$ on ξ_m under fixed ξ' is $2\pi k$ (σ_h is periodic function), then under additional assumption

$$\sigma(0, \dots, 0, -1) = \sigma(0, \dots, 0, +1)$$

(this property is usually called the transmission property), we'll obtain that variation of argument for the function $\sigma(\xi)$ under varying ξ_m from $-\infty$ to $+\infty$ under fixed ξ' (this variation of $\sigma(\xi)$ moves along the arc of big half-circumference on unit sphere) is also $2\pi k$. According to our assumptions on continuity of $\sigma(\xi)$ on the unit sphere, it will be the same number $2\pi k, k \in \mathbf{Z}$,

$$\lim_{h\to 0} \int_{-\pi h^{-1}}^{\pi h^{-1}} d\arg \sigma_h(\xi', \xi_m) = \int_{-\infty}^{+\infty} d\arg \sigma(\xi', \xi_m), \quad \forall \xi' \neq 0.$$

So both for Equations (1) and (2), the uniquely solvability condition is defined by the same number. This completes the proof. □

We see that both continual and discrete equations are solvable or unsolvable simultaneously and then we need to find good finite approximation for infinite system of linear algebraic equations for computer calculations. First steps in this direction were done in the paper Vasil'ev A and Vasil'ev V,[14,15] where the authors suggested to use fast Fourier transform.

3 | PART II: PSEUDO-DIFFERENTIAL OPERATORS

This part is devoted to developing previous results to more general operators and related equations. We give here our general concept of development and describe some principal results. For certain results, full proofs are given, and for some proofs, we recommend to a reader special papers. For considered general operators, we need appropriate discrete spaces, and we start from such a point. Results on a solvability for discrete pseudo-differential equations are proved in Vasil'ev A and Vasil'ev V,[32,33] and first results on an approximation rate are proved in Tarasova and Vasil'ev.[34]

3.1 | Discrete operators

3.1.1 | Discrete Sobolev–Slobodetskii spaces

Let $S(\mathbb{R}^m)$ be the Schwartz space of infinitely differentiable functions rapidly decreasing at infinity.

Let us denote $\zeta^2 = h^{-2}\sum_{k=1}^{m}(e^{-ih\cdot\xi_k} - 1)^2$ and introduce the following.

Definition 3. The space $H^s(h\mathbb{Z}^m)$ is a closure of the space $S(h\mathbb{Z}^m)$ with respect to the norm

$$||u_d||_s = \left(\int_{\hbar\mathbb{T}^m} (1 + |\zeta^2|)^s |\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2}. \quad (16)$$

We would like to note that a lot of properties for such spaces were studied in Frank.[35]

Further, let $D \subset \mathbb{R}^m$ be a domain and $D_d = D \cap h\mathbb{Z}^m$ be a discrete domain.

Definition 4. The space $H^s(D_d)$ consists of discrete functions from $H^s(h\mathbb{Z}^m)$ which supports belong to $\overline{D_d}$. A norm in the space $H^s(D_d)$ is induced by a norm of the space $H^s(h\mathbb{Z}^m)$. The space $H_0^s(D_d)$ consists of discrete functions u_d with a support in D_d, and these discrete functions should admit a continuation into the whole $H^s(h\mathbb{Z}^m)$. A norm in the $H_0^s(D_d)$ is given by the formula

$$||u_d||_s^+ = \inf ||\ell u_d||_s,$$

where infimum is taken over all continuations ℓ.

The Fourier image of the space $H^s(D_d)$ will be denoted by $\tilde{H}^s(D_d)$.

Such spaces were studied in detail in Frank.[35] Of course, all norms (16) are equivalent to the L_2-norm, but this equivalence depends on h. Let us note that all constants below in our considerations do not depend on h.

3.1.2 | Discrete pseudo-differential operators

Let $\tilde{A}_d(\xi)$ be a measurable periodic function in $\mathbb{R}^m$ with the basic cube of periods $\hbar\mathbb{T}^m$. Such functions are called symbols. As usual, we will define a discrete pseudo-differential operator by its symbol.

Definition 5. A discrete pseudo-differential operator A_d in a discrete domain D_d is called an operator of the following kind:

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^m} \int_{\hbar\mathbb{T}^m} \tilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \ \tilde{x} \in D_d,$$

An operator A_d is called an elliptic operator if

$$ess \inf_{\xi\in\hbar\mathbb{T}^m} \left|\tilde{A}_d(\xi)\right| > 0.$$

First as usual, we define the operator A_d on the dense set $S(h\mathbb{Z}^m)$ and then extend it on more general space.

Definition 6. The class E_α includes symbols satisfying the following condition:

$$c_1(1 + |\zeta^2|)^{\alpha/2} \le |A_d(\xi)| \le c_2(1 + |\zeta^2|)^{\alpha/2} \tag{17}$$

with universal positive constants c_1, c_2 non-depending on h and the symbol $A_d(\xi)$.

The number $\alpha \in \mathbb{R}$ is called an order of a discrete pseudo-differential operator A_d.

Remark 2. In Definitions 5 and 6, the number h (and respectively $\hbar$) is a parameter which can change. But the estimate (17) holds for arbitrary values of h with the same constants c_1, c_2. The symbol $A_d(\xi)$ can be defined on different domains $\hbar\mathbb{T}^m$ so that we say on a certain family of symbols in some sense.

In fact, the symbols $A_d(\xi)$ do not depend on h, and the domain $\hbar\mathbb{T}^m$ changes only for different h. One can include the notation $A_{d,h}(\xi)$, but it seems it will not bring a clarity.

3.2 | Discrete equations

We study the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \ \tilde{x} \in D_d, \tag{18}$$

assuming that we interested in a solution $u_d \in H^s(D_d)$ taking into account $v_d \in H_0^{s-\alpha}(D_d)$.

Main difficulty for this problem is related to a geometry of the domain D. Indeed, if $D = \mathbb{R}^m$, then the condition (17) guarantees the unique solvability for the Equation (18). We will consider here only so-called canonical domains and simplest discrete pseudo-differential operators with symbols non-depending on a spatial variable $\tilde{x}$. This fact is dictated by using in future the local principle. The last asserts that for a Fredholm solvability of the general Equation (18) with symbol $A_d(\tilde{x}, \xi)$ in an arbitrary discrete domain D_d, one needs to obtain invertibility conditions for so-called local representatives of the operator A_d, that is, for an operator with symbol $A_d(\cdot, \xi)$ in a special canonical domain.

Earlier authors have extracted some canonical domains, namely, $D = \mathbb{R}^m, \mathbb{R}^m_+, C^a_+$, where $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x', x_m), x_m > 0\}$, $C^a_+ = \{x \in \mathbb{R}^m : x_m > a|x'|, a > 0\}$. Methods for studying two last cases are related to special boundary value problems for holomorphic functions.[12,14,15,25–27]

Everywhere below we study the case $D = \mathbb{R}^m_+$.

3.2.1 | Solvability

Definition 7. Periodic factorization of an elliptic symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form

$$A_d(\xi) = A_{d,+}(\xi) A_{d,-}(\xi),$$

where the factors $A_{d,\pm}(\xi)$ admit an analytical continuation into half-strips $\hbar\Pi_\pm$ on the last variable ξ_m for almost all fixed $\xi' \in \hbar\mathbb{T}^{m-1}$ and satisfy the estimates

$$|A^{\pm 1}_{d,+}(\xi)| \le c_1(1 + |\hat{\zeta}^2|)^{\pm\frac{æ}{2}}, \ |A^{\pm 1}_{d,-}(\xi)| \le c_2(1 + |\hat{\zeta}^2|)^{\pm\frac{\alpha-æ}{2}},$$

with constants c_1, c_2 non-depending on h,

$$\hat{\zeta}^2 \equiv \hbar^2 \left(\sum_{k=1}^{m-1} (e^{-ih\xi_k} - 1)^2 + (e^{-ih(\xi_m + i\tau)} - 1)^2 \right), \ \xi_m + i\tau \in \hbar\Pi_{\pm}.$$

The number æ $\in \mathbb{R}$ is called an index of periodic factorization.

For an elliptic symbol $A_d(\xi)$, such periodic factorization exists always (see Vasil'ev A and Vasil'ev V and Eskin[14,24]). Using such a representation, we can describe a solvability of model elliptic pseudo-differential equations.[28,32]
Let ℓv_d be an arbitrary continuation of v_d into $H^{s-\alpha}(h\mathbb{Z}^m)$. For special case, we have the following result.

Theorem 3. *If the elliptic symbol $\tilde{A}_d(\xi) \in E_\alpha$ admits periodic factorization with index æ so that $|æ - s| < 1/2$, then the Equation (18) has unique solution in the space $H^s(D_d)$ for arbitrary right-hand side $v_d \in H^{s-\alpha}(D_d)$,*

$$\tilde{u}_d(\xi) = \tilde{A}_{d,+}^{-1}(\xi) P_h(\tilde{A}_{d,-}^{-1}(\xi) \widetilde{\ell v_d}(\xi)).$$

It is easy to see that the solution does not depend on choice of continuation ℓv_d.
Further, we consider more complicated case when the condition $|æ - s| < 1/2$ does not hold. There are two possibilities in this situation, and we consider one case which leads to typical boundary value problems.

Theorem 4. *Let $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$. Then a general solution of Equation (18) in Fourier images has the following form.*

$$\tilde{u}_d(\xi) = \tilde{A}_{d,+}^{-1}(\xi) X_n(\xi) P_h(X_n^{-1}(\xi) \tilde{A}_{d,-}^{-1}(\xi) \widetilde{\ell v_d}(\xi)) + \tilde{A}_{d,+}^{-1}(\xi) \sum_{k=0}^{n-1} \tilde{c}_k(\xi') \hat{\zeta}_m^k,$$

where $X_n(\xi)$ is an arbitrary polynomial of order n of variables $\hat{\zeta}_k = \hbar(e^{-ih\xi_k} - 1), k = 1, \ldots, m$, satisfying the condition (17) for $\alpha = n$, $\tilde{c}_k(\xi'), j = 0, 1, \ldots, n-1$, are arbitrary functions from $H^{s_k}(h\mathbb{T}^{m-1}), s_k = s - æ + k - 1/2$.
The a priori estimate

$$||u_d||_s \leq a \left(||v_d||_{s-\alpha}^+ + \sum_{k=0}^{n-1} [c_k]_{s_k} \right)$$

holds, where $[\cdot]_{s_k}$ denotes a norm in the space $H^{s_k}(h\mathbb{Z}^{m-1})$ and the constant a does not depend on h.

Theorem 5. *Let $æ - s = -n + \delta, n \in \mathbb{N}, |\delta| < 1/2$. Then the solution of Equation (18) can be represented on the form*

$$\tilde{u}_d(\xi) = \sum_{j=0}^{n} \tilde{b}_j(\xi') \left(\frac{e^{ih\xi_m} - 1}{h} \right)^{-j} A_{d,+}^{-1}(\xi) + \tilde{U}_d(\xi), \tag{19}$$

where

$$\tilde{U}_d(\xi) = \left(\frac{e^{ih\xi_m} - 1}{h} \right)^{-n} A_{d,+}^{-1}(\xi) (P_h(A_{d,-}^{-1}(\xi', \eta_m)(\widetilde{\ell v_d})(\xi', \eta_m) \left(\frac{e^{ih\eta_m} - 1}{h} \right)^n))(\xi', \xi_m),$$

$$\tilde{b}_j(\xi') = h \int_{-\hbar\pi}^{\hbar\pi} \left(\frac{e^{ih\xi_m} - 1}{h} \right)^j A_{d,-}^{-1}(\xi', \xi_m)(\widetilde{\ell v_d})(\xi', \xi_m) d\xi_m,$$

A priori estimates

$$||U_d||_s \leq c||v_d||_{s-\alpha}^+, \ ||b_j||_{s_j} \leq c||v_d||_{s-\alpha}^+, \ j = 0, 1, \ldots, n$$

hold.

Remark 3. One can show that the representation (19) is unique and the solution u_d belongs to $H^s(D_d)$ iff $\tilde{b}_j(\xi') \equiv 0, j = 0, 1, \ldots, n$.

3.2.2 | Discrete boundary value problems

Here, we consider only one case $æ - s = 1 + \delta, |\delta| < 1/2$ related to Theorem 4. Moreover, for simplicity, we put $v_d \equiv 0$. Then the kernel of the operator A_d includes only one arbitrary function so that we need only one additional condition. We give one example here with the discrete Dirichlet condition

$$\begin{cases} (A_d u_d)(\tilde{x}) = 0, \ \tilde{x} \in D_d, \\ u_d(\tilde{x}', 0) = g_d(x'), \ \tilde{x}' \in h\mathbb{Z}^{m-1}. \end{cases} \tag{20}$$

This problem has unique solution; more exactly, the following result is valid.

Theorem 6. *Let $æ - s = 1 + \delta, |\delta| < 1/2, s > 1/2$. Discrete boundary value problem (20) is uniquely solvable in the space $H^s(D_d)$ for arbitrary boundary function $g_d \in H^{s-1/2}(h\mathbb{Z}^{m-1})$.*

3.2.3 | Comparison

Such discrete boundary value problems can be used as discrete approximations for their continuous analogs. For example, the continuous analog of the discrete boundary value problem (20) is the following:

$$\begin{cases} (Au)(x) = 0, \ x \in \mathbb{R}^m_+, \\ u(x', 0) = g(x'), \ x' \in \mathbb{R}^{m-1}, \end{cases} \tag{21}$$

where A is a pseudo-differential operator with symbol $A(\xi)$ satisfying the condition

$$c_1(1 + |\xi|)^\alpha \le |A(\xi)| \le c_2(1 + |\xi|)^\alpha$$

with positive constants c_1, c_2 and factorization index æ with respect to the variable ξ_m.[24] Let us remind that explicit solution of the problem (21) can be represented by the formula

$$u(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\mathbf{R}^m} e^{i\tilde{x}\cdot\xi} b^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)d\xi,$$

where $A_+(\xi', \xi_m)$ is element of factorization of the symbol $A(\xi)$,

$$b(\xi') = \int\limits_{-\infty}^{+\infty} A_+^{-1}(\xi', \xi_m)d\xi_m,$$

under the condition

$$\inf_{\xi' \in \mathbb{R}^{m-1}} |b(\xi')| > 0. \tag{22}$$

Given function g, we construct its discrete approximation in the following way. We take the Fourier transform $\tilde{g}$, restrict it on $\hbar\mathbb{T}^{m-1}$, and do $(m-1)$-dimensional inverse discrete Fourier transform F_d^{-1}. It will be the function g_d. We construct the discrete pseudo-differential operator A_d in a similar way.[34] First we take factorization

$$A(\xi) = A_+(\xi) \cdot A_-(\xi)$$

and then we take restrictions $A_+(\xi), A_-(\xi)$ on $\hbar\mathbb{T}^m$ and periodically continue them onto $\mathbb{R}^m$. Thus, we obtain two factors of periodic factorization $A_{d,+}(\xi), A_{d,-}(\xi)$, and finally, we put

$$A_d(\xi) = A_{d,+}(\xi) \cdot A_{d,-}(\xi).$$

It permits to obtain the following comparison.

Theorem 7. *Let $æ > 1, s > m/2, g \in H^{s-1/2}(\mathbb{R}^{m-1})$. A comparison between solutions of problems (20) and (21) under the condition (22) is given in the following way:*

$$|u(\tilde{x}) - u_d(\tilde{x})| \le Ch^{æ-1},\ \tilde{x} \in h\mathbb{Z}^m.$$

Proof. We need to compare two integrals:

$$u(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\mathbf{R}^m} e^{i\tilde{x}\cdot\xi} b^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)d\xi$$

and

$$u_d(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi} b_d^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)d\xi$$

for $\tilde{x} \in h\mathbf{Z}^m$ taking into account that[27]

$$b_d(\xi') = \int\limits_{-\hbar\pi}^{+\hbar\pi} A_{d,+}^{-1}(\xi', \xi_m)d\xi_m,$$

Thus, we have

$$u(\tilde{x}) - u_d(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi}(b^{-1}(\xi) - b_d^{-1}(\xi'))\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)d\xi$$
$$+\frac{1}{(2\pi)^m} \int\limits_{\mathbf{R}^m\setminus\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi} b^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)dxii$$

because the functions $\tilde{g}, \tilde{g}_d$ and $A_+, A_{d,+}$ coincide in $\hbar\mathbf{T}^m$.

Now we estimate the second integral.

$$\left|\int\limits_{\mathbf{R}^m\setminus\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi} b^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)d\xi\right| \le const \int\limits_{\mathbf{R}^m\setminus\hbar\mathbf{T}^m} |\tilde{g}(\xi')||A_+^{-1}(\xi', \xi_m)|d\xi$$

$$\le const \int\limits_{\mathbf{R}^{m-1}\setminus\hbar\mathbf{T}^{m-1}} |\tilde{g}(\xi')| \left(\int\limits_{-\infty}^{-\hbar\pi} + \int\limits_{\hbar\pi}^{+\infty}\right) |A_+^{-1}(\xi', \xi_m)|d\xi_m d\xi'.$$

Further, we estimate

$$\left(\int\limits_{-\infty}^{-\hbar\pi} + \int\limits_{\hbar\pi}^{+\infty}\right) |A_+^{-1}(\xi', \xi_m)|d\xi_m \le const \int\limits_{\hbar\pi}^{+\infty} (1 + |\xi'| + |\xi_m|)^{-æ} d\xi_m$$
$$= \frac{const}{æ - 1}(1 + |\xi'| + \hbar\pi)^{1-æ} \le c_6 h^{æ-1}.$$

Now by Cauchy–Schwartz inequality, we have

$$\int\limits_{\mathbf{R}^{m-1}\setminus\hbar\mathbf{T}^{m-1}} |\tilde{g}(\xi')|d\xi' \le \left(\int\limits_{\mathbf{R}^{m-1}\setminus\hbar\mathbf{T}^{m-1}} |\tilde{g}(\xi')|^2(1 + |\xi'|)^{2s-1}d\xi'\right)^{1/2} \left(\int\limits_{\mathbf{R}^{m-1}\setminus\hbar\mathbf{T}^{m-1}} (1 + |\xi'|)^{-2s+1}d\xi'\right)^{1/2}.$$

Since $g \in H^{s-1/2}(\mathbf{R}^{m-1})$,[24] the first factor is less than $[g]_{s-1/2}$, and the second one tends to zero if $s > m/2$.

For the first integral, we use the estimate

$$|b^{-1}(\xi') - b_d^{-1}(\xi')| \le const \cdot h^{æ-1}$$

(see Tarasova and Vasil'ev[34])

Finally,

$$\left| \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi}(b^{-1}(\xi) - b_d^{-1}(\xi'))\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)d\xi \right|$$
$$\le const \cdot h^{æ-1} \int\limits_{\hbar\mathbf{T}^m} |\tilde{g}(\xi')||A_+^{-1}(\xi', \xi_m)|d\xi \le const \cdot h^{æ-1} \int\limits_{\hbar\mathbf{T}^{m-1}} \frac{|\tilde{g}(\xi')|}{(1 + |\xi'|)^{æ-1}} d\xi'$$

and further as above using Cauchy–Schwartz inequality. □

4 | CONCLUSION

Some elements of the discrete theory of pseudo-differential equations and related boundary value problems are presented in the paper. The results describe solvability conditions and certain approximation properties of the discrete equations. These considerations are related to simple examples of equations and boundary value problems. This approach can be transferred to more general situations. The authors work in this direction.

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CONFLICT OF INTEREST

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