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ON CERTAIN OPERATOR FAMILIES

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Abstract. In this paper, we propose an abstract scheme for the study of special operators and apply this scheme to examining elliptic pseudo-differential operators and related boundary-value problems on manifolds with nonsmooth boundaries. In particular, we consider cases where boundaries may contain conical points, edges of various dimensions, and even peak points. Using the constructions proposed, we present well-posed formulations of boundary-value problems for elliptic pseudo-differential equations on manifolds discussed in Sobolev–Slobodecky spaces.

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1. Introduction. In the theory of pseudo-differential operators and the corresponding equations and boundary-value problem, a key role is played by the notion of the symbol of a pseudo-differential operator. Pseudo-differential operators themselves, ellipticity conditions, and the Fredholm property are defined in terms of symbols. The symbols are considered as functions defined on certain geometric structures, for example, on smooth manifolds (cotangent bundles) in the case of pseudo-differential operators. All arithmetical operations in the algebra of operator (factorized by the ideal of compact operator) are inherited by the algebra of symbols, so that all attention is paid to the study of the algebra of symbols, which can be commutative (the scalar case) or not (the matrix case). The approach related to operator algebras is most popular in the last two or three decades (see [1, 7–9]), but there also exists "analytic" works (see [2, 4–6, 10]) whose main theme is the description of conditions for the Fredholm property of operators considered and the calculation of their indexes.

In this paper, we propose an operator approach to this problem based on considering special operators of the local type (see [11]); wide classes of pseudo-differential operators are of this type. Some preliminary results can be found in the author's works [12–15].

2. Operators of the local type and envelopes. We recall some general notions and ideas related to special classes of operators and based on constructions proposed by I. B. Simonenko (see [11]).

2.1. Basic notions. Let H_1 and H_2 be Hilbert (Banach) spaces consisting of functions defined on a compact manifold (perhaps, with boundary) M and $A : H_1 \to H_2$ be a linear bounded operator. Following [11], we introduce the following notion.

Definition 1. An operator A is called an *operator of the local type* if the operator $P_U A P_V$ is compact for any two compact disjoint sets $K_1, K_2 \subset M$, $K_1 \cap K_2 = \emptyset$, where P_K is the projector onto the set K; more precisely,

$$(P_K f)(x) = \begin{cases} f(x), & x \in K; \\ 0, & x \in M \setminus K \end{cases}$$

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for all $f(x) \in H_1(H_2)$.

Below we consider only operators of the local type.

We denote by $|||A|||$ the essential norm of the operator A,

$$|||A||| \equiv \inf ||A + T||,$$

where the infimum is taken over all compact operators $T : H_1 \to H_2$.

Definition 2. An operator $A_x : H_1 \to H_2$ is called a *local representative* of an operator A at a point $x \in M$ if for any $\varepsilon > 0$ there exists a neighborhood U of the point x on the manifold M, $x \in U \subset M$, such that the inequality

$$|||P_U(A - A_x)||| < \varepsilon$$

holds; notation $A \overset{x}{\sim} A_x$.

Definition 3. A *symbol* of an operator A is the operator-valued function $A(x) : M \to \{A_x\}_{x \in M}$ defined by its local representatives.

It is easy to verify (s [11]) that this definition of the symbol preserves all properties of the symbolic calculus. Namely, up to a compact term, the following assertions hold:

- the product and sum of two operators correspond to the product and sum of their local representatives;
- the adjoint operator corresponds to the adjoint local representative;
- the Fredholm property of an operator corresponds to the Fredholm property of the local representative.

By a Fredholm operator we mean a linear bounded operator with finite index. In [11], the following criterion of the Fredholm property is proved: *an operator A is a Fredholm operator is and only if the symbol $A(x)$ consists of Fredholm operators.*

Let $\{A_x\}_{x \in M}$ be a family of operators.

Definition 4. A family $\{A_x\}_{x \in M}$ is said to be *locally continuous* if for any $\varepsilon > 0$ and $x_0 \in M$, there exists a neighborhood $U \subset M$ of the point x_0 such that for any point $x \in U$, the following inequality holds:

$$|||P_U(A_x - A_{x_0})||| < \varepsilon.$$

Definition 5. An operator A is called an *enveloping operator* of a family $\{A_x\}_{x \in M}$ if

$$A \overset{x}{\sim} A_x \quad \forall x \in M.$$

In [11], the existence of a unique (up to a compact term) enveloping operator for any *locally continuous* family $\{A_x\}_{x \in M}$ is proved.

2.2. Transplantation. Let H_1' and H_2' be Hilbert spaces consisting of functions defined on $\mathbb{R}^m$ and $\widetilde{A} : H_1' \to H_2'$ be a linear bounded operator. Since M is a compact manifold, each point $x \in M$ possesses a neighborhood $U \ni x$ and a diffeomorphism $\omega : U \to D_x \subset \mathbb{R}^m$, $\omega(x) \equiv y$. We denote by S_ω the following (local) operator acting from H_k into H_k', $k = 1, 2$. For each function $u \in H_k$ vanishing outside U, we have

$$(S_\omega u)(y) = \begin{cases} u(\omega^{-1}(y)), & y \in D_x, \\ 0, & y \notin D_x. \end{cases}$$

The following definition corresponds to the notion of quasi-equivalence (see [11]).

Definition 6. A *local representative* of an operator $A : H_1 \to H_2$ at a point $x \in M$ is an operator $\widetilde{A} : H_1' \to H_2'$ such that for any $\varepsilon > 0$, there exists a neighborhood U_j of the point $x \in U_j \subset M$ for which the inequality

$$|||g_j A f_j - S_{\omega_j^{-1}} \hat{g}_j \widetilde{A} \hat{f}_j S_{\omega_j}||| < \varepsilon$$

holds for any pair of smooth functions f_j and g_j supported in $\overline{U}_j$; here $\hat{f}_j$ and $\hat{g}_j$ are their representations in local coordinates.

3. Ellipticity and Fredholm property. Starting from the above facts and taking into account further applications to pseudo-differential operators, we give the following definition.

Definition 7. An operator A is said to be *elliptic* if its symbol consists of invertible operators.

We consider the case where the family $\{A_x\}_{x \in M}$ is not locally continuous in the whole but possesses this property on some submanifolds $M_k \subseteq M$ of dimensions $k = 0, 1, \dots, m$. Thus, the submanifold M_0 is a finite union of various points of the boundary ∂M of the manifold M, $M_n \equiv M$ is the initial manifold, $M_{n-1} \equiv \partial M$, $M_k \subset M_{n-1}$, $k = 0, 1, \dots, n-2$. Let $\{A_x\}_{x \in M}$ be a family defined on M. We assume that this family is locally continuous on $M \setminus \bigcup\limits_{k=1}^{n-1} M_k$ and there exist the limits $\lim\limits_{x \to x_k \in M_k} A_x$, which, in general, do not coincide with A_{x_k}, and is locally continuous on each $M_k \setminus \bigcup\limits_{j=0}^{k-1} M_j$ and there exist the limits $\lim\limits_{x \to x_j \in M_j} A_x$, which, in general, do not coincide with A_{x_j}.

Now let A be an operator whose symbol is the family $\{A_x\}_{x \in M}$ described above.

Theorem 1. *An elliptic operator is always a Fredholm operator.*

Using a partition of unity on the manifold M, by an elliptic symbol $A(x)$, one can construct n operators A_j according to the number of singular submanifolds M_k, including the whole boundary ∂M and the manifold M itself.

Theorem 2. *The index of a Fredholm operator A can be calculated by the formula*

$$\operatorname{Ind} A = \sum_{j=1}^{n} \operatorname{Ind} A_j.$$

4. Pseudo-differential operators on conical bundles. We describe some applications of the above abstract schema to the study of pseudo-differential operators and equations on manifolds with nonsmooth boundaries. It is more convenient to do this in the context of vector bundles (see [3, 10]) since the definition of the cotangent bundle at nonsmooth points of the boundary meets some difficulties.

4.1. Conical bundles. A *conical bundle E* over a manifold M is a bundle in the usual sense (see [3, 10]) with the base M, fibers $\mathbb{R}^m$, and a local trivialization of the form $U \times \mathbb{R}^m$, where a neighborhood U is diffeomorphic to a certain cone in the m-dimensional space. These cones are different, depending on the location of a point of the base. In particular, they are cones of the form $\mathbb{R}^m$, $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : \ x = (x', x_m), \ x_m > 0\}$, or $W^k = \mathbb{R}^k \times C^{m-k}$ ($W^0 \equiv C^m$), where C^{m-k} is an acute convex cone in $\mathbb{R}^{m-k}$, which does not contain a whole straight line.

4.2. Pseudo-differential operator on a conical bundle. Now we present a construction of a pseudo-differential operator A on an m-dimensional compact manifold M with boundary, which is determined by a given function $A(x, \xi)$, $(x, \xi) \in E$.

Assume that $A(x,\xi)$ is continuously differentiable by $(x,\xi) \in E$ and satisfies the condition

$$c_1\big(1+|\xi|\big)^{\alpha} \le \big|a(x,\xi)\big| \le c_2\big(1+|\xi|\big)^{\alpha}, \tag{1}$$

where $\alpha \in \mathbb{R}$ is called the *order* of the operator. Here $a(x,\xi)$, $(x,\xi) \in U \times \mathbb{R}^m$, is the symbol $A(x,\xi)$, $(x,\xi) \in E$, written in local coordinates. The generating function $A(x,\xi)$ defined on the conical bundle E is called the *classical symbol* of the operator; we say that the classical symbol is *elliptic* if $A(x,\xi) \ne 0$ for all $(x,\xi) \in E$.

On the boundary ∂M of the manifold M, smooth submanifolds M_k (singularities) of dimension $0 \le k \le m-1$ are extracted. A local representative of the operator A at a point $x_0 \in M$ in a map $U \ni x_0$ is defined by the formula

$$(A_{x_0}u)(x) = \int\limits_{D_{x_0}} \int\limits_{\mathbb{R}^m} e^{i\xi\cdot(x-y)} a(x_0,\xi) u(y)\, d\xi dy, \quad x \in D_{x_0}, \tag{2}$$

and the structure of the canonical domain D_{x_0} has different type depending on the location of the point x_0 on the manifold M. The variants of the canonical domains D_{x_0} are $\mathbb{R}^m$, $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x', x_m),\ x_m > 0\}$, and $W^k = \mathbb{R}^k \times C^{m-k}$, where C^{m-k} is an acute convex cone in $\mathbb{R}^{m-k}$.

Definition 8. The family $\{A_x\}$ of the operators (2) is called the *symbol* of the pseudo-differential operator A.

By a given symbol, one can construct the pseudo-differential operator itself.

If M is a compact manifold, then a *special partition of unity* exists on it (see [10, 12]). This means that for any finite open covering $\{U_j\}_{j=1}^k$ of the manifold M, there exists a system of functions $\{\varphi_j(x)\}_{j=1}^k$, $\varphi_j(x) \in C^\infty(M)$, such that

(i) $0 \le \varphi_j(x) \le 1$,
(ii) $\operatorname{supp}\varphi_j \subset U_j$,
(iii) $\sum\limits_{j=1}^{k} \varphi_j(x) = 1$.

Thus, we have

$$f(x) = \sum_{j=1}^{k} \varphi_j(x) f(x)$$

for any function f defined on M.

On the manifold M, we fix two finite coverings and two partitions of unity corresponding to these coverings, $\{U_j, f_j\}_{j=1}^n$ and $\{V_j, g_j\}_{j=1}^n$, such that $\overline{U_j} \subset V_j$.

Definition 9. A *pseudo-differential operator* A on the manifold M is an operator representable in the form

$$A = \sum_{j=1}^{n} S_{\omega_j^{-1}} \hat{f}_j \cdot \tilde{A}_{x_j} \cdot \hat{g}_j S_{\omega_j} + T,$$

where $T : H_1 \to H_2$ is a compact operator, $x_j \in U_j$, and $\tilde{A}_{x_j}$ is a symbol from the family (2) at the point x_j.

Remark 1. Definition 9 is independent of the choice of an atlas, a partition of unity, and a local coordinate system in the following sense: after such a replacement, a compact operator can only be added to the local representative; we have seen this many times before (see [2, 10, 11]).

It is convenient to consider operators with symbols satisfying the condition (1) in Sobolev–Slobodetsky space $H^s(M)$ (see [2]); as their local versions, we take the spaces $H^s(D_{x_0})$. The space $H^s(M)$

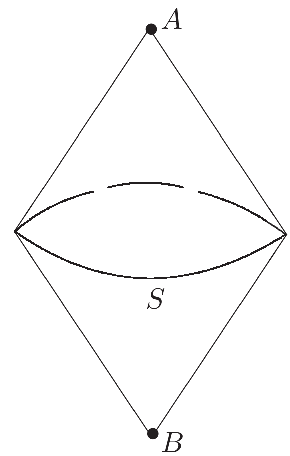

Fig. 1. Illustration to Theorem 2

is constructed by using a partition of unity (see [2]), and then an operator $A : H^s(M) \to H^{s-\alpha}(M)$ is a linear bounded operator of the local type.

Example 1. As an illustration to Theorem 2 and Definition 7, we describe the structure of a pseudo-differential operator on a simple manifold with nonsmooth boundary shown in Fig. 1.

The operator defined on such a manifold by the local formulas (2) consists of five different operators: one envelope A_1 corresponds to inner points of the "top," the second envelope A_2 to smooth points of the boundary, the third envelope A_3 to the edge, and the fourth and fifth envelopes A_4 and A_5 are separate operators corresponding to the conical points. To formulate the conditions of the Fredholm property, one must preliminary state such conditions (the invertibility in the latter two cases) for all five operators.

5. Local indexes and the Fredholm property. To formulate results on the Fredholm property of an elliptic (in the sense of Definition 7) pseudo-differential operator, we must introduce some additional characteristics of the classical symbol of an elliptic pseudo-differential operator (see [12, 13]).

We denote by $\overset{*}{C}{}^{m-k}$ the conjugate cone for C^{m-k}:

$$\overset{*}{C}{}^{m-k} = \{x \in \mathbb{R}^m : \ x \cdot y > 0 \ \forall y \in C^{m-k}\};$$

let $T(\pm\overset{*}{C}{}^{m-k})$ denote the radial tubular domain over the cone $\pm\overset{*}{C}{}^{m-k}$, i.e., a domain of the multidimensional complex space $\mathbb{C}^m$ of the form $\mathbb{R}^m \pm \overset{*}{C}{}^{m-k}$.

Let a function $a(\xi)$, $\xi \in \mathbb{R}^m$, satisfy the condition (1). Introduce the notation $\xi = (\xi'', \xi')$, $\xi'' = (\xi_1, \dots, \xi_k)$, $\xi' = (\xi_{k+1}, \dots, \xi_m)$.

Definition 10. The *k-wave factorization* of a function $a(\xi)$ with respect to the cone C^{m-k} is its representation in the form

$$a(\xi) = a_{\neq}(\xi) a_{=}(\xi),$$

where the factors $a_{\neq}(\xi)$ and $a_{=}(\xi)$ satisfy the following conditions:

(1) $a_{\neq}(\xi)$ and $a_{=}(\xi)$ are defined for all values $\xi \in \mathbb{R}^m$, except for, perhaps, points of the form $\mathbb{R}^k \times \partial\Big(\overset{*}{C}{}^{m-k} \cup \big(-\overset{*}{C}{}^{m-k}\big)\Big)$;

(2) $a_{\neq}(\xi)$ and $a_{=}(\xi)$ admit analytic continuations into the redial tubular domains $T(\overset{*}{C}{}^{m-k})$ and $T(-\overset{*}{C}{}^{m-k})$, respectively, for almost all $\xi'' \in \mathbb{R}^k$ satisfying the estimates

$$\left|a_{\neq}^{\pm 1}(\xi'', \xi' + i\tau)\right| \le c_1\bigl(1 + |\xi| + |\tau|\bigr)^{\pm \varkappa_k},$$
$$\left|a_{=}^{\pm 1}(\xi'', \xi' - i\tau)\right| \le c_2\bigl(1 + |\xi| + |\tau|\bigr)^{\pm (\alpha - \varkappa_k)}$$

for all $\tau \in \overset{*}{C}{}^{m-k}$.

The number $\varkappa_k \in \mathbb{R}$ is called the *index* of the k-wave factorization.

6. Ellipticity, Fredholm property, and boundary-value problems. Assume that the symbol of an operator A is a locally continuous on M_k, $k = 0, 1, \dots, m$, agreed family of operators. In particular, these conditions are always fulfilled if the function $A(x, \xi)$ defined on the conical bundle is continuously differentiable up to the boundary. Then, due to the envelope theorem (see [11]), by the operator symbol (2) we can construct n operators A_k, and all of them are Fredholm operators, then the original operator also possesses the Fredholm property with index described in Theorem 2.

We denote by $\varkappa_{n-1}(x)$ the factorization index of the function $A(x, \xi)$ at a point $x \in \partial M \setminus \bigcup_{k=0}^{m-2} M_k$ (see [2]) and by $\varkappa_k(x)$ the index of k-wave factorization with respect to the cone C_x^{m-k} at points $x \in M_k$, $k = 0, 1, \dots, n-2$. We assume that the function $\varkappa_k(x)$, $k = 0, 1, \dots, n-1$, can be continuously extended to $\overline{M_k}$. The last requirement is due to the fact that situations where $M_k \cap M_{k-1} \neq \emptyset$ are possible.

Remark 2. Similarly to [2], due to the uniqueness of the wave factorization (se [13]), one can verify that the functions $\varkappa_k(x)$, $k = 0, 1, \dots, m-1$, are independent of the choice of a local coordinate system.

Theorem 3. *Assume that a classical elliptic symbol $A(x, \xi)$ admits a k-wave factorization with respect to cones C^{m-k} with indexes $\varkappa_k(x)$, $k = 0, 1, \dots, m-2$, satisfying the condition*

$$\bigl|\varkappa_k(x) - s\bigr| < \frac{1}{2} \quad \forall x \in M_k, \quad k = 0, 1, \dots, m-1. \tag{3}$$

Then $A : H^s(M) \to H^{s-\alpha}(M)$ is a Fredholm operator.

Remark 3. If the ellipticity is violated on submanifolds M_k, one must consider modifications of the operator A based on boundary or coboundary operators (see [13, 15]). This occurs, in particular, if one of the condition (3) is violated.

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