# On the effect of transition from a model with concentrated parameters to a model with distributed parameters

**M V Polovinkina**

Department of higher mathematics and information technologies, Voronezh State University of Engineering Technologies, 19, Revolution Av., Voronezh, Russian Federation

E-mail: polovinkina-marina@yandex.ru

**Abstract.** We note from a general point of view that adding diffusion terms to ordinary differential equations, for example, to logistic ones, can in some cases improve sufficient conditions for the stability of a stationary solution. We give examples of models in which the addition of diffusion terms to ordinary differential equations changes the stability conditions of a stationary solution.

## 1. Introduction

Systems of differential equations simulate the growth of phenomena of various types, and for all such systems, studies of the stability of stationary solutions play an important role. These studies have a long history. In many cases, such models are based on ordinary differential equations. Although the theory of systems of ordinary differential equations has long been classical, interest in it does not fade away. In the last few decades, this is also due to the fact that such systems have found applications in modelling biological and social systems. From relatively recent works on mathematical biology, it is possible to indicate in this regard [1–6].

In the work [7], a model of the origin and development of currents in painting, based on equations of the same type, is considered.

In this paper, we consider a certain class of mathematical models with partial differential equations (models with distributed parameters), which are obtained from models with ordinary differential equations (models with concentrated parameters) by adding the so-called diffusion terms. The tendency of such sophistications of mathematical models can be traced in some works related to modelling the growth and distribution of populations, the growth and spread of infections, and the growth of tumors. In this regard, see first of all the monograph [8]. In the work [9], a diffusion model of a malignant tumor is presented.

The mathematical model of glioma growth is based on the classical definition of cancer as uncontrolled proliferation of cells with the potential for invasion and metastasis, simplified for gliomas, which practically do not metastasize. This model is governed by the equation (see [10])

$$\frac{\partial u}{\partial t} = \tau u\Big(1 - \frac{u}{s}\Big) + M\Delta u, \tag{1}$$

where $u(x,t)$ defines the concentration of malignant cells at location $x$ and time $t$, $M$ is the random motility coefficient defining the net rate of migration of the tumor cells, $\tau$ represents net proliferation rate of the tumor cells, $s$ is the limiting concentration of cells that a volume of tissue can hold,

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

is the Laplace operator. The term $M\Delta u$ is usually called the diffusion term.

We are interested in the stability of stationary solutions of diffusion models. This issue is discussed in the book [8]. This book states that adding diffusion terms can change the stability of a stationary solution both for the worse and for the better. For models of a certain type, we try to concretize sufficient conditions for the stability of stationary solutions.

## 2. Materials and methods

We consider the initial-boundary value problem for the system of partial differential equations:

$$\frac{\partial u_s}{\partial t} = \vartheta_s \Delta u_s + F_s(u), \quad x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n, \tag{2}$$

$$\left( \mu_s u_s + \eta_s \frac{\partial u_s}{\partial \overrightarrow{\nu}} \right) \Big|_{x \in \partial\Omega} = B_s(x), \ \mu_s^2 + \eta_s^2 > 0, \ \mu_s \geq 0, \ \eta_s \geq 0, \tag{3}$$

$$u_s(x, 0) = u_s^0(x), \ s = 1, \dots, m, \tag{4}$$

where $\Omega$ is a bounded domain with a piecewise smooth boundary $\Gamma = \partial\Omega$, $\overrightarrow{\nu}$ is a unit external normal vector to the boundary $\partial\Omega$ of the domain $\Omega$, $u = (u_1(x,t), \dots, u_m(x,t))$, $\vartheta_s \geq 0$, $B_s(x) \in C(\partial\Omega)$, $u_s^0(x) \in C(\overline{\Omega})$, $s = 1, \dots, m$, $\overline{\Omega} = \Omega \cup \partial\Omega$, $\Delta$ is the Laplace operator defined by the formula

$$\Delta v = \sum_{j=1}^{n} \frac{\partial^2 v}{\partial x_j^2}.$$

Let us consider a special kind of functions $F_s(u) = F_s(u_1, \dots, u_m)$ :

$$F_s(u) = \sum_{k=1}^{m} b_{sk}\, u_k + \sum_{\ell=1}^{m} \sum_{j=1}^{n} a_{s\ell j}\, u_\ell\, u_j + f_s(x), \ f_s \in C(\overline{\Omega}), \ a_{s\ell j} = a_{sj\ell}, \ \ell, \, j, \, s = 1, \dots, m. \tag{5}$$

Let $w = (w_1(x), \dots, w_m(x))$ be a stationary solution of system (2), that is, the solution of the system

$$\vartheta_s \Delta w_s + F_s(w) = 0, \ s = 1, \dots, m, \ x \in \Omega, \tag{6}$$

satisfying the boundary conditions

$$\left( \mu_s w_s + \eta_s \frac{\partial w_s}{\partial \overrightarrow{\nu}} \right) \Big|_{x \in \partial\Omega} = B_s(x), \ s = 1, \dots, m. \tag{7}$$

In this paper, we study the stability of a stationary solution of system (2). Let $z = z(x,t) = u(x,t) - w(x)$ be a vector of deviations from a stationary solution. We substitute $u = w + z$ in system (2). Then

$$\frac{\partial u_s}{\partial t} = \frac{\partial z_s}{\partial t} = \vartheta_s \Delta(w_s + z_s) + F_s(w + z), \ s = 1, \dots, m, \ x \in \Omega.$$

After identical transformations, we obtain the equality

$$\frac{\partial u_s}{\partial t} = \frac{\partial z_s}{\partial t} = \vartheta_s \Delta z_s + \sum_{k=1}^{m} b_{sk}\, z_k + \sum_{\ell=1}^{m} \sum_{j=1}^{n} a_{s\ell j}\, (2 w_\ell z_j + z_\ell\, z_j) +$$

$$+\vartheta_s \Delta w_s + F_s(w), \ s = 1, \ldots, m, \ x \in \Omega.$$

Taking into account (6), the last equality is converted to the form

$$\frac{\partial u_s}{\partial t} = \frac{\partial z_s}{\partial t} = \vartheta_s \Delta z_s + \sum_{k=1}^{m} b_{sk}\, z_k + \sum_{\ell=1}^{m}\sum_{j=1}^{n} a_{s\ell\, j}\, \left(2 w_\ell z_j + z_\ell\, z_j\right), \ s = 1, \ldots, m, \ x \in \Omega.$$

We multiply this equality by $z_s$ and integrate over the domain $\Omega$. We obtain:

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_\Omega z_s^2\, dx = \vartheta_s \int\limits_\Omega z_s \Delta z_s\, dx + \sum_{k=1}^{m}\int\limits_\Omega b_{sk}\, z_k z_s\, dx +$$

$$+ \int\limits_\Omega \left( 2\sum_{\ell=1}^{m}\sum_{j=1}^{n} a_{s\ell\, j}\, w_\ell z_j z_s + \sum_{\ell=1}^{m}\sum_{j=1}^{n} a_{s\ell\, j}\, z_\ell\, z_j z_s \right)\, dx.$$

Assuming that the deviations $z_s$ are small enough, we discard the monomials of degree higher than 2 of these deviations, then we get:

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_\Omega z_s^2\, dx = \vartheta_s \int\limits_\Omega z_s \Delta z_s\, dx + \int\limits_\Omega \left( \sum_{k=1}^{m} b_{sk}\, z_k z_s \, + \, 2\sum_{\ell=1}^{m}\sum_{j=1}^{n} a_{s\ell\, j}\, w_\ell z_j z_s \right)\, dx. \tag{8}$$

We apply the first Green formula for the Laplace operator to the first term on the right side of this equation. For two functions $f \in C^1(\overline{\Omega})$ and $g \in C^2(\Omega) \cap C^1(\overline{\Omega})$, the Green formula takes the form (see [11])

$$\int\limits_\Omega f \Delta g dx = -\int\limits_\Omega \nabla f \nabla g dx + \int\limits_\Gamma f \frac{\partial g}{\partial \overrightarrow{\nu}}\, d\Gamma,$$

where $\overrightarrow{\nu}$ is a unit external normal vector to $\Gamma$. Here $d\Gamma$ is an arc element of the boundary $\Gamma = \partial\Omega$. Substituting $f = z_s, \ g = z_s$ into (8), we obtain:

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_\Omega z_s^2\, dx = -\vartheta_s \int\limits_\Omega \mid \nabla z_s \mid^2 \ dx - \vartheta_s \int\limits_{\partial\Omega} g(z_s) d\Gamma +$$

$$+ \int\limits_\Omega \left( \sum_{k=1}^{m} b_{sk}\, z_k z_s \, + \, 2\sum_{\ell=1}^{m}\sum_{j=1}^{n} a_{s\ell\, j}\, w_\ell z_j z_s \right)\, dx, \ s = 1, \ldots, m, \tag{9}$$

where the second term on the right side of the equation is a surface (for $n \geq 3$) or contour (for $n = 2$) integral of the first kind over the boundary of the domain $\Omega$ or the sum of non-negative values at the ends of the interval $\Omega$ in the case of $n = 1$ ; $g(z_s) = 0$ for $\mu_s = 0$ or for $\eta_s = 0$; if $\eta_s \neq 0$, then $g(z_s) = \mu_s z_s^2/\eta_s$. In all cases $g(z_s) \geq 0$. Summing $m$ equalities (9), we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_\Omega \mid z \mid^2 \ dx = -\sum_{s=1}^{m}\vartheta_s \int\limits_\Omega \mid \nabla z_s \mid^2 \ dx - \sum_{s=1}^{m}\vartheta_s \int\limits_{\partial\Omega} g(z_s) d\Gamma +$$

$$+ \int\limits_\Omega \left( \sum_{s=1}^{m}\sum_{k=1}^{m} b_{sk}\, z_k z_s \, + \, 2\sum_{s=1}^{m}\sum_{\ell=1}^{m}\sum_{j=1}^{m} a_{s\ell\, j}\, w_\ell z_j z_s \right)\, dx,$$

or equivalently,

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\Omega} \mid z \mid^2 \, dx = -\sum_{s=1}^{m} \vartheta_s \int\limits_{\Omega} \mid \nabla z_s \mid^2 \, dx - \sum_{s=1}^{m} \vartheta_s \int\limits_{\partial\Omega} g(z_s) d\Gamma + \int\limits_{\Omega} \sum_{s=1}^{m}\sum_{k=1}^{m} \beta_{sk}\, z_k z_s \, dx, \quad (10)$$

where

$$\beta_{sk} = b_{sk} \, + \, 2\sum_{\ell=1}^{m} a_{s\ell\, k}\, w_\ell. \quad (11)$$

We put

$$\Theta_{sk} = (\beta_{sk} + \beta_{ks})/2. \quad (12)$$

Then equality (10) can be rewritten as

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\Omega} \mid z \mid^2 \, dx = -\sum_{s=1}^{m} \vartheta_s \int\limits_{\Omega} \mid \nabla z_s \mid^2 \, dx - \sum_{s=1}^{m} \vartheta_s \int\limits_{\partial\Omega} g(z_s) d\Gamma + \int\limits_{\Omega} \sum_{s=1}^{m}\sum_{k=1}^{m} \Theta_{sk}\, z_k z_s \, dx. \quad (13)$$

Transformation (12) is introduced for the transition from an unsymmetric quadratic form to a symmetric one. Obviously, the negative definiteness of a quadratic form

$$\sum_{s=1}^{m}\sum_{k=1}^{m} \Theta_{sk}\, z_k z_s \, dx \quad (14)$$

will ensure the negativity of the left side of equality (13), and, therefore, the stability of the stationary solution.

In the absence of diffusion terms, that is, when

$$\vartheta_s = 0, \;\; s = 1, \ldots, m, \quad (15)$$

the variables $x_1, \ldots, x_n$ are included in equations (2) as parameters whose derivatives are not contained in these equations. This is the case of a model with concentrated parameters. Let

$$\sum_{s=1}^{m} \vartheta_s^2 > 0, \quad (16)$$

that is, we proceed to the consideration of the diffusion model with distributed parameters. In this case, it is possible to weaken the sufficient condition for the stability of a stationary solution. For this purpose, we use the Steklov-Poincare-Friedrichs inequality (see [12] p. 150, [13] p. 62)

$$\int\limits_{\Omega} \mid \nabla z_s|^2 dx \geq \frac{1}{d^2}\int\limits_{\Omega} z_s^2 dx,$$

where $d = \operatorname{diam}\Omega$ is a diameter of the domain $\Omega$. Therefore,

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\Omega} \mid z \mid^2 \, dx \leqslant -\sum_{s=1}^{m} \frac{\vartheta_s}{d^2} \int\limits_{\Omega} z_s^2 \, dx - \sum_{s=1}^{m} \vartheta_s \int\limits_{\partial\Omega} g(z_s) d\Gamma + \int\limits_{\Omega} \sum_{s=1}^{m}\sum_{k=1}^{m} \Theta_{sk}\, z_k z_s \, dx. \quad (17)$$

Now we can assert that a sufficient condition for the stability of a stationary solution is the negative definiteness of the quadratic form

$$\sum_{s=1}^{m}\sum_{k=1}^{m} A_{sk}\, z_k z_s \,, \quad (18)$$

where

$$A_{sk} = \Theta_{sk} - \delta_{ks}\vartheta_s/d^2.$$

In order to demonstrate how the properties of the model change when introducing distributed parameters by adding diffusion terms, we consider the case

$$b_{sk} = 0,\ f(x) = 0,\ x \in \overline{\Omega},\ s,k = 1,\dots,m.$$

In this case, the vector $w = 0$ is a stationary solution of the system both for the case of concentrated parameters (15) and for the case of distributed parameters (16). However, the situations are fundamentally different. In the diffusionless case, the zero vector is not a stable solution. If all the equations of the system contain diffusion terms, that is,

$$\vartheta_s > 0,\ s = 1,\dots,m,$$

quadratic form (18) will take the form

$$-\frac{1}{d^2}\sum_{s=1}^{m} \vartheta_s\, z_s^2 \tag{19}$$

and, obviously, will be negatively defined, consequently, the zero solution will be stable.

Another interesting example is provided by the Hotelling equation

$$\frac{\partial u}{\partial t} = A(\xi - u)u + B\Delta u,$$

where $u$ is an unknown function, $u = u(x_1, x_2, t) \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ for any $t > 0$,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

is the Laplace operator, $A$, $B$, $\xi$ are given positive constants. This equation describes population growth and distribution. In this case, the values included in the equation have the following meaning: $x_1, x_2$ are the geographical coordinates, $A$ is the population growth rate, $B$ is the migration rate, $\xi$ is the coefficient of the saturated population density, $u$ is the population density, $t$ is the time parameter. This model takes into account migration processes. Population growth is modelled as a logistic process. Migration processes are described using Fourier's Law of Heat Conduction.

Let $w(x_1, x_2)$ be a stationary solution of the Hotelling equation, that is, the solution of the equation

$$A(\xi - w)w + B\Delta w = 0.$$

The above method leads to the conclusion that the condition

$$w > \frac{\xi}{2} - \frac{B}{2Ad^2}$$

is sufficient for the stability of the stationary solution $w(x_1, x_2)$ [14] (see also [15], where this result was generalized). It is interesting to note that in the diffusion case (when $B \neq 0$) the zero stationary solution can be both stable and instable, what is determined by the size of the domain $\Omega$.

## 3. Results and discussion

Let us consider the basic SIR model for the control of endemic infections (see [1]). This model assumes vaccination at birth at constant coverage $p$, which is reminiscent of a situation where a mandatory immunization program exists. The resulting model is as follows:

$$\frac{dS}{dt} = \epsilon(1-p) - \epsilon S - \beta SI, \tag{20}$$

$$\frac{dI}{dt} = \beta SI - \gamma I, \tag{21}$$

$$\frac{dR}{dt} = mp + \zeta I - mR, \tag{22}$$

where $S, I, R$ denote the fractions of individuals who are, respectively, susceptible to acquiring infection, infective, i.e. able to retransmit infection to others, and removed because of e.g. immunity acquired after recovery. The infective fraction I is also called the infection prevalence. The function $\beta(t)$ denotes the transmission rate which is typically time-dependent. The other demo-epidemiological parameters are: $\gamma = \epsilon + \zeta$, $\epsilon > 0$ which denotes both the birth and death rates, assumed identical to ensure that the population is stationary over time, and $\zeta > 0$ which is the rate of recovery from infection. The equality

$$S + I + R = 1 \tag{23}$$

allows to omit the third equation. In the most well-known case of constant transmission rate $\beta(t) = \beta$, the SIR model admits a disease-free equilibrium point DFE $= (1-p, 0, p)$, which is stable if $\beta(1-p) < \gamma$ and unstable otherwise.

Taking into account (23), we add the diffusion terms and consider the system

$$\frac{\partial S}{\partial t} = \epsilon(1-p) - \epsilon S - \beta SI + \vartheta_1 \Delta S, \tag{24}$$

$$\frac{\partial I}{\partial t} = \beta SI - \gamma I + \vartheta_2 \Delta I, \tag{25}$$

where $S = S(x_1, x_2, t) = S(x, t)$, $I = I(x_1, x_2, t) = I(x, t)$,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

is the Laplace operator.

We consider system (24)–(25) in the domain $\Omega \subset \mathbb{R}^2$ bounded by a piecewise smooth contour $\Gamma = \partial\Omega$. Let us introduce the notation $u_1 = u_1(x,t) = S$, $u_2 = u_2(x,t) = I$. We will impose additional conditions on the solution:

— boundary conditions

$$\left.\left(\mu_j u_j + \eta_j \frac{\partial u_j}{\partial \overrightarrow{\nu}}\right)\right|_{x\in\partial\Omega} = B_j(x),\ \mu_j^2 + \eta_j^2 > 0,\ \mu_j \geq 0,\ \eta_j \geq 0, \tag{26}$$

— initial conditions

$$u_j(x, 0) = w_j(x),\ j = 1, 2. \tag{27}$$

Here $B_j(x) \in C(\partial\Omega)$, $w_j(x) \in C^2(\Omega) \cap C(\overline{\Omega})$, $j = 1, 2$, $\overline{\Omega} = \Omega \cup \partial\Omega$.

Let $w = (w_1(x), w_2(x))$ be a stationary solution of system (24)–(25), i.e. the solution of the system

$$\epsilon(1-p) - \epsilon w_1 - \beta w_1 w_2 + \vartheta_1 \Delta w_1 = 0, \tag{28}$$

$$\beta w_1 w_2 - \gamma w_2 + \vartheta_2 \Delta w_2 = 0, \tag{29}$$

satisfying boundary conditions

$$\left.\left(\mu_j w_j + \eta_j \frac{\partial w_j}{\partial \overrightarrow{\nu}}\right)\right|_{x\in\partial\Omega} = B_j(x),\ \mu_j^2+\eta_j^2>0,\ \mu_j\geq 0,\ \eta_j\geq 0,\ j=1,2. \tag{30}$$

Let $z = z(x,t) = u(x,t) - w(x) = (z_1, z_2)$ be a vector of small deviations from the stationary solution. We substitute $u = w + z$ in system (24)–(25). Then equation (24) can be rewritten as

$$\frac{\partial u_1}{\partial t} = \frac{\partial S}{\partial t} = \frac{\partial z_1}{\partial t} = \epsilon(1-p) - \epsilon(w_1+z_1) - \beta(w_1+z_1)(w_2+z_2) + \vartheta_1\Delta(w_1+z_1).$$

After the obvious identity transformations, we obtain:

$$\frac{\partial z_1}{\partial t} = \epsilon(1-p) - \epsilon(w_1+z_1) - \beta w_1 w_2 - \beta z_1 w_2 - \beta w_1 z_2 - \beta z_1 z_2 + \vartheta_1\Delta w_1 + \vartheta_1 \Delta z_1.$$

Taking into account that the function $w_1$ satisfies equation (28), we get:

$$\frac{\partial z_1}{\partial t} = -\beta z_1(w_2+\epsilon) - \beta w_1 z_2 - \beta z_1 z_2 + \vartheta_1\Delta z_1. \tag{31}$$

Multiplying (31) by $z_1$, we obtain:

$$\frac{1}{2}\frac{\partial z_1^2}{\partial t} = -\beta z_1^2(w_2+\epsilon) - \beta w_1 z_1 z_2 - \beta z_1^2 z_2 + \vartheta_1 z_1 \Delta z_1\,.$$

Integrating this equality over the domain $\Omega$, we get:

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_\Omega z_1^2\,dx = \int\limits_\Omega \left(-\beta z_1^2(w_2+\epsilon) - \beta w_1 z_1 z_2 - \beta z_1^2 z_2\right)\,dx + \vartheta_1\int\limits_\Omega z_1\Delta z_1\,dx, \tag{32}$$

where $dx = dx_1 dx_2$.

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_\Omega z_1^2\,dx = \int\limits_\Omega \left(-\beta z_1^2(w_2+\epsilon) - \beta w_1 z_1 z_2 - \beta z_1^2 z_2\right)\,dx -$$

$$-\vartheta_1\int\limits_\Omega |\nabla z_1|^2\,dx - \vartheta_1\int\limits_{\partial\Omega} g_1\,d\Gamma. \tag{33}$$

In equality (33), the function $g_1(x)$ vanishes on $\Gamma$ when $\mu_1\eta_1 = 0$ or $g_1 = \mu_1 z_1^2/\eta_1$ when $\mu_1\eta_1 > 0$. Using Poincare-Steklov-Friedrichs inequality for (33), we get:

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_\Omega z_1^2\,dx \leqslant \int\limits_\Omega \left(-\beta z_1^2(w_2+\epsilon) - \beta w_1 z_1 z_2 - \beta z_1^2 z_2\right)\,dx -$$

$$-\frac{\vartheta_1}{d^2}\int\limits_\Omega z_1^2\,dx - \vartheta_1\int\limits_{\partial\Omega} g_1\,d\Gamma. \tag{34}$$

Let us proceed in the same way with equation (25). We obtain inequality

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_\Omega z_2^2\,dx \leqslant \int\limits_\Omega \left(\beta w_2 z_1 z_2 + \beta(w_1-\gamma)z_2^2 + \beta z_1 z_2^2\right)\,dx -$$

$$-\frac{\vartheta_2}{d^2}\int\limits_{\Omega} z_2^2\,dx - \vartheta_2 \int\limits_{\partial\Omega} g_2\,d\Gamma, \tag{35}$$

where, like before, the function $g_2(x)$ vanishes on $\Gamma$ when $\mu_2\eta_2 = 0$ or $g_2 = \mu_2 z_2^2/\eta_2$ when $\mu_2\eta_2 > 0$. Summing inequalities (34) and (35), we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\Omega} z^2\,dx \leqslant \int\limits_{\Omega}\sum_{k,j=1}^{2} A_{kj} z_k z_j\,dx - \int\limits_{\partial\Omega}\sum_{j=1}^{2}\vartheta_j g_j\,d\Gamma + \int\limits_{\Omega}\left(\beta z_1 z_2^2 - \beta z_1^2 z_2\right)\,dx, \tag{36}$$

where

$$A_{11} = -\beta(w_2+\epsilon) - \frac{\vartheta_1}{d^2}\,,\ A_{22} = \beta w_1 - \gamma - \frac{\vartheta_2}{d^2}\,, \tag{37}$$

$$A_{12} = (\beta w_2 - \beta w_1)/2. \tag{38}$$

The last term on the right side of (36) for small z does not affect the sign of the entire sum and can be omitted. Using the Sylvester criterion, we obtain the following conditions for the stability of the stationary solution:

$$A_{11} = -\left(\beta w_2 + \epsilon + \frac{\vartheta_1}{d^2}\right) < 0, \tag{39}$$

$$A_{11}A_{22} - A_{12}^2 = \left(-\beta w_2 - \epsilon - \frac{\vartheta_1}{d^2}\right)\left(\beta w_1 - \gamma - \frac{\vartheta_2}{d^2}\right) - \frac{1}{4}\beta^2\left(w_2 - w_1\right)^2 > 0. \tag{40}$$

These conditions are verifiable in practice with computer simulations. It should be noted that if, within the framework of this model, we consider the trivial stationary solution $w_1 = 1 - p$, $w_2 = 0$, conditions (39)–(40) can be rewritten as follows:

$$\frac{\vartheta_1}{d^2} + \epsilon > 0, \tag{41}$$

$$4\left(\frac{\vartheta_1}{d^2} + \epsilon\right)\left(\gamma + \frac{\vartheta_2}{d^2} - \beta(1-p)\right) - \beta^2(1-p)^2 > 0. \tag{42}$$

In a model with concentrated parameters, that is, when $\vartheta_1 = \vartheta_2 = \vartheta_3 = 0$, condition (42) is a consequence of the condition $\beta(1-p) < \gamma$, so it cannot be considered as an improvement of the result. In a model with distributed parameters, when $\vartheta_1\vartheta_2\vartheta_3 > 0$, condition (42), taking into account the nonnegativity of the parameters and the equality $\gamma = \epsilon + \zeta$, can be rewritten in the form

$$0 < \beta(1-p) < 2\sqrt{D_0} - 2\left(\epsilon + \frac{\vartheta_1}{d^2}\right), \tag{43}$$

where

$$D_0 = 4\left(\epsilon + \frac{\vartheta_1}{d^2}\right)^2 + 4\left(\epsilon + \frac{\vartheta_1}{d^2}\right)\left(\epsilon + \zeta + \frac{\vartheta_2}{d^2}\right)^2. \tag{44}$$

We must now find out whether condition (43) is improvement (weakening) of the condition $\beta(1-p) < \gamma$. This will be the case if the inequality

$$\gamma^2 - \Phi(d) < 0 \tag{45}$$

is satisfied, where

$$\Phi(d) = 6\epsilon\frac{\vartheta_1}{d^2} + 3\zeta\frac{\vartheta_1^2}{d^4} + 4\epsilon\frac{\vartheta_2}{d^2} + 4\frac{\vartheta_1\vartheta}{d^4} - 2\zeta\frac{\vartheta_1}{d^2}\,. \tag{46}$$

Since

$$\lim_{d\to 0+} \Phi(d) = +\infty,$$

condition (45) can be met for domains with a small diameter. For domains with a large diameter, this condition is not met. We do not presume to make final conclusions and to interprete their content. Let us only assume that for large areas, diffusion (spread of infection due to migration) has a small impact on the stability of the zero level of infection, while growth parameters have a decisive influence. However, it is possible that these parameters also depend on the diffusion conditions. In any case, we have to admit that the models of the growth and spread of diseases are in the active phase of development.

## References

[1] D'Onofrio A and Manfredi P 2020 The Interplay Between Voluntary Vaccination and Reduction of Risky Behavior: A General Behavior-Implicit SIR Model for Vaccine Preventable Infections *Current Trends in Dynamical Systems in Biology and Natural Sciences* SEMA SIMAI Springer Series **21** Springer Nature Switzerland AG 2020 185-203

[2] Seno H 2020 An SIS model for the epidemic dynamics with two phases of the human day-to-day activity *Journal of Mathematical Biology* **80** 2109–40

[3] Kabanikhina S I and Krivorotko O I 2020 Optimization Methods for Solving Inverse Immunology and Epidemiology Problems *Computational Mathematics and Mathematical Physics* 2020 **60** 4 580–9

[4] Aniji1 M, Kavitha1 N and Balamuralitharan S 2020 Approximate solutions for HBV infection with stability analysis using LHAM during antiviral therapy *Boundary Value Problems a Springer Open Journal* **80** https://doi.org/10.1186/s13661-020-01373-w

[5] Voropaeva O F and Tsgoev Ch A 2019 A Numerical Model of Inflammation Dynamics in the Core of Myocardial Infarction *Journal of Applied and Industrial Mathematics* **13** 2 372–83

[6] Afraimovich V, Young T, Muezzinoglu M K and Rabinovich M I 2011 Nonlinear Dynamics of Emotion-Cognition Interaction: When Emotion Does not Destroy Cognition? *Bull Math Biol* **73** 266–84

[7] Kolpak E P and Gavrilova A V 2019 Mathematical model of the emergence of cultural centers and trends in painting (in Russian) *Young Scientist* **22** 260 1–17

[8] Svirezhev Yu M and Logofet D O 1978 *Stability of biological communities* (in Russian) Moscow, Nauka

[9] Zhukova I V and Kolpak E P 2014 Mathematical models of malignant tumour (in Russian) *Vestnik Sankt-Peterburgskogo Universiteta. Seriya 10. Prikladnaya Matematika. Informatika. Protsessy Upravleniya* **3** 5–18

[10] Swanson K R, Rostomily R C and Alvord E C 2008 A mathematical modelling tool for predicting survival of individual patients following resection of glioblastoma: a proof of principle *Jr Br J Cancer* **98** 1 113–9 (Published online 2007 Dec 4 doi: 10.1038/sj.bjc.6604125)

[11] Gilbarg D and Trudinger N S 1983 *Elliptic Partial Differential Equations of Second Order* (Berlin-Heidelberg-New York-Tokyo, Springer-Verlag XIII 513 S)

[12] Mikhailov V P 1978 *Partial differential equatpons* Moscow, Mir Publisher

[13] Ladyzhenskaya O A 1985 *Boundary value problems of mathematical phisics* New York, Springer-Verlag Nauka, Moscow, 1973 (in Russian)

[14] Meshkov V Z, Polovinkin I P and Semenov M E 2002 On the stability of a stationary solution of the Hotelling equation (in Russian) *Appl. and Industrial Math. Rev.* **9** 1 226–7

[15] Gogoleva T N, Shchepina I N, Polovinkina M V and Rabeeakh S A 2019 On stability of a stationary solution to the Hotelling migration equation *J. Phys.: Conf. Ser. 1203 012041*