Article

HERMITE–HADAMARD TYPE INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS AND THEIR APPLICATIONS

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Abstract: In this paper we establish new generalizations of Hermite-Hadamard type inequalities. These inequalities are formulated in terms of modules of certain powers of proper functions. Generalizations for convex functions are also considered. As applications, some new inequalities for the digamma function in terms of trigamma function and inequalities involving special means of real numbers are given. The results include also estimates via arithmetic, geometric and logarithmic means. Examples are derived to demonstrate that some of our results of this paper are more exact than the existing ones and some improve several known results available in the literature. The constants in the derived inequalities are calculated, some of them are sharp. As a visual example graphs of some technically important functions are included in the text.

Keywords: Hermite–Hadamard inequality, digamma function, trigamma function, absolutely continuous mapping, convex function, arithmetic mean, geometric mean, logarithmic mean

MSC: 26D07, 26D10, 26D15, 26A33

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Contents

1. Introduction

The Hermite-Hadamard type inequalities are very important in many topics of mathematics and applications, its original version is defined in the following way [8,19]:

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b h(x)dx \leq \frac{h(a)+h(b)}{2}, \tag{1}$$

here a convex function h is defined on the interval $I \subseteq \mathbb{R} \to \mathbb{R}$ for real numbers $a, b \in I,\ a < b$.

The Hermite–Hadamard type inequalities (1) are an important instrument in such abstract and applied mathematical fields as mathematical analysis, function theory, optimization, control theory, theory of special means and different variants of entropy problems, interpolations and approximations, numerical methods including numerical integration, information theory, probability and statistics. The results of this article may be applied to integral inequalities for fractional interval-valued functions, and corresponding differential equations and optimization problems. Integral inequalities of Hermite–Hadamard type are also important in the transmutation theory for estimating different kinds of kernels for transmutational operators, cf. [28]. So the results of this paper are matched with the topic of this special issue "Analytical and Computational Methods in Differential Equations, Special Functions, Transmutations and Integral Transforms".

A considerable amount of works on this type of inequalities are known, and recently were developed new proofs, generalizations, refinements, computer and numerical applications and illustrations. As a result many authors have focused on Hermite-Hadamard type inequalities for various classes of convex functions and mappings, for instance, see [1–3,8–12,15–19,21,23–27,29**?** ,30] and the references cited therein.

From very recent important papers let us mention [16], in it connections with inclusion theory and fuzzy sets are studied with use of interval analysis. Namely, different kinds of convexity and non-convexity conditions leads to interesting classes of inequalities, including problems with inclusions. For studying fuzzy order relations an idea of logarithmic convexity is vital and fruitful. On this way various discrete forms of Hermite-Hadamard, Jensen and Schur inequalities are studied for fuzzy interval-valued functions based on using considerations for log convex settings. It leads to new ideas and approaches in fuzzy optimization problems, interval-valued functions and corresponding mathematical modelling. Also the connected notion of interval-valued preinvex functions is also exploited, as an example in [29] it is applied to the Riemann–Liouville fractional integrals in fractional calculus.

In [9] D.D. Dragomir and R.P. Agarwal among other important results proved the following inequality connected with the right–hand side of inequality (1), namely:

Theorem A If h is a differentiable function on an interval $[a, b]$, and $|h'|$ is a convex function on $[a, b]$, then the following inequality holds true:

$$\left|\frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(x)dx\right| \leq \frac{b-a}{8}\Big(|h'(a)| + |h'(b)|\Big). \tag{2}$$

In [17], Kirmaci proved the following result connected with the left part of the inequality (1). It states that:

Theorem B Under assumptions of the above Theorem A, the following holds true:

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(x)\,dx\right| \leq \frac{b-a}{8}\Big(|h'(a)| + |h'(b)|\Big). \tag{3}$$

Other interesting results in this direction were proved in [30], with several refinements and extensions of the Hermite–Hadamard and Jensen inequalities in n variables.

In this paper we establish some new Hermite-Hadamard type inequalities for a class of functions with convex derivatives under some conditions. As consequences, new inequalities involving the digamma and trigamma functions are obtained and some inequalities involving special means of real numbers are given. Analytical and numerical computation shows that the obtained results are better than the corresponding similar inequalities in (2) and (3).

2. New Hermite-Hadamard type inequalities for convex functions

The notion of convexity is very important and basic in mathematics. For results on convex functions we may mention references [7,8,13,14,19,22].

The following lemma is very useful to obtain results of this paper. The proof of it is based on an integration by parts, hence we omit the details.

Lemma 2.1. *Let h be an absolutely continuous function on an interval $[a,b]$ and its derivative $h' \in L_1[a,b]$, then the following holds true:*

$$\frac{1}{3}\left[h(a)+h\left(\frac{a+b}{2}\right)+h(b)\right]-\frac{1}{b-a}\int_a^b h(x)dx=(b-a)\left[\int_0^{\frac{1}{2}}(x-\frac{1}{3})h'(a+x(b-a))dx+\int_{\frac{1}{2}}^1(x-\frac{2}{3})h'(a+x(b-a))dx\right]. \tag{4}$$

Theorem 2.1. *Let h be an absolutely continuous function on an interval $[a,b]$ and its derivative $h' \in L_1[a,b]$, $|h'|^q$ is convex on $[a,b]$ for some $q \geq 1$, then the next holds true:*

$$\left|\frac{1}{3}\left[h(a)+h\left(\frac{a+b}{2}\right)+h(b)\right]-\frac{1}{b-a}\int_a^b h(x)dx\right| \leq (b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}} \times\left\{\left(\frac{111|h'(a)|^q}{1944}+\frac{|h'(b)|^q}{81}\right)^{\frac{1}{q}}+\left(\frac{|h'(a)|^q}{81}+\frac{111|h'(b)|^q}{1944}\right)^{\frac{1}{q}}\right\}. \tag{5}$$

Proof. From Lemma 2.1 we have

$$\begin{aligned}\left|\frac{1}{3}\left[h(a)+h\left(\frac{a+b}{2}\right)+h(b)\right]-\frac{1}{b-a}\int_a^b h(x)dx\right| &= (b-a)\left|\int_0^{\frac{1}{2}}(x-\frac{1}{3})h'(a+x(b-a))dx\right.\\ &\quad\left.+\int_{\frac{1}{2}}^1(x-\frac{2}{3})h'(a+x(b-a))dx\right|\\ &\leq (b-a)\left\{\int_0^{\frac{1}{2}}\left|x-\frac{1}{3}\right|\left|h'(a+x(b-a))\right|dx\right.\\ &\quad\left.+\int_{\frac{1}{2}}^1\left|x-\frac{2}{3}\right|\left|h'(a+x(b-a))\right|dx\right\}.\end{aligned} \tag{6}$$

Firstly, we assume that $q = 1$ and using the fact that the function $|h'|$ is convex on $[a, b]$, so we derive

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left|x - \frac{1}{3}\right| \left|h'(a + x(b-a))\right| dx + \int_{\frac{1}{2}}^1 \left|x - \frac{2}{3}\right| \left|h'(a + x(b-a))\right| dx \\
&\leq \int_0^{\frac{1}{2}} \left|x - \frac{1}{3}\right| \left((1-x)\left|h'(a)\right| + x\left|h'(b)\right|\right) dx + \int_{\frac{1}{2}}^1 \left|x - \frac{2}{3}\right| \left((1-x)\left|h'(a)\right| + x\left|h'(b)\right|\right) dx \\
&\leq \left|h'(a)\right| \left(\int_0^{\frac{1}{2}} (1-x)\left|x - \frac{1}{3}\right| + \int_{\frac{1}{2}}^1 (1-x)\left|x - \frac{2}{3}\right| dx\right) + \left|h'(b)\right| \left(\int_0^{\frac{1}{2}} x\left|x - \frac{1}{3}\right| + \int_{\frac{1}{2}}^1 x\left|x - \frac{2}{3}\right| dx\right) \\
&= \frac{5(|h'(a)| + |h'(b)|)}{72}.
\end{aligned} \tag{7}$$

Therefore, the desired inequality asserted by Theorem 2.1 in the case $q = 1$ holds true. 75
Now, suppose that $q > 1$. Further, we will use the Hölder integral inequality in the classical 76
settings for $L_p - L_q$ functions, about this inequality see e.g. the monograph [19]. So from 77
the Hölder integral inequality (with $p = \frac{q}{q-1}$), we get 78

$$\begin{aligned}
\int_0^{\frac{1}{2}} \left|x - \frac{1}{3}\right| \left|h'(a + x(b-a))\right| dx &= \int_0^{\frac{1}{2}} \left|x - \frac{1}{3}\right|^{1-\frac{1}{q}} \left(\left|x - \frac{1}{3}\right|^{\frac{1}{q}} |h'(a + x(b-a))|\right) dx \\
&\leq \left(\int_0^{\frac{1}{2}} \left|x - \frac{1}{3}\right| dx\right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \left|x - \frac{1}{3}\right| |h'(a + x(b-a))|^q dx\right)^{\frac{1}{q}} \\
&\leq \left(\frac{5}{72}\right)^{1-\frac{1}{q}} \left(|h'(a)|^q \int_0^{\frac{1}{2}} (1-x)\left|x - \frac{1}{3}\right| dx + |h'(b)|^q \int_0^{\frac{1}{2}} x\left|x - \frac{1}{3}\right| dx\right)^{\frac{1}{q}} \\
&\leq \left(\frac{5}{72}\right)^{1-\frac{1}{q}} \left(\frac{111|h'(a)|^q}{1944} + \frac{|h'(b)|^q}{81}\right)^{\frac{1}{q}}.
\end{aligned} \tag{8}$$

In the same way, we find

$$\int_{\frac{1}{2}}^1 \left|x - \frac{2}{3}\right| \left|h'(a + x(b-a))\right| dx \leq \left(\frac{5}{72}\right)^{1-\frac{1}{q}} \left(\frac{|h'(a)|^q}{81} + \frac{111|h'(b)|^q}{1944}\right)^{\frac{1}{q}}. \tag{9}$$

Keeping (6), (8) and (9) in mind, we get the result (5) asserted by Theorem 2.1. □ 79

Theorem 2.2. *Let h be an absolutely continuous function on an interval $[a, b]$ and its derivative* 80
$h' \in L_1[a, b]$, $|h'|^q$ is convex on $[a, b]$ for some $q > 1$, then the following holds true: 81

$$\left|\frac{1}{3}\left[h(a) + h\left(\frac{a+b}{2}\right) + h(b)\right] - \frac{1}{b-a}\int_a^b h(x)dx\right| \leq \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \left[|h'(a)|^q + |h'(b)|^q\right]^{\frac{1}{q}}. \tag{10}$$

with $\frac{1}{p} + \frac{1}{q} = 1.$ 82

Proof. As the function $|h'|^q$ is convex on $[a, b]$, we have

$$\int_0^{\frac{1}{2}} \left|h'(a + x(b-a))\right|^q dx \leq \frac{3|h'(a)|^q + |h'(b)|^q}{8},$$

and

$$\int_{\frac{1}{2}}^{1} \left|h'(a+x(b-a))\right|^q dx \le \frac{|h'(a)|^q + 3|h'(b)|^q}{8}.$$

Straightforward calculation yields

$$\int_0^{\frac{1}{2}} \left|x-\frac{1}{3}\right|^p dx = \int_{\frac{1}{2}}^{1} \left|x-\frac{2}{3}\right|^p dx = \frac{1+2^{p+1}}{6^{p+1}(p+1)}.$$

Applying the Hölder integral inequality, we get

$$\begin{aligned}\int_0^{\frac{1}{2}} \left|x-\frac{1}{3}\right|\left|h'(a+x(b-a))\right| dx &\le \left(\int_0^{\frac{1}{2}} \left|x-\frac{1}{3}\right|^p dx\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|h'(a+x(b-a))\right|^q dx\right)^{\frac{1}{q}} \\ &\le \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \left(\frac{3|h'(a)|^q + |h'(b)|^q}{8}\right)^{\frac{1}{q}},\end{aligned} \tag{11}$$

and

$$\begin{aligned}\int_{\frac{1}{2}}^{1} \left|x-\frac{2}{3}\right|\left|h'(a+x(b-a))\right| dx &\le \left(\int_{\frac{1}{2}}^{1} \left|x-\frac{2}{3}\right|^p dx\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|h'(a+x(b-a))\right|^q dx\right)^{\frac{1}{q}} \\ &\le \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \left(\frac{|h'(a)|^q + 3|h'(b)|^q}{8}\right)^{\frac{1}{q}}.\end{aligned} \tag{12}$$

Combining (6), (11) and (12) and making some elementary simplifications, the asserted result (10) follows. □ 83 84

In the next we define for brevity

$$\Xi(a,b) = \frac{1}{3}[h(a)+h(b)+h(A(a,b))] - \frac{1}{b-a}\int_a^b h(x)\,d\,x \ne 0 \quad \text{and} \quad A(a,b) = \frac{a+b}{2}.$$

By $A(a,b)$ we denote as usual an arithmetic mean of two non-negative numbers (a,b). 85

Theorem 2.3. *Under the conditions of Theorem 2.1, the following Hermite-Hadamard type inequality holds true:* 86 87

$$\begin{aligned}\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(x)dx\right| &\le (b-a)|\rho_1(a,b)|\left(\frac{5}{72}\right)^{1-\frac{1}{q}} \\ &\times \left\{\left(\frac{111|h'(a)|^q}{1944} + \frac{|h'(b)|^q}{81}\right)^{\frac{1}{q}} + \left(\frac{|h'(a)|^q}{81} + \frac{111|h'(b)|^q}{1944}\right)^{\frac{1}{q}}\right\},\end{aligned} \tag{13}$$

where

$$\rho_1(a,b) := 1 - 2\frac{A(h(a),h(b)) - h(A(a,b))}{3\,\Xi(a,b)}.$$

Furthermore, the next is true

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(x)dx\right| \le \frac{5(b-a)|\rho_1(a,b)|}{72}\Big[\left|h'(a)\right| + \left|h'(b)\right|\Big]. \tag{14}$$

Proof. With the aid of the formula (4), we thus get 88

$$
\begin{aligned}
\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(x)dx\right| &= (b-a)\left|\int_0^{\frac{1}{2}}(x-\frac{1}{3})h'(a+x(b-a))dx+\int_{\frac{1}{2}}^{1}(x-\frac{2}{3})h'(a+x(b-a))dx\right. \\
&\quad \left. -2\frac{A(h(a),h(b))-h(A(a,b))}{3(b-a)}\right| \\
&= (b-a)\left|\int_0^{\frac{1}{2}}(x-\frac{1}{3})h'(a+x(b-a))dx+\int_{\frac{1}{2}}^{1}(x-\frac{2}{3})h'(a+x(b-a))dx\right| \\
&\quad \times\left|1-2\frac{A(h(a),h(b))-h(A(a,b))}{3\,\Xi(a,b)}\right|.
\end{aligned} \tag{15}
$$

Obviously, by repeating the same calculations as in the proof of Theorem 2.1 with the help of (15) we achieve the required result (13). Finally, taking $q=1$ in (13) leads to the inequality (14). This completes the proof of Theorem 2.3. □

Remark 2.1. *We note that if $|\rho_1|<\frac{9}{5}$, then the inequality (14) is better than the inequality (3). It means that the absolute positive constant at the right-hand side of inequality (14) is smaller (better!) than the right-hand side of inequality (3), so under the above condition $|\rho_1|<\frac{9}{5}$ the inequality (14) is more exact than the inequality (3).*

Now, we present some examples to illustrate cases then the right-hand side of inequality (14) is better than the right hand side of inequality (3).

- *Let consider the function $h(x)=e^x$ and $[a,b]=[t,t+1],\ t\in\mathbb{R}$. Then we have*

$$\rho_1=\frac{3+3\sqrt{e}-3e}{4+\sqrt{e}-2e}\approx -0.98362..$$

Consequently, the inequality (14) is better than the inequality (3). It means that the positive constant in the right-hand side of inequality (14) is smaller (better!) than the positive constant in right-hand side of inequality (3).

- *Let take $h(x)=\psi(x)$ where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function and $[a,b]=[t,t+2],\ t>0$. It is known that the trigamma function $\psi'(x)$ is convex on $(0,\infty)$. Hence, by using the identity*

$$\psi(t+1)=\psi(t)+\frac{1}{t}, \tag{16}$$

we have

$$A(h(t),h(t+2))-h(A(t,t+2))=-\frac{1}{2t(t+1)}$$

and

$$\Xi(t,t+2)=\psi(t)+\frac{2}{3t}+\frac{1}{3(t+1)}-\frac{1}{2}\log(t^2+t).$$

Therefore,

$$\rho_1(t,t+2)=1+\frac{2}{6t(t+1)\psi(t)-3t(t+1)\log(t(t+1))+6t+4}.$$

Hence $F(t):=|\rho_1(t,t+2||<\frac{9}{5}$ for all $t>0$, see the figure 1 which verifies our claim. Consequently, for this case the right-hand side of inequality (14) is better than the right-hand side of inequality (3).

- *Next let $h(x)=\psi(x)$ and $[a,b]=[t,t+1],\ t>0$. Hence, for this case we have*

$$A(h(t),h(t+1))-h(A(t,t+1))=\psi(t)-\psi\left(t+\frac{1}{2}\right)+\frac{1}{2t},$$

and

$$\Xi(t,t+1)=\frac{2}{3}\psi(t)+\frac{1}{3}\psi\left(t+\frac{1}{2}\right)-\log(t)+\frac{1}{3t}.$$

Hence

$$\rho_1(t,t+1)=\frac{3t\psi(t+1/2)-3t\log(t)}{2t\psi(t)+t\psi(t+1/2)-3t\log(t)+1}.$$

By figure 1 we see that $G(t):=|\rho_1(t,t+1)|<\frac{9}{5}$, *which implies that (14) improves (3).* 104

Theorem 2.4. *Under the conditions of the Theorem 2.2 it follows*

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(x)dx\right|\le\frac{(b-a)|\rho_1(a,b)|}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left[|h'(a)|^q+|h'(b)|^q\right]^{\frac{1}{q}},\tag{17}$$

for $\frac{1}{p}+\frac{1}{q}=1$. *In particular, it holds*

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(x)dx\right|\le\frac{(b-a)|\rho_1(a,b)|}{12}\left[|h'(a)|^2+|h'(b)|^2\right]^{\frac{1}{2}}.\tag{18}$$

Proof. By using the same consequent steps as in the Theorem 2.2 with the help of the 105
formula (15) we obtain the inequality (17) asserted by Theorem 2.4, we choose to omit the 106
details involved. □ 107

Remark 2.2. *Under the conditions of the above Theorem, we see that the left hand side of inequality* 108
(18) is better than the inequality (3) if $|\rho_1(a,b)|<\frac{3}{2}$. *It means that the absolute positive constant at* 109
the right-hand side of inequality (18) is smaller (better!) than the right-hand side of inequality (3), 110
so under the above condition $|\rho_1(a,b)|<\frac{3}{2}$ *the inequality (18) is more exact than the inequality* 111
(3). 112

We note that the examples used in Remark 2.1 verifies also conditions $|\rho_1(a,b)|<\frac{3}{2}$, *it is* 113
illustrated by the figure 1 (see below). 114

Theorem 2.5. *Under the assumptions of Theorem 2.1, the next inequality holds true* 115

$$\begin{aligned}\left|\frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(x)dx\right|&\le|\rho_2(a,b)|(b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\\&\times\left\{\left(\frac{111|h'(a)|^q}{1944}+\frac{|h'(b)|^q}{81}\right)^{\frac{1}{q}}+\left(\frac{|h'(a)|^q}{81}+\frac{111|h'(b)|^q}{1944}\right)^{\frac{1}{q}}\right\},\end{aligned}\tag{19}$$

where

$$\rho_2(a,b)=:1-\frac{h(A(a,b))-A(h(a),h(b))}{3\,\Xi(a,b)}.$$

Proof. From Lemma 2.1, we have 116

$$\begin{aligned}\left|\frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(x)dx\right| &= (b-a)\bigg|\int_0^{\frac{1}{2}}(x-\frac{1}{3})h'(a+x(b-a))dx+\int_{\frac{1}{2}}^{1}(x-\frac{2}{3})h'(a+x(b-a))dx \\ &\quad -\frac{h(A(a,b))-A(h(a),h(b))}{3(b-a)}\bigg| \\ &= (b-a)\left|\int_0^{\frac{1}{2}}(x-\frac{1}{3})h'(a+x(b-a))dx+\int_{\frac{1}{2}}^{1}(x-\frac{2}{3})h'(a+x(b-a))dx\right| \\ &\quad \times\left|1-\frac{h(A(a,b))-A(h(a),h(b))}{3\,\Xi(a,b)}\right|.\end{aligned} \tag{20}$$

By repeating the same steps as in the proof of the Theorem 2.1 with the above relation, we derive the assertion of the Theorem 2.5. Exactly, these steps consist of using integral representation with derivative h' via arithmetic means (20) and following obvious integral estimates instead of (6), and after that using of the Hölder integral inequality [19] for cases $q = 1$ and $q > 1$ one by one. □

Remark 2.3. *If take $q = 1$ in the Theorem 2.5, we get*

$$\left|\frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(x)dx\right| \le \frac{5|\rho_2(a,b)|(b-a)}{72}\Big(|h'(a)|+|h'(b)|\Big). \tag{21}$$

We note that the obtained midpoint inequality (21) is better than the inequality (2) if $|\rho_2| < \frac{9}{5}$. It means that the absolute positive constant at the right-hand side of inequality (21)is smaller (better!) than the right-hand side of inequality (2), so under the above condition $|\rho_2| < \frac{9}{5}$ the inequality (21) is more exact than the inequality (2).

To support this, we consider the following example: set $h(x) = \psi(x)$ and $[a,b] = [t, t+2]$ $t > 0$. In this case, we have

$$h(A(t,t+2)) - A(h(t),h(t+2)) = \frac{1}{2t(t+1)}$$

and

$$\rho_2(t,t+2) = 1-\frac{1}{6t(t+1)\psi(t)-3t(t+1)\log(t(t+1))+6t+4}.$$

We set

$$H(t) = \frac{9}{5}-\left|1-\frac{1}{6t(t+1)\psi(t)-3t(t+1)\log(t(t+1))+6t+4}\right|.$$

Figure 1 illustrates that the right-hand side of inequality (21) is sharper than the right hand-side of inequality (2) on the interval $[t, t+2]$ where $t \in (0, 0.367217)$.

Theorem 2.6. *With the assumptions of Theorem 2.2, the following inequality*

$$\left|\frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(x)dx\right| \le \frac{(b-a)|\rho_2(a,b)|}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left[|h'(a)|^q+|h'(b)|^q\right]^{\frac{1}{q}}, \tag{22}$$

is true, with $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. Just repeat the same calculations as in the proof of the Theorem 2.2 with using (20). □

On setting $p = q = 2$ in (22), we get the following result as follows:

Corollary 1. *With the conditions of Theorem 2.2, we get*

$$\left| \frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(x)\,dx \right| \leq \frac{|\rho_2(a,b)|(b-a)}{12}\sqrt{|h'(a)|^2+|h'(b)|^2}. \tag{23}$$

Remark 2.4. *We note that the left hand side of inequality (23) is better than the left hand side of inequality (2) if $|\rho_2| < \frac{3}{2}$. It means that the absolute positive constant at the right-hand side of inequality (23)is smaller (better!) than the right-hand side of inequality (2), so under the above condition $|\rho_2| < \frac{3}{2}$ the inequality (23) is more sharp than the inequality (2).*

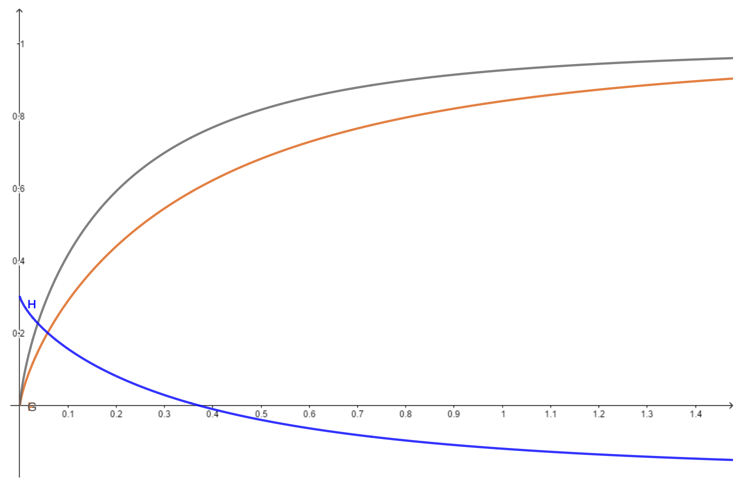

Figure 1. Graphs of functions F, G (from Remark 2.1) and H (from Remark 2.3)

3. Applications

3.1. Some new inequalities for digamma function in terms of trigamma function

Our aim in this section is to establish new inequalities involving the digamma and trigamma functions.

Proposition 2. *For $t > 0$, the next inequality holds true:*

$$\left| \psi(t) - \log(\sqrt{t(t+1)}) + \frac{3t+2}{3t(t+1)} \right| \leq \frac{5}{72}\left(2\psi'(t) - \frac{2t^2+2t+1}{t^2(t+1)^2} \right). \tag{24}$$

Proof. Upon setting $h(x) = \psi(x)$ and $[a,b] = [t, t+2]$ in Theorem 2.1 ($q = 1$) and using (16) we get

$$\left| 3\psi(t) + \frac{2}{t} + \frac{1}{t+1} - \frac{3}{2}\log(t(t+1)) \right| \leq \frac{5}{24}\Big(|\psi'(t+2)| + |\psi'(t)| \Big). \tag{25}$$

Again, by using (16) we have

$$\psi'(t+2) = \psi'(t) - \frac{1}{t^2} - \frac{1}{(t+1)^2}.$$

In view of the above relations and straightforward calculation we derive the desired inequality (24) asserted by Proposition 2. □

Proposition 3. *For any* $t > 0$, *the next inequality holds true:*

$$\left|2\psi(t)+\psi\left(t+\frac{1}{2}\right)-3\log(t)+\frac{1}{t}\right| \leq \frac{5}{24}\left(2\psi'(t)+\frac{1}{t}\right). \tag{26}$$

Proof. We set $h(x) = \psi(x)$ and $[a, b] = [t, t+1]$, $t > 0$ in Theorem 2.1. The details involved are derived by a straightforward calculation. □

3.2. Applications of Hermite–Hadamard type inequalities to linear combinations of some special means

The study of different kinds of means is fulfilled e.g. in [4–6,13,19,20]. It is one of the most important notion in mathematics and applications.

With aid of some results in Section 3, our aim in this section is to derive some new inequalities involving combinations of special means and its powers.

For arbitrary positive real numbers a, b we define as usual

1. The arithmetic and geometric means:

$$A(x,y) = \frac{x+y}{2}, G(x,y) = \sqrt{ab}.$$

2. The logarithmic mean:

$$L(a,b) = (b-a)/(\log(b)-\log(a)), a \neq b.$$

3. The generalized logarithmic mean:

$$L_n(a,b) = \left[(b^{p+1}-a^{p+1})/((p+1)(b-a))\right]^{\frac{1}{p}},\ p \in \mathbb{R}\setminus\{-1,0\}, a \neq b.$$

In fact the logarithmic mean and generalized logarithmic mean are special cases of means introduced by Tibor Radó. T. Radó also received most important inequalities for them, cf. [19**? ?**].

Proposition 4. *Let* $r \in (0,1]$ *and* $a, b \in \mathbb{R}$ *such that* $0 < a < b$. *Then the following inequality*

$$\begin{aligned}\left|\frac{2}{3}A(a^r,b^r)+\frac{1}{3}A^r(a,b)-L_r^r(a,b)\right| &\leq r(b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}} \\ &\times\left\{\left(\frac{111a^{q(r-1)}}{1944}+\frac{b^{q(r-1)}}{81}\right)^{\frac{1}{q}}+\left(\frac{a^{q(r-1)}}{81}+\frac{111b^{q(r-1)}}{1944}\right)^{\frac{1}{q}}\right\},\end{aligned} \tag{27}$$

holds true for all $q \geq 1$.

Proof. The claim follows from Theorem 2.1 with $q = 1$ and $h(x) = x^r$, $r \in (0,1]$. □

Proposition 5. *Under the assumptions of Proposition 4, the following inequalities holds:*

$$\left|\frac{2}{3}A(a^r,b^r)+\frac{1}{3}A^r(a,b)-L_r^r(a,b)\right| \leq \frac{r(b-a)}{6}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}}A^{\frac{1}{q}}\left(a^{q(r-1)},b^{q(r-1)}\right), \tag{28}$$

where $\frac{1}{p}+\frac{1}{q} = 1$ $(p > 1)$.

Proof. The claim follows from Theorem 2.2 with $h(x) = x^r$, $\alpha \in (0,1]$. □

Setting $h(x) = \frac{1}{x^r}, r \in (0,1]$ in Theorem 2.1, we deduce the following inequality:

Proposition 6. *With the conditions of Proposition 4, we get*

$$\left|\frac{2}{3}A(a^{-r},b^{-r})+\frac{1}{3}A^{-r}(a,b)-L_{-r}^{-r}(a,b)\right|\leq r(b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}} \times\left\{\left(\frac{111\;a^{-q(r+1)}}{1944}+\frac{b^{-q(r+1)}}{81}\right)^{\frac{1}{q}}+\left(\frac{a^{-q(r+1)}}{81}+\frac{111\;b^{-q(r+1)}}{1944}\right)^{\frac{1}{q}}\right\}, \tag{29}$$

holds true for all $q\geq 1$.

Remark 3.1. *If we set* $r=1$ *in (29) we get*

$$\left|\frac{2}{3}A(a^{-1},b^{-1})+\frac{1}{3}A^{-1}(a,b)-L_{-1}^{-1}(a,b)\right|\leq (b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}} \times\left\{\left(\frac{111\;a^{-2q}}{1944}+\frac{b^{-2q}}{81}\right)^{\frac{1}{q}}+\left(\frac{a^{-2q}}{81}+\frac{111\;b^{-2q}}{1944}\right)^{\frac{1}{q}}\right\}, \tag{30}$$

where $q\geq 1$.

Letting $f(t)=\frac{1}{t^r}, r\in(0,1]$ in Theorem 2.2 leads to the following inequality:

Proposition 7. *With the assumptions of Proposition 4, the following inequality is valid:*

$$\left|\frac{2}{3}A(a^{-r},b^{-r})+\frac{1}{3}A^{-r}(a,b)-L_{-r}^{-r}(a,b)\right|\leq \frac{r(b-a)}{6}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}}A\left(a^{-q(r+1)},a^{-q(r+1)}\right) \tag{31}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

In particular, we get

$$\left|\frac{2}{3}A(a^{-1},b^{-1})+\frac{1}{3}A^{-1}(a,b)-L_{-1}^{-1}(a,b)\right|\leq \frac{(b-a)}{6}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}}A\left(a^{-2q},a^{-2q}\right), \tag{32}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

4. Conclusion

In this paper, we establish new Hermite-Hadamard type inequalities for a class of functions with some convexity conditions on derivatives. As consequences we obtained some new inequalities for the digamma function in terms of the trigamma function. Some applications to special means of real numbers are also given. Analytical and numerical computation shows that some of the obtained results are better than the similar known results.

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