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ANALYSIS OF THE ELECTRIC FIELD IN A LASER BY THE MULTIPOLE METHOD

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Abstract. The multipole method is modified such that it may be applicable to analysis of electric field in the laser of special design. An optimal form of electrodes in the device under consideration is found. Main characteristics of the field are obtained in terms of closed formulae. Data of numerical study which confirm high effectiveness and accuracy of this method are given.

Key words: boundary value problem, multipole method, calculation of electric field in the laser.

Introduction

The multipole method suggested in [1], [2] and developed in [3]- [15] is based on the use of functions Ω_p, $p = 1, 2, \ldots$, which satisfy identically a given equation in initial domain g (e.g., the Laplace equation), satisfy homogeneous boundary condition at curve $\gamma \subset \partial g$, and constitute a complete and minimum system at the complementary arc $\Gamma = \partial g \backslash \gamma$. A solution of a boundary value problem is presented as a sequence of linear combinations of functions Ω_p. Those functions are boundary multipoles for an extension G of initial domain g over arc Γ; the concept of the boundary multipoles is meant in the sense of [3]. It is important that these functions can be expressed by the simple formula in the case of boundary value problems for the Laplace equation. Thus, if a Dirichlet problem in g with zero condition at γ is considered, then functions Ω_p will be given as follows: $\Omega_p = \operatorname{Im} F^p$, where F is a conformal mapping of the above extension G onto the upper half-plane.

The multipole method was substantiated and investigated in [3]. The obtained theoretical estimates show the method enables to calculate effectively the problem solution and all its derivatives both in domain g and at arc γ even if it has complex shape, contains geometrical singularities or infinities. It should be emphasized , that this method, according to the above estimates, gives the convergence in C^n-norm with arbitrary n in domain, including a part of its boundary, while traditional methods (e.g., finite element method) yield approximation only in W_1^2-norm (energy norm), and the error of gradient increases when approaching the boundary. The performed numerical experiments confirmed high efficiency of the multipole method; e.g., when the Dirichlet problem was being solved for the Poisson equation in L-type domain with rounded re-entrant corner [5], [7], [9], the use of only 40 degrees of freedom (i.e. functions Ω_p) ensured the accuracy 10^{-8} in C-norm for gradient near the rounded corner.

The present work is devoted to modification of this method and to its application to a difficult engineering problem which arises when designing a gas laser of special structure.

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According to the general principle of laser operation, an active medium which would amplify electromagnetic waves passing through, must be created in the laser [16].

The most effective process of an active medium creation in a gas laser is implemented by a glow electric discharge of sufficient intensity maintained in the gas mixture. Discharge conditions, the speed of this process, and, therefore, effectiveness of laser generation are characterized by some parameters among which electric field intensity E plays a vital role [16]. The laser operation is highly sensitive to the change of E, so very accurate analysis of the field is essentially important.

The laser under consideration is of a complex design what makes this analysis rather difficult. The main feature of this device is in the special electrode structure suggested by researchers from P.N. Lebedev Physical Institute of the Russian Academy of Sciences (Prof. A.N. Lobanov and his colleagues). Namely, both the anode and cathode inserted into the gas medium are composed of isolated sections provided with a system which enables to vary a potential on each of them independently. Thus, a variable distribution of potential can be fed at the bottoms of the electrodes; there is a constant potential at their side facings. The lasers of such a structure have a supplementary possibility which is that the electric field can be tuned for the most efficient laser operation, owing to redistribution of the potential and variation of the bottoms form.

The experience of above mentioned researches from P.N.Lebedev Physical Institute of the Russian Academy of Sciences showed that application of various numerical methods to the evaluation of field in this laser encountered considerable obstacles and did not yield satisfying results. And all of their efforts to obtain field intensity E, which is a differential characteristic, failed. In the present work we apply the multipole method that enables to get all the characteristics required with high accuracy and efficiency.

A boundary value problem which describes the electric potential in the gas mixture under consideration is stated in Sect. 1. The solution of the boundary value problem is constructed with the help of the multipole method in Sect. 2. Sect. 3 is dedicated to finding such a distance between the electrodes and a form of their bottoms that the given maximum constant field would be kept at the bottoms for a constant potential preassigned at them. Sect. 4 contains general representation of the main field characteristics and data of specific implementation.

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1. Statement of the problem

1.1. Domains and Boundary Value Problem. The analysis of discharge conditions which take place in the laser gas mixture can be reduced to solving a Dirichlet problem for the Laplace equation in two-dimensional unbounded domain g. This domain is the upper half-plane from that a half-strip with a curved bottom removed.

To describe domain g accurately, we introduce an auxiliary domain G. Let G in complex plane $w = u + iv$ be half-plane $\mathcal{H} = \{w : v > 0\}$ without two parallel rays:

$$G = \mathcal{H} \backslash \{w : u = \pm a; \quad v \geq b\},$$

where a and b are positive numbers. The initial points of rays will be denoted by $B = a + ib$ and $B' = -a + ib$. The domain G boundary contains three infinities A, M and A' reached as $|w| \to \infty$ with $u > a$ for $A, -a < u < a$ for M and $u < -a$ for A', see Fig. 1.

Now we join points B and B' by a Jordan smooth curve Γ which lies in G except for the endpoints. This curve divides domain G into two subdomains, one of them with boundary $(ABB'A'A)$ will be considered as domain g, see Fig. 2.

Domain g corresponds physically to the half section of a volume with gas mixture while the half-strip represents the electrode section. Let a certain potential $\phi(w)$ be distributed at electrode bottom Γ and certain constant potential values ϕ_1 and ϕ_2 be preset at the electrode side facings $(B'A')$ and (AB), respectively. Besides, zero potential is presumed at u-axis $(A'A)$ what follows from antisymmetry of the field with respect to this axis. Then sought potential Φ excited in the gas mixture in view of a small density of a volume charge throughout the system is described by the following boundary value problem:

$$\begin{aligned}
\Delta\Phi(w) &= 0, && w \in g, && (1.1)\\
\Phi(w) &= 0, && w \in (A'A), && (1.2)\\
\Phi(w) &= \phi_1, && w \in (B'A'), && (1.3)\\
\Phi(w) &= \phi_2, && w \in (AB), && (1.4)\\
\Phi(w) &= \phi(w), && w \in \Gamma. && (1.5)
\end{aligned}$$

Here, boundary function ϕ is continuous at Γ and joins continuously with boundary values at sides $(B'A')$ and (AB), i.e. $\phi(B') = \phi_1$, $\phi(B) = \phi_2$. Function $\Phi(w)$ is a bounded solution of problem (1.1)-(1.5) that belongs to $C^2(g) \cap C\,(\bar{g} \backslash (A \cup A'))$.

Note bottom Γ in practice is chosen smoothly joined with the electrode side facings $(B'A')$ and (AB). Otherwise, a field concentration will occur near the junction of the electrode bottom with the side facings what can cause the unstable discharge and electron breakdown [16].

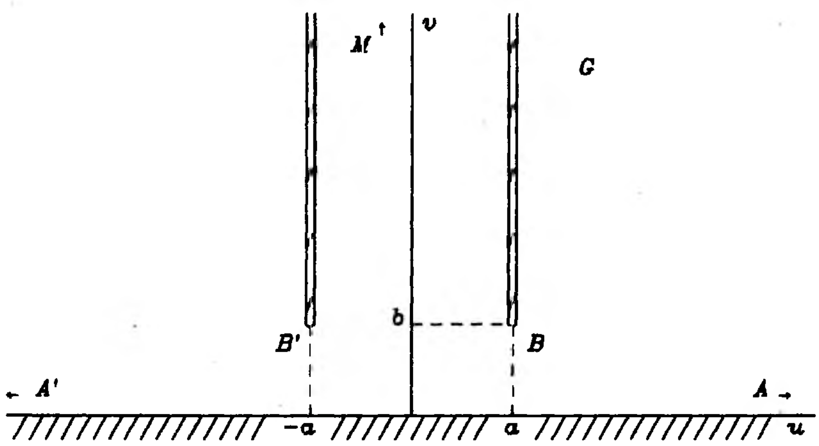

Fig. 1. Domain G.

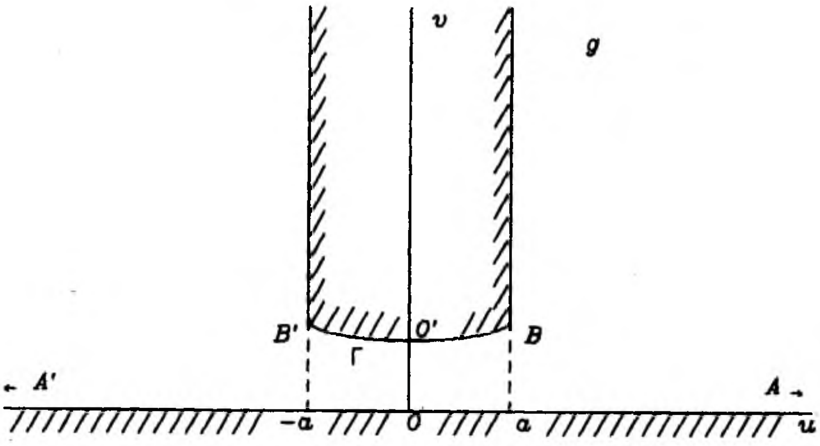

Fig. 2. Domain g.

1.2. Conformal Mapping of Auxiliary Domain G. When the boundary value problem (1.1)-(1.5) being solved by the multipole method, conformal mapping $z = F(w)$ of domain G onto the half-plane $\mathcal{H}$ plays a vital part. However, it is difficult to find an effective global analytical representation for this mapping what makes us turn to an inverse mapping (of $\mathcal{H}$ onto G) denoted by $f(z)$ that can be found in this case with reasonable facility. We define the following correspondence between three boundary points of one domain and those of the other: points A, M, and A' of ∂G correspond to points 1, ∞, and -1 of $\partial\mathcal{H}$, i.e. in terms of mapping f

$$f(1) = A, \qquad f(\infty) = M, \qquad f(-1) = A'.$$

We remark that mapping f satisfies relation $f(-\bar{z}) = -\bar{f}(z)$, $z \in \mathcal{H}$, following from the Schwarz reflection principle [17], [18]. Then points $B = a + ib$ and $B' = -a + ib$ will correspond respectively to real points $z = k^{-1}$ and $z = -k^{-1}$ for a certain $k \in]0, 1[$ which will be determined below.

Function $f(z)$ may be expressed by Schwarz-Christoffel integral [17], [18]

$$w = f(z) = C \int\limits_0^z (1 - \zeta^2)^{-3/2}(k^{-2} - \zeta^2)d\zeta. \tag{1.6}$$

To find the unknown real k and C, we shall use conditions resulting from the above definition of conformed mapping $f(z)$:

$$\operatorname{Re} f(x) = a, \qquad x > 1; \qquad f(k^{-1}) = a + ib. \tag{1.7}$$

Let x be a real number from interval $] - 1, 1[$. Taking integral (1.6) along real segment $[0, x]$, getting, thus,

$$f(x) = C\left(\frac{k^{-2} - 1}{\sqrt{1 - x^2}}\,x + \arcsin x\right),$$

and analytically continuing this function from real interval $]-1,1[$ to upper half-plane $\mathcal{H}$, we obtain the explicit formula for the required mapping

$$f(z) = C\,i\left(\frac{k^{-2}-1}{\sqrt{z^2-1}}\,z + \ln(z+\sqrt{z^2-1})\right) + \frac{\pi C}{2};$$

we consider the main branch of logarithm: $\operatorname{Im}\ln z \in]-\pi,\pi[$, $z\in\mathcal{H}$. This expression in accordance with conditions (1.7) follows $C = 2a/\pi$ and parameter $k = k(a/b)$ is a root of transcendental equation

$$\frac{\sqrt{1-k^2}}{k^2} + \ln\frac{1+\sqrt{1-k^2}}{k} = \frac{\pi\, b}{2\, a}.$$

A graph of k as a function of dimensionless parameter a/b is drawn in Fig. 3.

Now we can write out final expression for mapping $f(z)$:

$$w = f(z) = \frac{2ai}{\pi}\left(\frac{k^{-2}-1}{\sqrt{z^2-1}}\,z + \ln(z+\sqrt{z^2-1})\right) + a;$$

Thus, inverse mapping $f(z)$ is completely defined, and the required mapping $F(w)$ can be obtained by inversion of it. A very convenient local inversing procedure will be presented in Subsect. 2.4.

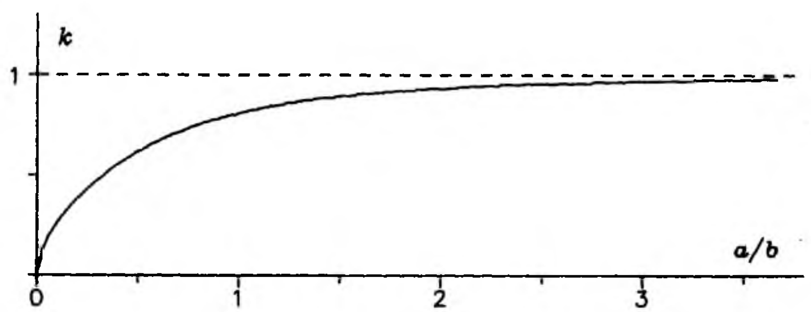

Fig. 3. Parameter k versus ratio a/b.

2. Solution of boundary value problem

2.1. Reduction of Problem. We introduce function $U_0(w)$ via conformal mapping $F(w)$ defined at the end of the previous section

$$U_0(w) = \frac{1}{\pi}\operatorname{Im}\left(\phi_1 \ln\left[1+F(w)\right] - \phi_2 \ln\left[1-F(w)\right]\right). \tag{2.1}$$

Function $U_0(w)$ is obviously a bounded harmonic in G and continuous in $\bar{G}\backslash(A \cup M \cup A')$ one, and satisfies the following boundary conditions at ∂G:

$$\begin{array}{rcllcl} U_0(w) & = & 0, & w & \in & (A'\,A), \\ U_0(w) & = & \phi_1, & w & \in & (M\,B'\,A'), \\ U_0(w) & = & \phi_2, & w & \in & (A\,B\,M). \end{array}$$

Let us present solution $\Phi(w)$ of original problem (1.1) – (1.5) in the form

$$\Phi(w) = U_0(w) + U(w). \tag{2.2}$$

Taking into account the above properties of function U_0 and the inclusion $\Phi \in C\left(\bar{g}\backslash(A \cup A')\right)$, we find $U(w)$ is a classical solution of the Dirichlet problem

$$\Delta U(w) = 0, \qquad w \in g, \tag{2.3}$$

$$U(w) = 0, \qquad w \in \gamma = \partial g \backslash \Gamma, \tag{2.4}$$

$$U(w) = \phi(w) - U_0(w), \qquad w \in \Gamma. \tag{2.5}$$

Function $U(w)$ will be constructed with the help of the multipole method.

2.2. Multipole Method for Solving Problem. The multipole method which is an analytical-numerical one for solving elliptic boundary value problems in complex-shaped domains was theoretically studied, advanced and generalized in [3]- [15] and some other works, and was successfully used for solving a number of theoretical and applied problems.

The basis of the method is an application of functions $\Omega_p(w)$ expressed by the formula

$$\Omega_p(w) = \mathrm{Im}\,[F(w)]^p, \qquad p \in N.$$

One can interprete function $\Omega_p(w)$ for every natural p in the electrostatic sense [3]: it represents electric potential excited in domain G by pth order boundary multipole which is located in point M. These are harmonic in g functions, which satisfy zero boundary condition at arc $\gamma = (B'\,A'\,A\,B)$ and constitute complete and minimum system in $L_2(\Gamma)$. The required solution of problem (2.3) – (2.5) can be obtained as the limit

$$U(w) = \lim_{K \to \infty} U^K(w) \tag{2.6}$$

of sequence of functions $U^K(w)$ which are $\Omega_p(w)$ linear combinations

$$U^K(w) = \sum_{p=1}^{K} a_p^K\, \Omega_p(w),$$

where coefficients a_n^K are defined by the condition that $L_2(\Gamma)$-norm of difference of $U^K(w)$ and boundary function $\phi(w) - U_0(w)$ should be minimum

$$\|U^K - \phi + U_0\|_{L_2(\Gamma)} \longmapsto \min\,. \tag{2.7}$$

Sequence $U^K(w)$ converges uniformly to solution $U(w)$ of problem (2.3) – (2.5) everywhere in every compact lying into set $g \cup \gamma$ and, moreover, admits differentiation of any order in set $g \cup \gamma \backslash (A \cup A')$; the differentiated sequence converges uniformly in every compact lying into the latter set [3].

Besides, an expansion in the multipole system

$$U(w) = \sum_{n=1}^{\infty} a_p\, \Omega_p(w), \qquad a_p = \lim_{K \to \infty} a_p^K$$

is an analog of Taylor series in the sense that it can be differentiated any times and it converges with exponential speed everywhere in its convergence set

$$\{w = f(z);\ |z| < R,\ \mathrm{Im}\, z \geq 0\}, \qquad R = \min_{w \in \Gamma} |F(w)|.$$

We remark that the set represents the union of a certain subdomain of g and a part of arc $(B'\,A'\,A\,B)$ adjacent to it.

Condition (2.7) results in linear algebraic system

$$\sum_{p=1}^{K} C_{pq} a_p^K = H_q, \qquad q = 1, 2, \ldots, K, \tag{2.8}$$

where C_{pq} are elements of Gram's matrix for system $\{\Omega_p(w)\}_{p=1}^{\infty}$, and H_p is a projection of boundary function $\phi - U_0$ onto Ω_p:

$$C_{pq} = \int_{\Gamma} \Omega_p(w) \Omega_q(w) |dw|, \qquad H_p = \int_{\Gamma} \Omega_p(w)[\phi(w) - U_0(w)] |dw|.$$

By virtue of the fact that function system $\{\Omega_p(w)\}_{p=1}^{\infty}$ is complete and minimum in $L_2(\Gamma)$ as it has been said, algebraic system (2.8) is uniquely solvable.

Thus, summing up the preceding, we can write out the expression for the approximate solution of problem (1.1) – (1.5)

$$\Phi^K(w) = \frac{1}{\pi}\, \mathrm{Im}\, (\phi_1 \ln[1 + F(w)] - \phi_2 \ln[1 - F(w)]) + \sum_{p=1}^{K} a_p^K \Omega_p(w)$$

converging to $\Phi(w)$ in the closed domain $\bar{g}$ as $K \to \infty$.

2.3. Expansion of Solution in Orthonormal System. It may be convenient to use a representation for $\Phi(w)$ in the form of series in an orthonormal system. Following [3], we introduce orthonormal function system $\{\omega_p(w)\}$ which can be obtained from system $\Omega_p(w)$ by the Schmidt orthogonalization process, see e.g. [19]

$$\omega_p(w) = \frac{1}{\sqrt{\mathrm{Det}_p \mathrm{Det}_{p-1}}} \sum_{n=1}^{p} A_{np} \Omega_n(w),$$

where Det_p is determinant of matrix $\{C_{mn}\}_{m,n=1}^{p}$ while A_{np} is the algebraic complement of element C_{np} in this matrix. Then exact solution $\Phi(w)$ is presented as follows

$$\Phi(w) = \frac{1}{\pi}\,\mathrm{Im}\,\left(\phi_1 \ln\left[1+F(w)\right] - \phi_2 \ln\left[1-F(w)\right]\right) + \sum_{p=1}^{\infty} h_p \omega_p(w), \tag{2.9}$$

where

$$h_p = \int_{\Gamma} \omega_p(w)[\phi(w) - U_0(w)]|dw|.$$

According to [3], series (2.9) converges everywhere in $\bar{g}$ and admits any order differentiation in $g \cup \gamma \backslash (A \cup A')$.

2.4. Constructing of Mapping $z = F(w)$. As we have already said, conformal mapping $z = F(w)$ of the extended domain G onto half-plane $\mathcal{H}$ is an important, apparatus for the multipole method. However, the problem of constructive representation for this mapping in the general case is difficult. Therefore, when the multipole method being implemented, one has to use the inverse of $F(w)$ denoted by $f(z)$ which can usually be found easier. If $f(z)$ is determined, then the required mapping could be constructed by inversing $f(z)$. It can be done with the help of the method of successive approximations which is based on the Newton's method and gives local inversing procedure [20]. Here we formulate this method for a conformal mapping in the general case.

Let function $w = \psi(z)$ accomplish a conformal mapping of domain G_1 onto domain G_2, and the inverse image $z_0 \in G_1$ for a certain $w_0 \in G_2$ is known,

$$f(z_0) = w_0.$$

Now we introduce some objects required for the formulation and proof of the method. The distance of point z_0 to the domain G_1 boundary will be denoted by R_{z0}; the disk

$$\mathcal{D}_{z0}(r) = \{z : |z - z_0| < r\}$$

for every $r \in]0, R_{z0}]$ lies obviously into domain G_1. Let us define the function

$$W(z_1, z_2) = \frac{\psi(z_1) - \psi(z_2)}{\psi'(z_0)(z_1 - z_2)} - 1$$

which is holomorphic in G_1 with respect to the both variables by virtue of the fact that $\psi(z)$ is holomorphic in G_1 and $\psi'(z_0) \neq 0$. A maximum absolute value of W with respect to z_1 and z_2 from the closed disk $\bar{\mathcal{D}}_{z0}(r)$, $r \in]0, R_{z0}]$ will be denoted by $Q(r)$,

$$Q(r) = \max|W(z_1, z_2)|, \qquad z_1, z_2 \in \bar{\mathcal{D}}_{z0}(r);$$

and maximum absolute value of $|W|$ of z_1 from the same disk, when $z_2 = z_0$, will be denoted by $q(r)$

$$q(r) = \max|W(z_1, z_0)|, \qquad z_1 \in \bar{\mathcal{D}}_{z0}(r).$$

It is clear that $Q(0) = q(0) = 0$. Now we will state some other properties of the intoduced functions.

Lemma 2.1. *If function $\psi(z)$ is not linear, then for every $r \in]0, R_{z0}]$*

1. *The following inequality takes place*

$$0 < q(r) < Q(r); \tag{2.10}$$

2. *Functions $Q(r)$ and $q(r)$ increase strictly;*

3. *Functions $Q(r)$ and $q(r)$ are majorized by convergent series*

$$Q(r) \leq \frac{1}{|\psi'(z_0)|} \sum_{n=1}^{\infty} \frac{|\psi^{(n+1)}(z_0)|}{n!} r^n < \infty; \tag{2.11}$$

$$q(r) \leq \frac{1}{|\psi'(z_0)|} \sum_{n=1}^{\infty} \frac{|\psi^{(n+1)}(z_0)|}{(n+1)!} r^n < \infty. \tag{2.12}$$

□ Using the principle of maximum for holomorphic function [17], [18], it is not difficult to verify that function $|W(z_1, z_0)|$ over the disk $\mathcal{D}_{z0}(r)$ reaches its maximum value at the disk boundary, while the maximum of $|W(z_1,\, z_2)|$ is reached when both points z_1, z_2 belongs to the disk boundary. In other words, the relations take place

$$Q(r) = |W(z_0 + re^{i\alpha_1}, z0 + re^{i\alpha_2})|, \qquad q(r) = |W(z_0 + re^{i\alpha_3}, z0)|$$

for certain $\alpha_j = \alpha_j(r) \in \; [0,\, 2\pi[,\; j = 1, 2, 3$. This follows the validity of inequality (2.10) and the strict monotony of the functions $Q(r)$ and $q(r)$. Further, expanding function $W(z_1, z_2)$

$$W(z_1, z_2) = \frac{1}{\psi'(z_0)} \sum_{n=2}^{\infty} \frac{\psi^{(n)}(z_0)}{n!} \frac{(z_1 - z_0)^n - (z_2 - z_0)^n}{(z_1 - z_0) - (z_2 - z_0)},$$

placing $z_j = z_0 + re^{i\alpha_j}$ into this expression and estimating its modulus, we obtain the inequality

$$Q(r) \leq \frac{1}{|\psi'(z_0)|} \sum_{n=2}^{\infty} \frac{|\psi^{(n)}(z_0)|}{n!} \left| \frac{e^{i\alpha_1 n} - e^{i\alpha_2 n}}{e^{i\alpha_1} - e^{i\alpha_2}} \right| r^{n-1}.$$

Taking into account

$$\left| \frac{e^{i\alpha_1 n} - e^{i\alpha_2 n}}{e^{i\alpha_1} - e^{i\alpha_2}} \right| \leq \sum_{m=0}^{n-1} \left| e^{i\alpha_1 m}\, e^{i\alpha_2 (n-m-1)} \right| = n,$$

we obtain the required estimate (2.11). To get (2.12), one can operate analogously. The lemma is proved. ■

Note if function $\psi(z)$ is linear, then $Q(r) = q(r) = 0$ for every r.

Let us define numbers R^1_{z0} and r_{z0} as follows. If $Q(r) < 1$ for all r from $[0, R_{z0}]$, then $R^1_{z0} = R_{z0}$, otherwise, by R^1_{z0} will be denoted the point at which function Q takes value 1. Then $r_{z0} \in [0,\, R^1_{z0}]$ is a point at which function $r[1 - q(r)]$ reaches its maximum,

$$r_{z0}[1 - q(r_{z0})] = \max \left(r[1 - q(r)] \right), \qquad r \in [0,\, R^1_{z0}].$$

Proposition 2.1. *The function sequence*

$$\Psi_0(w) \equiv z_0, \qquad \Psi_{n+1}(w) \equiv \Psi_n(w) - \frac{\psi(\Psi_n(w)) - w}{\psi'(z_0)}, \qquad n = 0, 1, 2, \ldots \tag{2.13}$$

converges to function $z = \Psi(w)$, the inverse of $\psi(z)$, uniformly inside disk $\mathcal{D}_{w0}(r_{w0})$, where

$$r_{w0} = r_{z0}[1 - q(r_{z0})]\,|\psi'(z_0)|.$$

If $|w - w_0| = d < r_{w0}$, then the convergence rate of successive approximations (2.13) is estimated as follows

$$|\Psi(w) - \Psi_n(w)| \le \frac{Q_0^n}{1 - Q_0}\,\frac{d}{|\psi'(z_0)|}, \tag{2.14}$$

for

$$Q_0 = Q(d^*) < 1, \qquad d^* = \frac{d}{[1 - q(r_{z0})]|\psi'(z_0)|}. \tag{2.15}$$

□ Let d be an arbitrary number from the interval $]0, r_{w0}[$. We define quantity d^* by relation (2.15); it is clear that $0 < d^* < r_{z0}$. Now we will prove that for w such that $|w - w_0| = d$, all the approximations $\Psi_n(w)$ belong to disk $\mathcal{D}_{z0}(d^*)$. As to $\Psi_0(w) \equiv z_0$, it belongs obviously to this disk; let $\Psi_n(w)$ be known for a certain n to lie into this disk, we show that $\Psi_{n+1}(w)$ is from $\mathcal{D}_{z0}(d^*)$ too. Indeed, denoting $z = \Psi_n(w)$, we write

$$|\Psi_{n+1}(w) - z_0| = \left|z - \frac{\psi(z) - w}{\psi'(z_0)} - z_0\right| \le \left|z - z_0 - \frac{\psi(z) - \psi(z_0)}{\psi'(z_0)}\right| + \left|\frac{w_0 - w}{\Psi'(z_0)}\right|;$$

the former summand is, by definition of q, not greater than $q(d^*)|z - z_0|$ and, thus, it is less than $q(d^*)d^*$, the latter summand equals to $d/|\psi'(z_0)| = [1 - q(r_{z0})]d^*$, that follows

$$|\Psi_{n+1}(w) - z_0| < q(d^*)d^* + [1 - q(r_{z0})]d^* < d^*.$$

By virtue of the principle of mathematical induction, we obtain that $\Psi_n(w) \in \mathcal{D}_{z0}(d^*)$, $n = 0, 1, 2 \ldots$

Further, according to the definition of function $Q(r)$, the following inequality takes place

$$\left|z_1 - z_2 - \frac{\psi(z_1) - \psi(z_2)}{\psi'(z_0)}\right| < Q(d^*)|z_1 - z_2|, \qquad Q(d^*) < 1,$$

for every z_1, z_2 from $\mathcal{D}_{z0}(d^*)$. We can place $z_1 = \Psi_n(w)$, $z_2 = \Psi_{n-1}(w)$ in this formula, what, taking into account (2.13), follows

$$|\Psi_{n+1}(w) - \Psi_n(w)| \le Q(d^*)|\Psi_n(w) - \Psi_{n-1}(w)|,$$

and for every $N \ge n$

$$\sum_{m=n}^{N} |\Psi_{m+1}(w) - \Psi_m(w)| < \frac{[Q(d^*)]^n}{1 - Q(d^*)}\left|\frac{\psi(z_0) - w}{\psi'(z_0)}\right|.$$

This means sequence $\Psi_n(w)$ converges uniformly in every closed disk $\mathcal{D}_{w0}(d)$, $d < r_{w0}$ to function $\Psi(w)$, with $\psi \circ \Psi(w) = w$. Tending the upper limit N in the last inequality to the infinity, we obtain estimate (2.14). According to Weierstrass' theorem [18], $\Psi(w)$ constructed in this manner is a holomorphic function in the open disk $\mathcal{D}_{w0}(r_{w0})$. The proposition is proved. ■

3. Optimum Electrode Shape

3.1. Formulation of Problem. We remind domain g from the class under consideration is uniquely defined by arc Γ (the electrode bottom shape) and parameters a (the electrode half-width) and b (the altitude of electrode sides over $(A'A)$-axis).

We will consider the following statement: let a certain constant potential $\phi_0 > 0$ be preset at the whole electrode surface $(A\,B\,B'\,A')$ with the electrode width $2a$ being given. It is required to find bottom Γ and parameter b that provide a constant magnitude of electric intensity along the whole bottom, equal to the preset value E_0.

The arc Γ shape satisfying this statement is an optimum one in the sense that any other arc with the same endpoints contains a some place where the magnitude of electric intensity exceeds E_0.

To be specific, we shall solve the following boundary value problem in domain g with free boundary arc Γ

$$\begin{aligned} \Delta\Phi(w) &= 0, & w &\in g, & (3.1)\\ \Phi(w) &= 0, & w &\in (A'\,A), & (3.2)\\ \Phi(w) &= \phi_0, & w &\in (A\,B\,B'\,A'), & (3.3)\\ |\mathrm{grad}\,\Phi(w)| &= E_0, & w &\in \Gamma = (B\,B'), & (3.4) \end{aligned}$$

quantities a, ϕ_0 and E_0 are assumed to be preset.

An explicit analytical expression for Γ as a complex-valued parametrical function $\Gamma(t) = \Gamma_1(t) + i\Gamma_2(t)$ will be found below in Subsect. 3.3. The corresponding potential $\Phi(w)$ can be obtained by means of the method presented in Sect. 2.

3.2. Preliminary Notes. The problem (3.1)-(3.4) of constructing an optimal rounding curve Γ is solved below by the hodograph method [21]- [24].

Problem (3.1)-(3.4) is reduced to the question of the existence of such a conformal mapping $\zeta = \Psi(w)$ of domain g onto strip

$$\{\zeta : 0 < \mathrm{Im}\,\zeta < \phi_0\}$$

that points A and A' would be mapped into the right and left infinities of the strip, respectively, and

$$|\Psi'(w)| = E_0$$

for every point w of the unknown arc Γ. Then function $\Phi(w)$ which corresponds to statement (3.1)-(3.4) will be expressed as follows

$$\Phi(w) = \mathrm{Im}\,\Psi(w).$$

Function $\Psi(w)$ is called complex potential; its derivative is well known to be related to field intensity $E(w)$ by the formula

$$\bar{E}(w) = i\,\Psi'(w),$$

where $E(w)$ is the complex conjugate of $E(w)$. A general representation of function $\Psi(w)$ for problem (1.1)-(1.5) will be given in Subsect. 4.1.

Existence and uniqueness of arc Γ which would bring into being the mapping, can be proved with the help of the general variation principle [21], [25]. Note arc Γ and, therefore, the whole domain g are, in our case, symmetrical with respect to v-axis. The segment of this axis joining the middle point O' of Γ and the origin of coordinates O divides g into two symmetrical subdomains $g^+ = \{w \in g : u > 0\}$ and $g^- = \{w \in g : u < 0\}$, see Fig. 2.

Note conformal mapping $\zeta = \Psi(w)$ has one degree of freedom that we will fix with condition $\Psi(O) = 0$. It is not difficult to demonstrate, using the reflection principle already mentioned, that this unique mapping transforms domain g^+ into half-strip

$$\{\zeta : 0 < \operatorname{Re}\zeta; \quad 0 < \operatorname{Im}\zeta < \phi_0\} \tag{3.5}$$

with the correspondence of boundary points

$$\Psi(O) = 0, \qquad \Psi(A) = \infty, \qquad \Psi(O') = i\,\phi_0. \tag{3.6}$$

It is obvious that $(A\,B\,B'\,A')$ is mapped into line $\operatorname{Im}\zeta = \phi_0$, and real axis $(A'\,A)$ is mapped into real axis $\operatorname{Im}\zeta = 0$. The image of point B is a certain point $\beta + i\phi_0$; positive quantity β which depends on the problem parameters will be determined below. Along with normalization (3.6) for Ψ we shall use another correspondence which follows from (3.6):

$$\Psi(A) = \infty, \qquad \Psi(B) = \beta + i\phi_0, \qquad \Psi(O') = i\phi_0 \tag{3.7}$$

with unknown β.

Let us consider now a conformal mapping conditioned by the derivative of function $\Psi(w)$. The domain g^+ image under the mapping $W = \Psi'(w)$ is readily verified to be a sector

$$\{W : |W| < E_0; \quad -\pi/2 < \arg W < 0\} \tag{3.8}$$

in the complex plane W (so-called hodograph plane); boundary points are transformed as follows:

$$\Psi'(A) = 0, \qquad \Psi'(B) = -iE_0, \qquad \Psi'(O') = E_0, \tag{3.9}$$

and $\Psi'(0)$ is a real number from interval $]0,\, E_0[$.

We introduce now function $W = \Omega(\zeta)$ which accomplishes a conformal mapping of half-strip (3.5) from complex potential plane ζ onto sector (3.8) in the hodograph plane with the following normalization

$$\Omega(\infty) = 0, \qquad \Omega(\beta + i\phi_0) = -iE_0, \qquad \Omega(i\phi_0) = E_0. \tag{3.10}$$

Then, from definition of $\Psi(w)$ and $\Omega(\zeta)$, and in view of correspondences (3.7), (3.9), (3.10), we get $\Psi'(w) = \Omega \circ \Psi(w)$, whence it follows, owing to $\zeta = \Psi(w)$, the equality

$$dw = \frac{d\zeta}{\Omega(\zeta)}. \tag{3.11}$$

Integrating right-hand side of (3.11) along straight segment $[t+i\phi_0,\ \beta+i\phi_0]$, when $0 \le t \le \beta$, in the complex potential plane and the left-hand side along the corresponding part of arc Γ in initial plane w, we obtain the expression for the required arc in terms of parametrical function

$$\Gamma(t) = a + ib - \int\limits_t^\beta \frac{dx}{\Omega(x+i\phi_0)}, \qquad t \in [0,\ \beta], \tag{3.12}$$

where the parameter values $t = 0$ and $t = \beta$ correspond to points O' and B of arc Γ.

Therefore, the mapping which satisfies relations (3.10) remains to be found, then everyone can determine the arc sought, applying formula (3.12). Unknown parameters b and β will be found in Subsect. 3.3.

3.3. Representation of Optimum Bottom. Mapping $\Omega(\zeta)$ is obtained effortlessly

$$\Omega(\zeta) = E_0 \left(\sqrt{1 + \frac{\cosh^2 \dfrac{\pi\zeta}{2\phi_0}}{\sinh^2 \dfrac{\pi\beta}{2\phi_0}}} - \frac{\cosh \dfrac{\pi\zeta}{2\phi_0}}{\sinh \dfrac{\pi\beta}{2\phi_0}} \right). \tag{3.13}$$

Replacing ζ by $x + i\phi_0$ in this relation and substituting the result into (3.12), we find the following formula

$$\Gamma(t) = a + ib - \frac{1}{E_0} \int\limits_t^\beta \left(\sqrt{1 - \frac{\sinh^2 \dfrac{\pi x}{2\phi_0}}{\sinh^2 \dfrac{\pi\beta}{2\phi_0}}} + i \frac{\sinh \dfrac{\pi x}{2\phi_0}}{\sinh \dfrac{\pi\beta}{2\phi_0}} \right) dx. \tag{3.14}$$

The imaginary part of (3.14) can be written via elemental functions while the real part, by means of substitution of integration variable $x = (2\phi_0/\pi)\,\mathrm{arsinh}\, x'$, can be reduced to expression in terms of the incomplete elliptic integrals [26]:

$$\Gamma(t) = a + ib - \frac{2\phi_0}{\pi E_0 \tau} \left[E_1(\theta, \tau) - E_2(\theta, \tau) + i \left(\sqrt{1-\tau^2} \cosh \frac{\pi t}{2\phi_0} - 1 \right) \right]. \tag{3.15}$$

The following designations are accepted in the last formula:

$$E_1(\theta, \tau) = \int\limits_0^\theta \frac{dx}{\sqrt{1 - \tau^2 \sin^2 x}}, \qquad E_2(\theta, \tau) = \int\limits_0^\theta \sqrt{1 - \tau^2 \sin^2 x dx}$$

are the first and second type incomplete elliptic integrals (since letters F and E which are their commonly accepted designations are already engaged in our paper, we have to use unconventional designations), and

$$\theta = \theta(t) = \arccos \frac{\sinh \dfrac{\pi t}{2\phi_0}}{\sinh \dfrac{\pi\beta}{2\phi_0}}, \qquad \tau = \tanh \frac{\pi\beta}{2\phi_0}. \tag{3.16}$$

Note $0 \le \theta \le \pi/2$, $\quad 0 \le \tau < 1$, and expression $E_1(\theta,\tau) - E_2(\theta,\tau)$ can be represented in the form of series

$$E_1(\theta,\tau) - E_2(\theta,\tau) = \sum_{n=1}^{\infty} \frac{(1/2)_{n-1}}{(n-1)!} I_n(\theta)\tau^{2n}, \tag{3.17}$$

here and below $(\alpha)_m$ denotes the Pochhammer's symbol [27]; for values

$$I_n(\theta) \equiv \int_t^{\theta} \sin^{2n} x \, dx$$

a recurrence formula is valid

$$I_0(\theta) = \theta, \quad I_n(\theta) = -\frac{\sin^{2n-1}\theta\cos\theta}{2n} + \left(1 - \frac{1}{2n}\right) I_{n-1}(\theta). \tag{3.18}$$

Series (3.17) converges for every θ, owing to the fact that τ is always less than 1 and quantities $I_n(\theta)$ can be estimated as follows

$$I_n(\theta) \le I_n(\pi/2) = \frac{\pi}{2}\frac{(1/2)_n}{n!} = O\left(n^{-1/2}\right), \qquad n \to \infty.$$

Thus, we have found function (3.15) which describes sought arc Γ, and it remains only to determine quantities β and b in this expression. For this purpose we consider two representations for the middle point O' of arc Γ. On the one hand, as it has been remarked, $O' = \Gamma(0)$, hence, placing $t = 0$ in (3.12), we find

$$O' = a + ib - \int_0^{\beta} \frac{dx}{\Omega(x + i\phi_0)},$$

On the other hand, O' can be got by integrating the left-hand side of (3.11) along the straight segment $[O,\, O']$ of v-axis (while the right-hand side is integrated along the corresponding straight segment $[0,\, i\phi_0]$),

$$O' = \frac{i}{E_0}\int_0^{\phi_0} \frac{dx}{\Omega(ix)}.$$

Inserting already known function Ω defined by formula (3.13) into the two last relations, separating real and imaginary parts and equating them, we obtain two equations which define the required quantities completely:

$$a - \frac{1}{E_0}\int_0^{\beta} \sqrt{1 - \frac{\sinh^2 \dfrac{\pi x}{2\phi_0}}{\sinh^2 \dfrac{\pi\beta}{2\phi_0}}}\, dx = 0, \tag{3.19}$$

and

$$\frac{1}{E_0}\int_0^{\phi_0}\left(\sqrt{1-\frac{\cos^2\dfrac{\pi x}{2\phi_0}}{\sinh^2\dfrac{\pi\beta}{2\phi_0}}}+\frac{\cos\dfrac{\pi x}{2\phi_0}}{\sinh\dfrac{\pi\beta}{2\phi_0}}\right)dx=b-\frac{2\phi_0}{\pi E_0}\tanh\frac{\pi\beta}{4\phi_0}. \tag{3.20}$$

Equation (3.19) is an implicit expression for β which is modified in terms of hypergeometrical function $F \equiv_2 F_1$, see e.g. [26], [27], as follows:

$$\frac{\tau}{2}F(1/2,3/2;2;\tau^2)=\frac{aE_0}{\phi_0}, \tag{3.21}$$

where τ has been defined in formula (3.16).

In view of equation (3.21) we can draw a conclusion that auxiliary parameter $\tau\in(0,1)$ depends only on dimensionless quantity $\lambda=aE_0/\phi_0$ that can be varied in the range $(0,\infty)$, and the required parameter β depends on λ and ϕ_0.

We remark that for great values of $\lambda = aE_0/\phi_0$ it would be convenient for finding parameter β to use the following equation, equivalent to (3.21),

$$\frac{\sqrt{1-\epsilon}}{\pi}\sum_{n=0}^{\infty}\frac{(1/2)_n(3/2)_n}{(n!)^2}[\mu_n-\ln\epsilon]\epsilon^n=\frac{aE_0}{\phi_0}; \tag{3.22}$$

here small parameter ϵ is related to β as follows

$$\epsilon=1-\tau^2=\cosh^{-2}\frac{\pi\beta}{2\phi_0}, \tag{3.23}$$

and the recurrence formula for coefficients μ_n is valid

$$\mu_0=2(\ln 4-1),\qquad \mu_n=\mu_{n-1}-\frac{2}{n(2n-1)(2n+1)}.$$

Using expressions (3.21) and (3.22), we find asymptotic behavior of parameter β in the limiting cases $\lambda\to 0$ and $\lambda\to\infty$ for every fixed ϕ_0:

$$\beta=\frac{4\phi_0}{\pi}\lambda+O(\lambda^2),\qquad \lambda\to 0$$

and

$$\beta=\phi_0\lambda-\frac{2(\ln 8-1)}{\pi}+O\left(\lambda e^{-\pi\lambda}\right),\qquad \lambda\to\infty.$$

Besides, it is not difficult to demonstrate that β as a function of λ (for a fixed ϕ_0) increases monotonically while its derivative $\frac{d\beta}{d\lambda}$ will decrease monotonically from $4\phi_0/\pi$ when $\lambda=0$, down to ϕ_0 when $\lambda\to\infty$. The graph of β as a function of aE_0/ϕ_0 for different values of potential ϕ_0 is presented in Fig. 4.

Finally, we are coming to a determination of parameter b. We return to equation (3.20) where all the quantities, except for b, have already been found. Inserting (3.13) with $\zeta = ix$ into (3.20), we obtain

$$\frac{1}{E_0}\int\limits_0^{\phi_0}\left(\sqrt{1-\frac{\cos^2\dfrac{\pi x}{2\phi_0}}{\sinh^2\dfrac{\pi\beta}{2\phi_0}}}+\frac{\cos\dfrac{\pi x}{2\phi_0}}{\sinh\dfrac{\pi\beta}{2\phi_0}}\right)dx = b-\frac{2\phi_0}{\pi E_0}\tanh\frac{\pi\beta}{4\phi_0}.$$

We rearrange the last formula and have finally a sought expression for b

$$b=\frac{\phi_0}{E_0\tau}\left(F(-1/2,1/2,1;1-\tau^2)+\pi/2\right).$$

Thus, all the sought parameters of domain g which satisfy the statement (3.1)-(3.4) are determined completely. The next subsection deals with a study of characteristics of the obtained optimum bottom. As to solution $\Phi(w)$ of problem (3.1)-(3.4), it can be constructed according to the method outlined above, in Sect. 2. A graphical representation of this solution will be given below for a certain set of the domain g parameters, see Subsect. 4.2, Example 3.

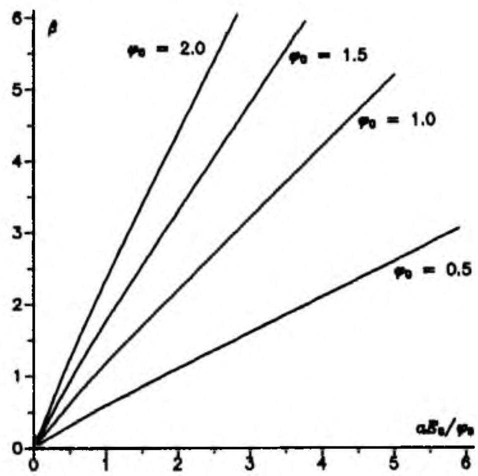

Fig. 4. Parameter β versus aE_0/ϕ_0 for different ϕ_0.

3.4. Electrode Curvature and Field Distribution. It is of interest to find such characteristics of the optimum electrode form as its bottom curvature and electric field distribution at side facings (the field magnitude is constant at the electrode bottom, according to condition (3.4)).

We will search for curvature K and field magnitude $|E| = |\mathrm{grad}\,\Phi|$ as functions of coordinate $s\in(-\infty,\infty)$, which is the arc length measured from point O' along electrode contour with values $s=-\infty$, $s=0$, $s=\infty$ corresponding to points A', O' and A. We will find first a relation between parameter t used in representation of arc Γ (see the previous subsection) and new coordinate s. To do this, we will use equality (3.11): integrating absolute value of its left- and right-hand sides along the corresponding straight segments $[0,s]$ and

$[i\phi_0, t+i\phi_0]$, we obtain just the arc length. Further, taking into account expression (3.12), we find

$$s(t) = \int_0^t \frac{dx}{|\Omega(x+i\phi_0)|}, \tag{3.24}$$

where

$$\begin{aligned} |\Omega(x+i\phi_0)| &= \frac{1}{E_0}, && \text{if } |x| \le \beta, \\ |\Omega(x+i\phi_0)| &= \frac{1}{E_0}\left(\sqrt{\frac{\sinh^2 \dfrac{\pi x}{2\phi_0}}{\sinh^2 \dfrac{\pi\beta}{2\phi_0}} - 1} + \frac{\sinh \dfrac{\pi x}{2\phi_0}}{\sinh \dfrac{\pi\beta}{2\phi_0}}\right), && \text{if } |x| > \beta. \end{aligned}$$

Taking integral (3.24), we come to the final expression for $s(t)$: if $|t| \le \beta$,

$$s(t) = \frac{t}{E_0}, \tag{3.25}$$

while if $|t| > \beta$,

$$s(t) = \frac{\beta}{E_0} + \frac{2\phi_0}{\pi}\sqrt{\frac{\epsilon}{1-\epsilon}}\left[\cosh\frac{\pi t}{2\phi_0}\left(1+\sqrt{1-\frac{\epsilon^{-1}-1}{\sinh^2\dfrac{\pi t}{2\phi_0}}}\right) - \frac{1+E_2(\theta(t),\sqrt{\epsilon})}{\sqrt{\epsilon}}\right]. \tag{3.26}$$

Note values $|t| \le \beta$ and, therefore, $|s| < \beta/E_0$ correspond to points at the electrode bottom Γ; values $t < -\beta$ (i.e. $s < -\beta/E_0$) and $t > \beta$ (i.e. $s > \beta/E_0$) correspond to the left $(B'\,A')$ and right $(A\,B)$ electrode side facings. We remind that quantity $\epsilon < 1$, the solution of equation (3.22), is related to β by formula (3.23), parameter $\theta(t)$ has been introduced by equality (3.16). Besides, the second type elliptic integral $E_2(\theta,\sqrt{\epsilon})$, see Subsect. 3.3., can be expanded into a Taylor series in terms of powers of small parameter ϵ:

$$E_2(\theta,\sqrt{\epsilon}) = \sum_{n=0}^{\infty}\frac{(-1/2)_n}{n!} I_n(\theta)\epsilon^n,$$

where factors $I_n(\theta)$ are subject to recurrence formula (3.18).

We remark also that expression (3.25) for arc coordinate s follows, in particular, the whole length of the electrode bottom Γ equals to $2\beta/E_0$.

Now we start finding curvature $K(s)$ and field magnitude $|E|$ at the electrode contour. The curvature of the electrode side facings is obviously equal to zero, i.e. $K(s) = 0$ for $|s| > \beta/E_0$. Curvature $\hat{K}(t)$ of an arc Γ set parametrically $\Gamma(t) = \Gamma_1(t) + i\Gamma_2(t)$ is expressed by the well-known formula

$$\hat{K}(t) = \frac{|\Gamma_1'(t)\Gamma_2''(t) - \Gamma_1''(t)\Gamma_2'(t)|}{|\Gamma'(t)|^3}.$$

Curvature of the same arc parametrized with the coordinate s will be found by the relation

$$K(s) = \hat{K} \circ t(s),$$

where, in our case, $t(s) = E_0 s$ for $|s| < \beta/E_0$. Using formula (3.14) for $\Gamma(t)$, we obtain

$$K(s) = \frac{\pi E_0}{2\phi_0} \cosh \frac{\pi E_0 s}{2\phi_0} \left(\sinh^2 \frac{\pi\beta}{2\phi_0} - \sinh^2 \frac{\pi E_0 s}{2\phi_0} \right)^{-1/2}, \quad |s| < \beta/E_0. \tag{3.27}$$

Here we observe the singular behavior of the bottom curvature at the points of the junction with side facings. Namely, it is not difficult to verify that when coordinate s approaches β/E_0 from the left (i.e. $w \to B$ as $w \in \Gamma$), the following asymptotic relation is valid

$$K(s) = \frac{1}{2} \left(\frac{\pi E_0}{\phi_0} \operatorname{cotanh} \frac{\pi\beta}{2\phi_0} \right)^{1/2} \left(\frac{\beta}{E_0} - s \right)^{-1/2} + O\left(\frac{\beta}{E_0} - s \right)^{1/2}, \ s \to \beta/E_0 - 0;$$

and a similar relation takes place for $s \to -\beta/E_0 + 0$ (i.e. $w \to B'$ as $w \in \Gamma$).

To obtain an expression for field intensity magnitude $|\hat{E}(t)|$ along the electrode contour at a point corresponding to a certain value of parameter t, we observe first that this magnitude coincides with the modulus of mapping $\Omega(\zeta)$ when $\zeta = t + i\phi_0$, according to Subsect. 3.2. Using expression (3.13) for Ω, we find

$$|\hat{E}(t)| = E_0, \qquad \text{if } |t| \le \beta; \tag{3.28}$$

$$|\hat{E}(t)| = E_0 \left(\frac{\sinh \frac{\pi t}{2\phi_0}}{\sinh \frac{\pi\beta}{2\phi_0}} - \sqrt{\frac{\sinh^2 \frac{\pi t}{2\phi_0}}{\sinh^2 \frac{\pi\beta}{2\phi_0}} - 1} \right), \quad \text{if } |t| > \beta. \tag{3.29}$$

One can determine field magnitude at a point which lies in the electrode contour at distance s (measured along the contour) from point O'. For this purpose, parameter t in formula (3.29) should be substituted by arc coordinate s in accordance with relation (3.26).

Figures 5 and 6 give the graphs of the electrode curvature (3.27) and field magnitude (3.28), (3.29) at the electrode contour versus arc coordinate s.

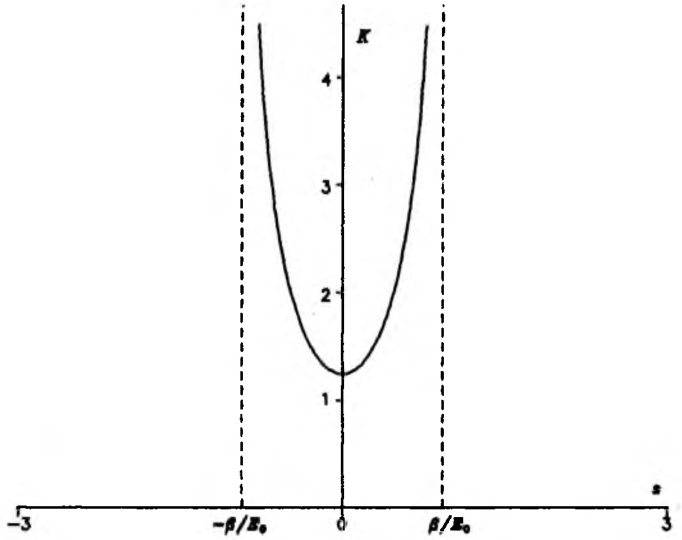

Fig. 5. Curvature K of optimum electrode contour versus arc coordinate s.

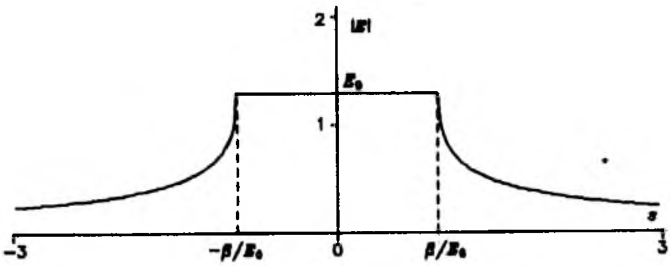

Fig. 6. Field magnitude $|E|$ at the contour versus arc coordinate s.

4. Main Field Characteristics

4.1. Analytical Representations. The multipole method enables also to obtain analytical representations for any derivative of the boundary value problem (1.1)-(1.5) solution $\Phi(w)$ as well as for the harmonic conjugate of this solution. Thus, function $\tilde{\Phi}(w)$, such that function $\Psi(w) = \tilde{\Phi}(w) + i\Phi(w)$ is holomorphic and $\tilde{\Phi}(0) = 0$ (this function is unique in view of simple connectivity of domain g [2, 3]), can be expressed similarly as (2.1):

$$\tilde{\Phi}(w) = \tilde{U_0}(w) + \tilde{U}(w),$$

here the formulae for $\tilde{U_0}$ and $\tilde{U}(w)$, analogous to (2.2) and (2.6), take place

$$\tilde{U_0}(w) = \frac{1}{\pi}\left(\phi_1 \ln|1 + F(w)| - \phi_2 \ln|1 - F(w)|\right),$$

and

$$\tilde{U}(w) = \lim_{K\to\infty} \sum_{p=1}^{K} a_p^K \tilde{\Omega}_p(w), \qquad \tilde{\Omega}_p(w) = \mathrm{Re}\,[F(w)]^p;$$

the last limit exists for every $w \in g$. Summing the functions $\tilde{\Phi}(w)$ and $i\Phi(w)$, we find expression for complex potential

$$\Psi(w) = \frac{1}{\pi}\left(\phi_1 \ln[1+F(w)] - \phi_2 \ln[1-F(w)]\right) + \lim_{K\to\infty} \sum_{p=1}^{K} a_p^K\,[F(w)]^p. \tag{4.1}$$

Furthermore, the derivative of holomorphic function $\Psi(w)$ with respect to complex variable w can be readily obtained because the multipole method admits differentiation of any order. Differentiating (4.1), we get

$$\Psi'(w) = F'(w)\left(\frac{1}{\pi}\left[\frac{\phi_1}{1+F(w)} + \frac{\phi_2}{1-F(w)}\right] + \lim_{K\to\infty} \sum_{p=1}^{K} p\,a_p^K [F(w)]^{p-1}\right).$$

Modulus of the last function coincides with the field intensity magnitude $|E(w)| = |\Psi'(w)|$.

4.2. Specific Implementation. The specific implementation for the obtained solution was performed for various sets of domain g parameters (quantities a, b and forms of arc Γ) and various distributions $\phi(w)$ of boundary potential. For range of ratio a/b from 0.2 to 5, for sufficiently smooth arcs Γ and distributions $\phi(w)$ chosen in accordance with physical reasons, it was sufficient to use 20 multipoles Ω_p in order to reach global relative error for field intensity E less than 10^{-3} everywhere in closed domain $\bar{g}$.

Figures 7-9 demonstrate numerical results for Examples 1–3, respectively. For these three examples are presented:

a) equipotentials $\{w : \Phi(w) = \mathrm{const}\}$,

b) lines of force $\{w : \tilde{\Phi}(w) = \mathrm{const}\}$,

c) lines of equal intensity magnitude $\{w : |E(w)| = \mathrm{const}\}$.

Example 1. The solution of problem (1.1)-(1.5) with the following input parameters is considered: $a = 1$, $b = 0.9$; arc Γ is specified as a graph of dependence

$$v(u) = 0.9 - 0.2(1-u^2)^{1/2}.$$

Potential distribution at arc Γ was prescribed as function of u-coordinate

$$\phi(u) = 0.75(u+0.2)^2 + 0.62;$$

note the continuity condition for the potential along the electrode contour follows: $\phi_1 = 1.1$, $\phi_2 = 1.7$.

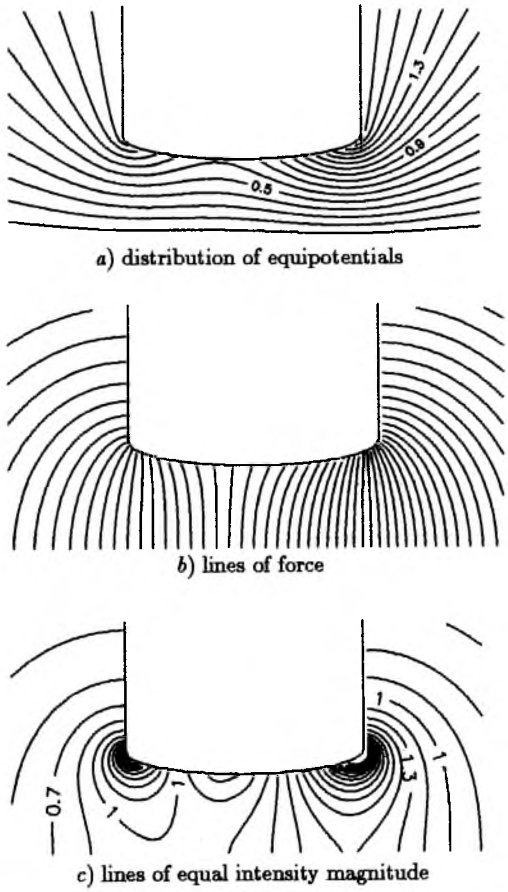

Fig. 7. Illustrations for Example 1.

Example 2. The solution of problem (1.1)-(1.5) with the following input parameters is under consideration: $a = 1$, $b = 1$; arc Γ is specified as a graph

$$v(u) = 1 - 0.4(1-u^2)^{1/4} + 0.15(1-u^2)\exp(-5u/6).$$

Potential distribution at arc Γ was prescribed as follows

$$\phi(u) = -0.175u^3 + 0.525u + 0.35 + 1.35\cos^4\frac{\pi}{2}u$$

and $\phi_1 = 0.8$, $\phi_2 = 1.5$.

Example 3. The solution of optimum problem (3.1)-(3.4) with parameters $a = 0.8$, $\phi_0 = 1$, $E_0 = 1.3$ is under consideration.

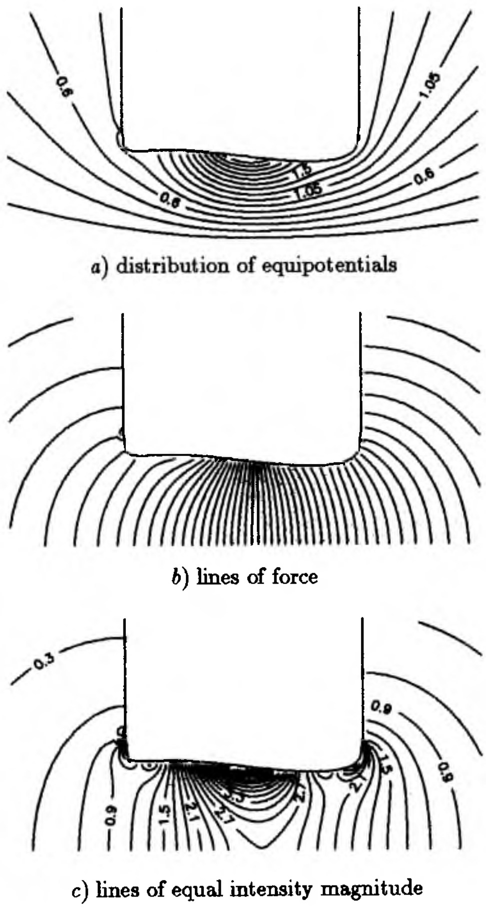

Fig. 8. Illustrations for Example 2.

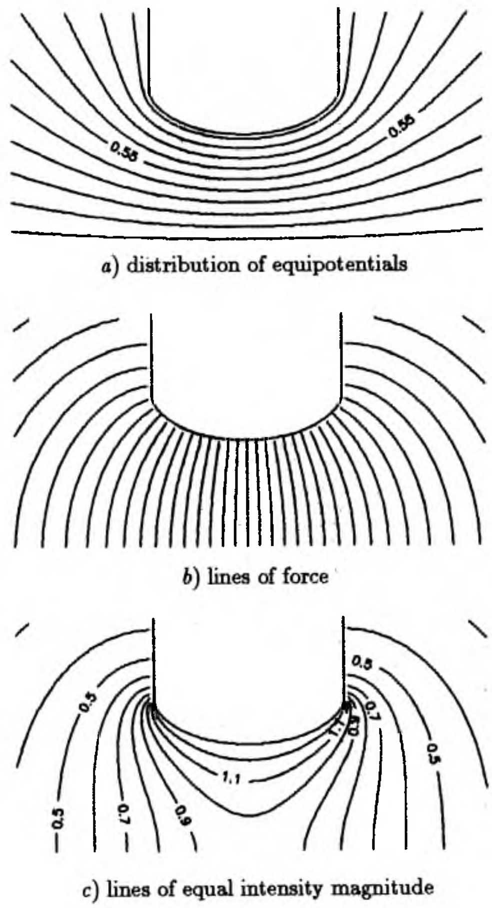

Fig. 9. Illustrations for Example 3.

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ИССЛЕДОВАНИЕ ЭЛЕКТРИЧЕСКОГО ПОЛЯ В ЛАЗЕРЕ С ПОМОЩЬЮ МЕТОДА МУЛЬТИПОЛЕЙ

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Аннотация. Работа посвящена модификации метода мультиполей и его применению к исследованию электрического поля в лазере специальной конструкции. Найдена оптимальная форма электродов в этом приборе. Для основных характеристик поля найдены явные формулы. Полученные численные результаты подтверждают высокую эффективность и точность используемого метода.

Ключевые слова: краевые задачи, метод мультиполей, расчет электрического поля в лазере.