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INITIAL-BOUNDARY VALUE PROBLEMS FOR TWO DIMENSIONAL KAWAHARA EQUATION

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Abstract. In this paper we study initial-boundary value problems on a half-strip with different types of boundary conditions for the generalized two-dimensional Kawahara equation with nonlinearity of higher order. The solutions are considered in weighted at infinity Sobolev spaces. The use of weighted spaces is crucial for the study. We establish results on global existence and uniqueness in classes of weak and strong solutions, as well as large-time decay of week and strong solutions under small input data.

Keywords: Two-Dimensional Kawahara Equation, Solvability of the Initial Bundary Value Problem, Dissipation of Solutions at Infinity

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НАЧАЛЬНО-КРАЕВЫЕ ЗАДАЧИ ДЛЯ ДВУХМЕРНОГО УРАВНЕНИЯ КАВАХАРЫ

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Аннотация. В работе были изучены начально-краевые задачи с разными типами граничных условий для двухмерной модификации уравнения Кавахары с высокой нелинейностью. Уравнение рассматривалось на полу-полосе конечной ширины. Были получены результаты о существовании и единственности сильных и слабых решений поставленных задач и о диссипации решений на бесконечности. Решения рассматривались в весовых пространствах Соболева.

Ключевые слова: двухмерное уравнение Кавахары, разрешимость начально-краевой задачи, диссипация решений на бесконечности

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1. Introduction. In the following paper we consider initial-boundary value problems for two dimensional Kawahara equation:

$$u_t - (u_{xxxx} + u_{yyyy})_x + b(u_{xx} + u_{yy})_x + au_x + (g(u))_x = f(t, x, y), \tag{1}$$

posed on a domain $\Pi_T^+ = (0, T) \times \Sigma_+$, where $\Sigma_+ = \mathbb{R}_+ \times (0, L) = \{(x, y) : x > 0, 0 < y < L\}$ is a half-strip of a given width L and $T > 0$ is arbitrary for equation (1), with the initial condition:

$$u(0, x, y) = u_0(x, y), \quad (x, y) \in \Sigma_+, \tag{2}$$

and boundary conditions:

$$u(t,0,y) = u_x(t,0,y) = 0, \quad (t,y) \in B_T = (0,T) \times (0,L), \tag{3}$$

and boundary conditions for $(t,x) \in \Omega_{T,+} = (0,T) \times \mathbb{R}_+$ of one of the following two types:

$$\begin{aligned} a).\ u(t,x,0) = u(t,x,L) = u_{yy}(t,x,0) = u_{yy}(t,x,L) = 0, \\ b).\ u_y(t,x,0) = u_y(t,x,L) = u_{yyy}(t,x,0) = u_{yyy}(t,x,L) = 0. \end{aligned} \tag{4}$$

The assumptions on the function $g(u)$ are specified later; a, b are arbitrary real constants. Results on global existence are bases on estimates which are the analogues of the following conservation laws for the initial value problem

$$\iint_{\mathbb{R}^2} u^2 dxdy = const, \quad \iint_{\mathbb{R}^2} (u_{xx}^2 + u_{yy}^2 + bu_x^2 + bu_y^2 - 2g^*(u))dxdy = const,$$

where

$$g^*(u) \equiv \int_0^u g(\theta)d\theta.$$

The equation (1) is a two-dimensional version of the Kawahara equation:

$$u_t - u_{xxxxx} + bu_{xxx} + au_x + uu_x = 0.$$

Obtained in [10], it describes the propagation of long nonlinear waves in weakly dispersive media. Kawahara equation (also known as fifth-order Korteweg–de Vries equation) is a modification of the well-known Korteweg–de Vries equation (KdV):

$$u_t + u_{xxx} + au_x + uu_x = 0,$$

which also has the two-dimensional form, so called Zakharov – Kuznetsov equation:

$$u_t + u_{xxx} + u_{xyy} + au_x + u^2 u_x = 0.$$

In this paper we establish global existence and uniqueness of solutions to initial-boundary value problems (1) – (4) and large-time decay under small input data.

Through the years there was a wide variety of investigations dedicated to various aspects of the Kawahara equation and some of its modifications. The initial value problem and initial-boundary value problems are considered, for instance, in [5, 11, 1, 9]. However, two-dimensional modifications of Kawahara equation are studied considerably less. Kawahara equation has a another two-dimensional modification known as Kawahara – Zakharov – Kuznetsov:

$$u_t - u_{xxxxx} + u_{xxx} + u_{xyy} + au_x + uu_x = 0.$$

For the first time an initial-boundary value problem for this equation was considered in [12]. The author obtained global existence, uniqueness of regular solutions and large-time decay for the small initial data. Those results were extended for the three-dimensional case of the Kawahara equation in [13]. Recently, in [14] author studied smoothness properties of solutions of a two-dimensional Kawahara equation.

Our methods are similar to those given in [3], where the author studied the initial-boundary value problems for the Kawahara – Zakharov – Kuznetsov equation on a half-strip. Previously, the author also obtained similar results for Zakharov – Kuznetsov equation in [6, 7, 8]. However, in our case we studied a different form of two-dimensional Kawahara equation given by (1).

Introduce function spaces $\widetilde{H}_+^k$ taking into account boundary conditions (4). For any multi-index $\nu = (\nu_1, \nu_2)$, let $\partial^\nu = \partial_x^{\nu_1}\partial_y^{\nu_2}$ and $\widetilde{H}_+^0 = L_{2,+}$ for $k \geq 1$ the space $\widetilde{H}_+^k$ consists of functions $\varphi(x)$ such that $\partial^\nu \varphi \in L_{2,+}$ if $\nu_1 + \nu_2 \leq k$ and in case (a)

$$\partial_y^{2m}\varphi\big|_{y=0} = \partial_y^{2m}\varphi\big|_{y=L} = 0, \quad \forall m \in [0, k/2),$$

and in case (b)

$$\partial_y^{2m+1}\varphi\big|_{y=0} = \partial_y^{2m+1}\varphi\big|_{y=L} = 0, \quad \forall m \in [0, (k-1)/2).$$

Now, let us give the definition of the admissible weight function.

Definition 1.1. *The function $\psi(x)$ is called admissible weight function if φ is an infinitely smooth positive function on $\overline{\mathbb{R}}_+$, such that for each $j \in \mathbb{N}$ and $\forall x \geq 0$*

$$|\psi^{(j)}(x)| \leq c(j)\psi(x).$$

Introduce the following

$$\lambda^+(u;T) = \sup_{x_0 \geq 0} \int_0^T \int_{x_0}^{x_0+1} \int_0^L u^2 dydxdt. \tag{5}$$

We construct solutions to the considered problems in space $X_\omega^{k,\psi(x)}(\Pi_T^+)$ for two cases for $k = 0$ (weak solutions), $k = 2$ (strong solutions) and for admissible weight $\psi(x)$, such that $\psi'(x)$ are also admissible weight functions, consisting of functions $u(t,x,y)$, such that

$$u \in C_\omega([0,T]; \widetilde{H}_+^{k,\psi(x)}) \cap L_2(0,T; \widetilde{H}_+^{k+2,\psi'(x)}).$$

Further, we denote $X_\omega^{0,\psi(x)}(\Pi_T^+)$ as $X_\omega^{\psi(x)}(\Pi_T^+)$. Introduce the notion of weak solutions to the considered problems, define special function spaces of smooth functions. Let $\widetilde{S}(\overline{\Sigma}_+)$ be a space of infinitely smooth on $\overline{\Sigma}_+$ function $\varphi(x,y)$ such that $(1+x)^n|\partial^\alpha\varphi(x,y)| \le c(n,\alpha)$ for any n, multi-index $\alpha, (x,y) \in \overline{\Sigma}_+$ and $\partial_y^{2m}\varphi\big|_{y=0} = \partial_y^{2m}\varphi\big|_{y=L} = 0$ for case (a) and $\partial_y^{2m+1}\varphi\big|_{y=0} = \partial_y^{2m+1}\varphi\big|_{y=L} = 0$ for case (b) for any m.

Definition 1.2. *Let $u_0 \in L_{2,+}$, $f \in L_1(0,T;L_{2,+})$. The function $u \in L_\infty(0,T;L_{2,+})$ is called a weak solution of problem (1) – (4), if for any $\varphi \in C^\infty([0,T];\widetilde{S}(\overline{\Sigma}))$, such that $\varphi\big|_{t=T} = \varphi\big|_{x=0} = \varphi_x\big|_{x=0} = \varphi_{xx}\big|_{x=0} = 0$, the following relation is satisfied:*

$$\iiint_{\Pi_T^+} (u\varphi_t - u\varphi_{xxxxx} - u\varphi_{yyyyx} + bu\varphi_{xxx} + bu\varphi_{yyx} + au\varphi_x + g(u)\varphi_x + f\varphi)\,dtdxdy + \iint_{\Sigma_+} u_0\varphi\big|_{t=0}dxdy = 0. \quad (6)$$

Now let us introduce the main results. The first two theorems establish global existence and uniqueness of weak and strong solutions respectably.

Theorem 1.1. *Let $u_0 \in L_{2,+}^{\psi(x)}$, $f \in L_1(0,T;L_{2,+}^{\psi(x)})$ for certain admissible weight function $\psi(x)$, such that $\psi'(x)$ is also an admissible weight function. Let $g \in C^1(\mathbb{R})$ and for certain constants $p \in [0,4)$ and $c > 0$*

$$|g'(u)| \le c|u|^p \quad \forall u \in \mathbb{R}, \quad (7)$$

and if $p > 1$ the function ψ for certain constants n and $c > 0$ satisfies an inequality $\psi(x) \le c(1+x)^n\psi'(x)$. Then there exists a weak solution to problem (1) – (4) $u \in X_\omega^{\psi(x)}(\Pi_T^+)$; moreover $\lambda^+(u_{xx};T) + \lambda^+(u_{yy};T) < +\infty$. In addition, if $p \le 3$ in (7) and for certain positive c_0

$$(\psi'(x))^{p+1}\psi^{p-1}(x) \ge c_0 \quad \forall x \ge 0, \quad (8)$$

then this solution is unique in $X_\omega^{\psi(x)}(\Pi_T^+)$.

Remark 1.1. *The exponential weight $\psi(x) \equiv e^{2\alpha x}$ $\forall\alpha > 0$ and the power weight $\psi(x) \equiv (1+x)^{2\alpha x}$ $\forall\alpha \ge \frac{1}{4}(1+\frac{1}{p})$, $p > 0$, satisfy the hypothesis of the Theorem 1.1 (including uniqueness). If $u_0 \in L_{2,+}$, $f \in L_1(0,T;L_{2,+})$, there exists a weak solution $u \in C_\omega([0,T];L_{2,+})$, $\lambda^+(u_{xx}) + \lambda^+(u_{yy}) < +\infty$.*

Theorem 1.2. *Let $u_0 \in \widetilde{H}_+^{2,\psi(x)}$, $f \in L_2(0,T;\widetilde{H}_+^{2,\psi(x)})$ for certain admissible weight function $\psi(x)$, such that $\psi'(x)$ is also an admissible weight function, $u_0(0,y) \equiv u_{0x}(0,y) \equiv 0$. Let $g \in C^2(\mathbb{R})$ and verifies condition (8) for $p \in [0,4)$. Then there exists a strong solution to problem (1) – (4) $u \in X_\omega^{1,\psi(x)}(\Pi_T^+)$; moreover $\lambda^+(u_{xxxx};T) + \lambda^+(u_{yyyy};T) + \lambda^+(u_{xxyy};T') < +\infty$. In addition, if for certain constants $q \ge 0$ and $c > 0$*

$$|g''(u)| \le c|u|^q \quad \forall u \in \mathbb{R}, \quad (9)$$

and for certain positive c_0 and $r \in (2,4]$

$$\psi'(x)^{r-2}\psi^{rq+2}(x) \ge c_0 \quad \forall x \ge 0, \quad (10)$$

then this solution is unique in $X_\omega^{2,\psi(x)}(\Pi_T^+)$.

Remark 1.2. *The exponential weight $\psi(x) \equiv e^{2\alpha x}$ $\forall\alpha > 0$ and the power weight $\psi(x) \equiv (1+x)^{2\alpha x}$ $\forall\alpha > 0$, satisfy the hypothesis of the Theorem 1.2 (including uniqueness). If $u_0 \in \widetilde{H}_+^2$, $u_0(0,y) \equiv u_{0x}(0,y) \equiv 0$, $f \in L_2(0,T;\widetilde{H}_+^2)$, there exists a weak solution $u \in C_\omega([0,T];\widetilde{H}_+^2)$, $\lambda^+(u_{xxxx}) + \lambda^+(u_{yyyy}) + \lambda^+(u_{xxyy};T') < +\infty$.*

Next, we introduce two theorems on large-time decay of weak and strong solutions.

Theorem 1.3. *Let the function $g \in C^1(\mathbb{R})$ satisfies inequality (7) for $p \in (0,3]$. Then there exists $L_0 > 0$, $\alpha_0 > 0$ and $\epsilon_0 > 0$ such that for any $L \in (0,L_0]$, $\alpha \in (0,\alpha_0]$ and $\beta = \pi^4/(8L^4)$, such that if $u_0 \in L_{2,+}^{e^{2\alpha x}}$, $\|u_0\|_{L_{2,+}} \le \epsilon_0$, $f \equiv 0$, the corresponding unique solution $u(t,x,y)$ to problem (1) – (4) in the case a). from the space $X_\omega^{e^{2\alpha x}}(\Pi_T^+)$ $\forall T > 0$ satisfies an inequality:*

$$\|e^{\alpha x}u(t,\cdot,\cdot)\|_{L_{2,+}}^2 \le e^{-\alpha\beta t}\|e^{\alpha x}u_0\|_{L_{2,+}}^2 \quad \forall t \ge 0. \quad (11)$$

Theorem 1.4. *Let the function $g \in C^2(\mathbb{R})$ satisfies inequality (7) for $p \in [1,4]$ and inequality (9) for $q = p-1$. Then there exists $L_0 > 0$, $\alpha_0 > 0$ and $\epsilon_0 > 0$, such that for any $L \in (0,L_0]$, $\alpha \in (0,\alpha_0]$ and $\beta = \pi^4/(8L^4)$, such that if $u_0 \in \widetilde{H}^{1,e_+^{2\alpha x}}$ for $\alpha \in (0,\alpha_0]$, $\|u_0\|_{L_{2,+}} \le \epsilon_0$, $u_0(0,y) \equiv u_{x0}(0,y) \equiv 0$, $f \equiv 0$ the corresponding unique solution $u(t,x,y)$ to problem (1) – (4) in the case a). from the space $X_\omega^{1,e^{2\alpha x}}(\Pi_T^+)$,$\forall T > 0$ satisfies an inequality*

$$\|e^{\alpha x}u(t,\cdot,\cdot)\|_{\widetilde{H}_+^1}^2 \le c(\|u_0\|_{\widetilde{H}_+^{2,e^{2\alpha x}}}, \alpha, \beta)e^{-\alpha\beta t} \quad \forall t \ge 0. \quad (12)$$

2. Preparations. In this section we establish some preliminary results. First, introduce the following notations: let $\eta(x)$ be a cutoff function, η is an infinitely smooth non-decreasing function on $\mathbb{R}$ such that $\eta(x) = 0$ for $x \le 0$, $\eta(x) = 1$ for $x \ge 1$, $\eta(x) + \eta(1-x) \equiv 1$; let $S_{exp}(\overline{\Sigma}_+)$ be a space of infinitely smooth functions $\varphi(x, y)$ on $\overline{\Sigma}_+$, such that $e^{nx}|\partial^{\nu}\varphi(x, y)| \le c(n, \nu)$ for any n, multi-index ν,$(x, y) \in \overline{\Sigma}_+$; let $\widetilde{S}_{exp}(\overline{\Sigma}_+)$ be a subspace of $S_{exp}(\overline{\Sigma}_+)$, consisting of functions, on the boundaries $y = 0$,$y = L$ verifying the same conditions as in the definition of the space $\widetilde{S}_{exp}(\overline{\Sigma}_+)$. This space is dense in $\widetilde{H}^k_+$.

Further, we drop limits of integration in integrals with respect to x and y over the whole half-strip Σ_+ and and with respect to x over the half-line $\mathbb{R}_+$. The following interpolating inequalities are very important for our next steps.

Lemma 2.1. *Let $\psi_1(x)$,$\psi_2(x)$ be two admissible weight functions, $q \in [2, +\infty]$*

$$s = s_0(q) = \frac{1}{4} - \frac{1}{2q},$$

then for every function satisfying $(|\varphi_{xx}| + |\varphi_{yy}| + |\varphi|)\psi_1^{1/2}(x) \in L_{2,+}$, $\varphi\psi_2^{1/2}(x) \in L_{2,+}$, $\varphi(0, y) \equiv 0$, $\varphi(x, 0)\varphi_y(x, 0) = \varphi(x, L)\varphi_y(x, L) \equiv 0$, the following inequality holds:

$$\|\varphi\psi_1^s\psi_2^{1/2-s}\|_{L_{q,+}} \le c\|(|\varphi_{xx}| + |\varphi_{yy}| + |\varphi|)\varphi_1^{1/2}\|^{2s}_{L_{2,+}} \|\varphi\psi_2\|^{1-2s}_{L_{2,+}}, \tag{13}$$

where the constant c depends on L, q and the properties of the functions ψ_i; if, in addition, $\varphi\big|_{y=0} = 0$ or $\varphi\big|_{y=L} = 0$ then this constant is uniform with respect to L.

Proof. Without loss of generality, assume that φ is a smooth, decaying at $+\infty$ function (for example $\varphi \in S_{exp}(\overline{\Sigma}_+)$).

First, uniformly with respect to L we establish the following:

$$\iint (\varphi_x^2 + \varphi_y^2)\psi_1^{1/2}\psi_2^{1/2}dxdy \le c(\iint (\varphi_{xx}^2 + \varphi_{yy}^2 + \varphi^2)\psi_1 dxdy)^{1/2}(\iint \varphi^2\psi_2 dxdy)^{1/2}. \tag{14}$$

In fact, boundary conditions on the function φ yield that

$$\iint (\varphi_x^2 + \varphi_y^2)\psi_1^{1/2}\psi_2^{1/2}dxdy = -\iint (\varphi_{xx} + \varphi_{yy})\psi_1^{1/2}\varphi\psi_2^{1/2}dxdy - \iint \psi\varphi_x(\psi_1^{1/2}\psi_2^{1/2})'dxdy.$$

Since ψ_i are admissible weight functions, we get

$$\iint (\varphi_x^2 + \varphi_y^2)\psi_1^{1/2}\psi_2^{1/2}dxdy \le \sqrt{2}(\iint (\varphi_{xx}^2 + \varphi_{yy}^2)\psi_1 dxdy)^{1/2}(\iint \varphi^2\psi_2 dxdy)^{1/2}$$
$$+c(\iint \varphi_x^2\psi_1^{1/2}\psi_2^{1/2}dxdy)^{1/2}(\iint \varphi^2\psi_1 dxdy)^{1/4}(\iint \varphi^2\psi_2 dxdy)^{1/4},$$

whence (14) follows.

Next, we use the following interpolating inequality from [1] in the case of the domain $\Omega = \Sigma_+$

$$\|f\|_{L_\infty(\Omega)} \le c(\|f_{xx}\|_{L_1(\Omega)} + \|f_{yy}\|_{L_1(\Omega)} + \|f\|_{L_1(\Omega)}), \tag{15}$$

and apply it to the function $f \equiv \varphi^2\psi_1^{1/2}\psi_2^{1/2}$, then

$$\|\varphi\psi_1^{1/4}\psi_2^{1/4}\|^2_{L_\infty(\Sigma_+)} \le c\iint [|(\varphi^2\psi_1^{1/2}\psi_2^{1/2})_{xx}| + |(\varphi^2\psi_1^{1/2}\psi_2^{1/2})_{yy}| + \varphi^2\psi_1^{1/2}\psi_2^{1/2}]dxdy. \tag{16}$$

Here,

$$(\varphi^2\psi_1^{1/2}\psi_2^{1/2})_{xx} = 2(\varphi\varphi_{xx} + \varphi_x^2)\psi_1^{1/2}\psi_2^{1/2} + 4\varphi\varphi_x(\psi_1^{1/2}\psi_2^{1/2})' + \varphi^2(\psi_1^{1/2}\psi_2^{1/2})'',$$
$$\iint |\varphi\varphi_{xx}|\psi_1^{1/2}\psi_2^{1/2}dxdy \le (\iint \varphi_{xx}^2\psi_1 dxdy)^{1/2}(\iint \varphi^2\psi_2 dxdy)^{1/2},$$

and since ψ_i are admissible weight functions

$$\iint |\varphi\varphi_x(\psi_1^{1/2}\psi_2^{1/2})'|dxdy \le c(\iint \varphi_x^2\psi_1^{1/2}\psi_2^{1/2}dxdy)^{1/2}(\iint \varphi^2\psi_1 dxdy)^{1/4}$$
$$(\iint \varphi^2\psi_2 dxdy)^{1/4},$$
$$\iint \varphi^2|(\psi_1^{1/2}\psi_2^{1/2})''|dxdy \le c\iint \varphi^2\psi_1^{1/2}\psi_2^{1/2} \le c(\iint \varphi^2\psi_1 dxdy)^{1/2}$$

$$\Big(\iint \varphi^2\psi_2 dxdy\Big)^{1/2}.$$

Other terms in the right-hand side of (16) are estimated in a similar way and with the use of (14) inequality (13) in the case $q=+\infty$ follows.

If $q\in(2,+\infty)$, then with the use of the (14) for $q=+\infty$

$$\|\varphi\psi_1^s\psi_2^{1/2-s}\|_{L_{q,+}} = \Big(\iint |\varphi|^{q-2}\psi_1^{\frac{q-2}{4}}\psi_2^{\frac{q-2}{4}}\varphi^2\psi_2 dxdy\Big)^{1/q} \le \|\varphi\psi_1^{1/4}\psi_2^{1/4}\|_{L_{\infty,+}}^{(q-2)/q}\|\varphi\psi_2^{1/2}\|_{L_{2,+}}^{2/q}$$
$$\le c\|(|\varphi_{xx}|+|\varphi_{yy}|+|\varphi|)\psi_1^{1/2}\|_{L_{2,+}}^{2s}\|\varphi\psi_2^{1/2}\|_{L_{2,+}}^{1-2s}.$$

Finlay, if, for instance, $\varphi\big|_{y=L}=\varphi\big|_{y=0}=0$, extend the function φ by zero to the quarter–plate $\mathbb{R}_+\times\mathbb{R}_+$ and carry out the same argument with the use of (15) for $\Omega=\mathbb{R}_+\times\mathbb{R}_+$ and (14) for $L=+\infty$, then estimate (13) becomes uniform with respect to L.□

Further we also use an interpolating inequality, following from the one in [4].

Lemma 2.2. *Let $\psi_1(x)$, $\psi_2(x)$ be two admissible weight functions, such that $\psi_1(x)\le c_0\psi_2(x)$, $\forall x\ge 0$ for certain constant $c_0>0$, $q\in[2,+\infty)$*

$$s=s_1(q)=\frac{1}{2}-\frac{1}{2q}, \tag{17}$$

then there exists a constant $c>0$, such that for any function $\varphi(x,y)$ verifying $\varphi_{xx}\psi_1^{1/2}(x)$, $\varphi_{yy}\psi_1^{1/2}(x)\in L_2(\Sigma_+)$, $\varphi\psi_2^{1/2}(x)\in L_2(\Sigma_+)$, if $|\nu|=1$ the following inequality holds:

$$\|\partial^\nu\varphi\psi_1^s\psi_2^{1/2-s}\|_{L_{2,+}} \le c\|(|\varphi_{xx}|+|\varphi_{yy}|)\psi_1^{1/2}\|_{L_{2,+}}^{2s}\times\|\varphi\psi_2^{1/2}\|_{L_{2,+}}^{1-2s}+c\|\varphi\psi_2^{1/2}\|_{L_{2,+}}. \tag{18}$$

We use next two lemmas from [3].

Lemma 2.3. *Let $\psi(x)$ be an admissible weight function, then there exists a constant c depending on the properties of the function ψ, such that for any function $\varphi(x,y)$ verifying $\varphi_{xx},\varphi\in L_{2,+}^{\psi(x)}$ the following inequalities hold:*

$$\iint\varphi_x^2\psi dxdy\le c\Big[\iint\varphi_{xx}^2\psi dxdy\Big]^{1/2}\Big[\iint\varphi^2\psi dxdy\Big]^{1/2}+c\iint\varphi^2\psi dxdy. \tag{19}$$

$$\int_0^L\varphi_x^2\big|_{x=0}dxdy\le c\Big[\iint\varphi_{xx}^2\psi dxdy\Big]^{3/4}\Big[\iint\varphi^2\psi dxdy\Big]^{1/4}+c\iint\varphi^2\psi dxdy. \tag{20}$$

Introduce anisotropic Sobolev spaces with smoothness properties only with respect to x. Let $H_+^{(k,0)}$ be a space of functions $\varphi(x,y)\in L_{2,+}$ such that $\partial_x^j\varphi\in L_{2,+}$ for $j\le k$ endowed with the natural norm $\|\varphi\|_{H_+^{(k,0)}}=(\sum_{j=0}^k\|\partial_x^j\varphi\|_{L_{2,+}}^2)^{1/2}$. Let $H_+^{(-m,0)}=\{\varphi(x,y)=\sum_{j=0}^m\varphi_j(x,y):\forall\varphi_j\in L_{2,+}\}$, endowed with the natural norm $\|\varphi\|_{H_+^{(-m,0)}}=(\sum_{j=0}^m\|\varphi_j\|_{L_{2,+}}^2)^{1/2}$.

Lemma 2.4. *If $\varphi\in H_+^{(k,0)}$, $\partial_x^n\varphi\in H_+^{(-m,0)}$ for $n\ge k+m$, $\partial_x^{k+1\varphi\in L_{2,+}}$ and for certain constant $c=c(k,m,n)$*

$$\|\partial_x^{k+1}\varphi\|_{L_{2,+}}\le c(\|\partial_x^n\varphi\|_{H_+^{(-m,0)}}+\|\varphi\|_{H_+^{(k,0)}}). \tag{21}$$

For the large-time decay results we need the Steklov inequality in the following form:

$$\int_0^L f^2(y)dy\le\frac{L^2}{\pi^2}\int_0^L(f'(u))^2dy. \tag{22}$$

where $f\in H_0^1(0,L)$.

Let $\psi_l(y)$, $l\in\mathbb{N}$, be the orthonormal in $L_2(0,L)$ system of the eigenfunctions for the operator $(-\psi'')$ on the segment $[0,L]$ with corresponding boundary conditions $\psi(0)=\psi(L)=0$ in the case (a) and $\psi'(0)=\psi'(L)=0$ in the case (b), λ_l be the corresponding eigenvalues. Such systems are well known and can be written in trigonometric functions.

Besides equation (1) we consider a linear equation:

$$u_t-(u_{xxxx}+u_{yyyy})_x+b(u_{xx}+u_{yy})_x+au_x=f(t,x,y), \tag{23}$$

with initial and boundary conditions (2) – (4). Weak solutions to this problem are understood similarly to Definition 1.2.

Lemma 2.5. *Let $u_0\in\widetilde{S}_{exp}(\overline{\Sigma}_+)$, $f\in C^\infty([0,T];\widetilde{S}_{exp}(\overline{\Sigma}_+))$. Set $\widetilde{\Phi}_0(x,y)\equiv u_0(x,y)$ and for $j\ge1$*

$$\widetilde{\Phi}_j\equiv\partial_t^{j-1}f(0,x,y)+(\partial_x^5+\partial_x\partial_y^4-b\partial_x^3-b\partial_x\partial_y^2-a\partial_x)\widetilde{\Phi}_{j-1}(x,y), \tag{24}$$

and let $\widetilde{\Phi}_j(0,y) = \widetilde{\Phi}_{jx}(0,y) = 0$ for any j.Then there exists a unique solution to problem (23), (2) – (4) $u \in C^\infty([0,T];\widetilde{S}_{exp}(\overline{\Sigma}_+))$.

Proof. Consider the corresponding initial value problem. Let $\Sigma = \mathbb{R} \times (0, L)$ and $\widetilde{S}(\overline{\Sigma}))$ be a space of infinitely smooth on $\overline{\Sigma}$ functions $\phi(x,y)$ such that $(1+|x|)^n|\partial^\alpha\varphi(x,y)| \le c(n,\alpha)$ for any n, multi-index α, $(x,y) \in \overline{\Sigma}$ and on the boundaries $y = 0$, $y = L$ verifying the same conditions as in the definition if the space $\widetilde{S}(\overline{\Sigma}_+)$. Extend the functions u_0 and f to the whole strip such that $u_0 \in \widetilde{S}(\overline{\Sigma})$, $f \in C([0,T];\widetilde{S}(\overline{\Sigma}))$ and consider problem (23) (in $\Pi_T = (0,T) \times \Sigma$), (2) (in Σ), (4) (in $\Omega_T = (0,T) \times \mathbb{R}$). Then with the use of the Fourier transform for the variable x and the Fourier series for the variable y a solution to problem (23), (2) – (4) can be written as the following:

$$u(t,x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \sum_{l=1}^{+\infty} e^{i\xi x}\psi_l(y)\hat{u}(t,\xi,l)d\xi,$$

where

$$\hat{u}(t,\xi,l) = \hat{u_0}(\xi,l)e^{i(\xi^5+\xi\lambda_l^2+b\xi^3+b\xi\lambda_l-a\xi)t} + \int_0^t \hat{f}(\tau,\xi,l)e^{i(\xi^5+\xi\lambda_l^2+b\xi^3+b\xi\lambda_l-a\xi)(t-\tau)}d\tau,$$
$$\hat{u_0}(\xi,l) \equiv \iint_\Sigma e^{-i\xi x}\psi_l(y)u_0(x,y)dxdy, \tag{25}$$
$$\hat{f}(t,\xi,l) \equiv \iint_\Sigma e^{-i\xi x}\psi_l(y)f(t,x,y)dxdy.$$

According to the properties of the u_0 and f this solution $u \in C^\infty([0,T];\widetilde{S}(\overline{\Sigma}))$.

Next, let $v \equiv \partial_x^k\partial_y^l u$ for some k, l. Then the function v satisfies an equation of (23) type, where f is replaced by $\partial_x^k\partial_y^l f$. Let $m \ge 5$, $\psi(x) \equiv x^m$ (note that this function is not an admissible weight function). Multiplying this equation by $2v(t,x,y)\psi(x)$ and integrating over Σ_+, we get

$$\frac{d}{dt}\int v^2\psi dxdy + \iint (5v_{xx}^2 + v_{yy}^2)\psi' dxdy + b\iint (3v_x^2 + v_y^2)\psi' dxdy$$
$$= \iint 5v_x^2\psi''' dxdy + \iint (-\psi^{(5)} + b\psi''' + a\psi')v^2 dxdy + 2\iint \partial_x^k\partial_y^l f v\psi dxdy, \tag{26}$$

where

$$\iint v_x^2\psi' dxdy = -\iint v_{xx}v\psi' dxdy + \frac{1}{2}\iint v^2\psi''' dxdy,$$
$$\iint v_y^2\psi' dxdy = -\iint v_{yy}v\psi' dxdy,$$
$$\iint v_x^2\psi''' dxdy = -\iint v_{xx}v\psi''' dxdy - \iint v_x v\psi^{(4)} dxdy.$$

Note, that $\psi''' \le \sqrt{6\psi'\psi^{(5)}}$, $\psi^4 \le \sqrt{2\psi'\psi^{(5)}}$.

From the equality above we get

$$-3b\iint v_x^2\psi' dxdy \le \iint v_{xx}^2\psi' dxdy + \frac{9b^2}{4}\iint v^2\psi' dxdy + \frac{3b}{2}\iint v^2\psi''' dxdy,$$
$$-b\iint v_y^2\psi' dxdy \le \iint v_{yy}^2\psi' dxdy + \frac{b^2}{4}\iint v^2\psi' dxdy,$$
$$\iint v_x^2\psi''' dxdy \le \iint v_{xx}^2\psi' dxdy + 8\iint v^2\psi^{(5)} dxdy.$$

Equally (26) yields

$$\frac{d}{dt}\int v^2\psi dxdy \le c(a,b)\iint (\psi^{(5)} + \psi''' + \psi')v^2 dxdy + 2\iint \partial_x^k\partial_y^l f v\psi dxdy. \tag{27}$$

Fix $\alpha > 0$ and let $n \ge 5$. For any $m \in [5,n]$ multiplying the corresponding inequality (27) by $(2\alpha)^m/(m!)$ and summing by m we obtain that for

$$z_n \equiv \iint \sum_{m=0}^{n} \frac{(2\alpha x)^m}{m!} v^2(t,x,y)dxdy,$$

due to the special choice of the function ψ, inequalities

$$z_n'(t) \le cz_n(t) + c, \quad z_n(0) \le c,$$

hold uniformly with respect to n, whence it follows that

$$\sup_{t\in[0,T]} \iint e^{2\alpha x} v^2 dxdy < \infty.$$

Thus, $u \in C^\infty([0,T]; \widetilde{S}_{exp}(\overline{\Sigma}_+))$. We will use the following notation $\omega(t,x,y)$ for the constructed solution of the initial value problem.

Let $\mu_0(t,y) \equiv -\omega(t,0,y)$, $\mu_1(t,y) \equiv -\omega_x(t,0,y)$. Note that the functions $\mu_j \in C^\infty(\overline{B}_T)$ and satisfy boundary conditions (4), and the compatibility conditions from the hypothesis of the lemma ensure that $\partial_t^l \mu_j(0,y) \equiv 0$, $\forall l$. Consider in Π_T^+ an initial-boundary value problem:

$$u_t - (u_{xxxx} + u_{yyyy})_x + b(u_{xx} + u_{yy})_x + au_x = 0, \tag{28}$$

$$u\big|_{t=0} = 0, u\big|_{x=0} = \mu_0(t,y), u_x\big|_{x=0} = \mu_1(t,y), \tag{29}$$

with boundary conditions (4).

Let $\Psi(t,x,y) \equiv \mu_0(t,y)\eta(1-x)+\mu_1(t,y)x\eta(1-x)$, $F(t,x,y) \equiv -\Psi_t+(\Psi_{xxxx}+\Psi_{yyyy})_x-b(\Psi_{xx}+\Psi_{yy})_x-a\Psi_x = 0$, $U(t,x,y) \equiv u(t,x,y) - \Psi(t,x,y)$, then the problem (28), (29), (4) is equivalent to problem (23), (2) – (4) for the function U, $u_0 \equiv 0$, $f \equiv F$. It is obvious, that $F \in C^\infty([0,T]; \widetilde{S}_{exp}(\overline{\Sigma}_+))$ and $\partial_t^l F(0,x,y) \equiv 0$, $\forall l$.

Apply the Galerkin method. Let $\{\varphi_j(x) : j = 1,2,3...\}$ be a set of linearly independent functions complete in the space $\{\varphi \in H^5(\mathbb{R}_+) : \varphi(0) = \varphi'(0) = 0\}$. Seek an approximate solution of the last problem in the form $U_k(t,x,y) = \sum_{j,l=1}^{k} c_{kjl}(t)\varphi_j(x)\psi_l(y)$ via conditions:

$$\begin{aligned} &\iint (U_{kt} - (U_{kxxxx} + U_{kyyyy})_x + b(U_{kxx} + U_{kyy})_x + aU_{kx})\varphi_i(x)\psi_m(y)dxdy \\ &- \iint F\varphi_i\psi_m dxdy = 0, \quad i,m = 1,...,k, \quad t \in [0,T], c_{kjl}(0) = 0. \end{aligned} \tag{30}$$

Multiplying (30) by $2c_{kim}(t)$ and summing with respect to i, m, we find that

$$\frac{d}{dt}\iint U_k^2 dxdy + \int_0^L U_{kxx}^2\big|_{x=0} dy = 2\iint FU_k dxdy, \tag{31}$$

thus

$$\|U_k\|_{L_\infty(0,T;L_{2,+})} \le \|F\|_{L_1(0,T;L_{2,+})}. \tag{32}$$

Multiplying (30) by $c'_{kim}(0)$, putting $t = 0$ and summing with respect to i, m, we obtain that $U_{kt}\big|_{t=0}$. Then differentiating (30) with respect to t, multiplying by $2c'_{kim}(t)$ and summing with respect to i, m, we find (similar to (32)) that

$$\|U_{tk}\|_{L_\infty(0,T;L_{2,+})} \le \|F\|_{L_1(0,T;L_{2,+})}. \tag{33}$$

Since $\psi_m^{(2n)}(y) = (-\lambda_m)^n\psi_m(y)$ it follows from (30) that for any n and l similary to (32) and (33)

$$\|\partial_t^l\partial_y^n U_k\|_{L_\infty(0,T;L_{2,+})} \le \|\partial_t^l\partial_y^n F\|_{L_1(0,T;L_{2,+})}. \tag{34}$$

Estimate (34) provides existence of a weak solution $U(t,x,y)$ to the considered problem such that $\partial_t^l\partial_y^n U \in C([0,T]; L_{2,+})$, for all l, n in the sense of the corresponding integral equality of the corresponding integral equality of (6) type for $g \equiv 0$, $f \equiv F$, $u_0 \equiv 0$. Note, that the traces of the function U satisfy conditions (2) for $u_0 \equiv 0$ and (4).

It follows from the corresponding equality of (6) type that since

$$U_{xxxxx} = U_t - U_{yyyyx} + b(U_{xx} + U_{yy})_x + aU_x - F, \tag{35}$$

$\partial_t^l\partial_y^n U_{xxxxx} \in C([0,T]; H_+^{(-3,0)})$ $\forall l,n$ therefore, the application of inequality (21) (for $\varphi \equiv \partial_t^l\partial_y^n U$, $k = 0$, $m = 3$) yields that $\partial_t^l\partial_y^n U_x \in C([0,T]; L_{2,+})$, $\forall l,n$ then the application twice of (35) and (21) (for $k = 1, m = 2$ and $k = 2, m = 1$) yields that $\partial_t^l\partial_y^n U_{xxx} \in C([0,T]; L_{2,+})$, $\forall l,n$. And again from (35) follows that $\partial_t^l\partial_y^n U_{xxxxx} \in C([0,T]; L_{2,+})$, $\forall l,n$, the function U satisfies (23) in Π_T^+ and its traces satisfy (2). For any natural m differentiating corresponding equation (23) $5(m-1)$ times and using induction with respect to m, we derive that $\partial_t^l\partial_x^{5m}\partial_y^n U \in C([0,T]; L_{2,+})$.

As a result, the solution to the problem (28), (29), (4) is constructed such that $\partial_t^l\partial_x^m\partial_y^n u \in C([0,T]; L_{2,+})$, $\forall l,m,n$. From now on in the proof we use notation $v(t,x,y)$ for this solution.

The function $u(t,x,y) + v(t,x,y)$ is the solution to problem (23), (2) – (4) such that $\partial_t^l\partial_x^m\partial_y^n u \in C([0,T]; L_{2,+})$, $\forall l,m,n$. Let $\widetilde{u}(t,x,y)\eta(x-1)$. The function $\widetilde{u}$ solves an initial value problem in the strip Σ of (23), (2), (4) type, where the functions f, u_0 are substituted by corresponding functions $\widetilde{f}$, $\widetilde{u}_0$ from the same classes and the obtained result at the beginning of the proof for the initial value problem together with the obvious uniqueness provide that $\widetilde{u} \in C^\infty([0,T]; \widetilde{S}_{exp}(\overline{\Sigma}_+))$ and so $u \in C^\infty([0,T]; \widetilde{S}_{exp}(\overline{\Sigma}_+))$.□

Lemma 2.6. *A generalized solution to problem (23), (2) – (4) is unique in the space $L_2(\Pi_T^+)$.*
Proof. This lemma is a corollary of the following result on existence of smooth solutions to the corresponding adjoint problem. □

In Π_T^+ consider an initial-boundary value problem for an equation:

$$u_t + (u_{xxxx} + u_{yyyy})_x - b(u_{xx} + u_{yy})_x - au_x = f(t,x,y), \tag{36}$$

with initial condition (2), boundary conditions: (4) and boundary conditions

$$u\big|_{x=0} = u_x\big|_{x=0} = u_{xx}\big|_{x=0} = 0. \tag{37}$$

Lemma 2.7. *Let $u_0 \in \widetilde{S}(\overline{\Sigma}_+)$, $f \in C^\infty([0,T]; \widetilde{S}(\overline{\Sigma}_+))$ and $\widetilde{\Phi}_j(0,y) = \widetilde{\Phi}_{jx}(0,y) \equiv 0$ for any j, where here in the definition of the corresponding functions $\widetilde{\Phi}_j$ in comparison with (24) the sign before the second term in the right-hand side is changed. Then there exists a unique solution to problem (36), (2), (37), (4), $u \in C^\infty([0,T]; \widetilde{S}(\overline{\Sigma}_+))$.*
Proof. Extend the functions u_0 and f to the whole strip and consider problem (36), (2), (4), construct its solution $\omega \in C^\infty([0,T]; \widetilde{S}(\overline{\Sigma}_+))$ in a similar way with the only obvious difference in (25).

Let $\mu_0(t,y) \equiv -\omega(t,0,y)$, $\mu_1 \equiv -\omega_x(t,0,y)$, $\mu_2 \equiv -\omega_{xx}(t,0,y)$. Note that the functions $\mu_j \in C^\infty(\overline{B}_T)$ and satisfy boundary conditions (4). Moreover,the compatibility conditions form the hypothesis of the lemma ensure that $\partial_t^l \mu_j(0,y) \equiv 0$, $\forall l$. In Π_T^+. Consider an initial-boundary value problem:

$$u_t + (u_{xxxx} + u_{yyyy})_x - b(u_{xx} + u_{yy})_x - au_x = 0, \tag{38}$$

$$u\big|_{t=0} = 0, u\big|_{x=0} = \mu_0(t,y), u_x\big|_{x=0} = \mu_1(t,y), u_{xx}\big|_{x=0} = \mu_2(t,y), \tag{39}$$

and with boundary conditions (4).

Let $\Psi(t,x,y) \equiv \mu_0(t,y)\eta(1-x) + \mu_1(t,y)x\eta(1-x) + \mu_2(t,y)x^2\eta(1-x)/2$, $F(t,x,y) \equiv -\Psi_{xxxxx} - \Psi_{xxxxy} + b\Psi_{xxx} + b\Psi_{xyy} + a\Psi_x - \Psi_t$, $U(t,x,y) \equiv u(t,x,y) - \Psi(t,x,y)$, then problem (38), (39), (4) is equivalent to problem (36), (2), (37), (4) for the function U, $u_0 \equiv 0$, $f \equiv F$. Obviously $F \in C^\infty([0,T]; \widetilde{S}(\overline{\Sigma}_+))$ and $\partial_t^l F(0,x,y) \equiv 0$, $\forall l$.

Let $\{\varphi_j(x) : j = 1,2,3,...\}$ be the same set of functions as in the proof of Lemma 2.5. Seek an approximate solution in the form $U_k(t,x,y) = \sum_{j,l=1}^k c_{kjl}(t)\varphi_j(x)\psi_l(y)$ via conditions:

$$\begin{aligned} \iint [U_{tk}\varphi_i\psi_m - U_k(\varphi_i^{(5)}\psi_m + \varphi_i^{(4)}\psi_m' - b\varphi_i'''\psi_m - b\varphi_i''\psi_m' - a\varphi_i\psi_m])dxdy \\ - \iint F\varphi_i\psi_m dxdy = 0, i, m = 1,2,3,...,k, t \in [0,T], \end{aligned} \tag{40}$$

$c_{kjl}(0) = 0$. Multiplying (40) by $2c_{kim}(t)$ and summing with respect to i, m, we derive equality (31), which implies estimate (32). Similarly we get (34), which provide existence of a weak solution $U(t,x,y)$ to the considered problem such that $\partial_t^l \partial_y^n U \in C([0,T]; L_{2,+})$, $\forall l, n \geq 0$ in the following sense: for any function $\phi \in L_\infty(0,T; \widetilde{H}_+^4)$, such that ϕ_t, ϕ_{xxxxx}, $\phi_{yyyyx} \in L_\infty(0,T; L_{2,+})$, $\phi\big|_{t=T} = \phi\big|_{x=0} = \phi_x\big|_{x=0} = 0$ the following equality is satisfied:

$$\iiint_{\Pi_T^+} [U(\phi_t + (\phi_{xxxx} + \phi_{yyyy})_x - b(\phi_{xx} + \phi_{yy})_x - a\phi_x) + F\phi]dxdydt = 0.$$

Then also similarly to the proof of Lemma 2.5 we obtain a solution to problem (38), (39), (4) v such that $\partial_t^l \partial_x^m \partial_y^n v C([0,T]; L_{2,+})$, $\forall l$, m, n.

Similar to Lemma 2.5 we show that the function $u \equiv w + v$ is the desired solution. □

Remark 2.1. *In further lemmas of this section we first consider smooth solutions constructed in Lemma 2.5 and then pass to the limit on the basis of obtained estimates.*

Lemma 2.8. *Let $\psi(x)$ be admissible weight function, such that $\psi'(x)$ is also an admissible weight function, $u_0 \in L_{2,+}^{\psi(x)}$, $f \equiv f_0 + f_{1x}$, where $f_0 \in L_1(0,T; L_{2,+}^{\psi(x)})$, $f_1 \in L_{4/3}(0,T; L_{2,+}^{\psi^{3/2}(x)(\psi'(x))^{-1/2}})$. Then there exist a unique weak solution to problem (23), (2) – (4) form the space $X^{\psi(x)}(\Pi_T^+)$ and a function $\mu_2 \in L_2(B_T)$ such that for any function $\phi \in L_\infty(0,T; \widetilde{H}_+^4)$, ϕ_t, ϕ_{xxxxx}, $\phi_{yyyyx} \in L_\infty(0,T; L_{2,+})$ $\phi\big|_{t=T} = \phi\big|_{x=0} = \phi_x\big|_{x=0} = 0$ the following equality holds:*

$$\begin{aligned} \iiint_{\Pi_T^+} (u\phi_t - u\phi_{xxxxx} - u\phi_{yyyyx} + bu\phi_{xxx} + bu\phi_{yyx} + au\phi_x + f_0\phi - f_1\phi_x)dtdxdy \\ + \iint_{\Sigma_+} u_0\phi\big|_{t=0}dxdy - \iint_{B_T} \mu_2\phi_{xx}\big|_{x=0}dydt = 0. \end{aligned} \tag{41}$$

Moreover, for a.e. $t \in (0;T]$

$$\|u\|_{X^{\phi(x)}(\Pi_t^+)} + \|\mu_2\|_{L_2(B_t)} \leq c(T), \tag{42}$$

and for a. e. $t \in (0;T]$

$$\frac{d}{dt}\iint u^2\psi\,dxdy + \psi(0)\int_0^L \mu_2^2\big|_{x=0}dy + \iint [5u_{xx}^2 + u_{yy}^2 + 3bu_x^2 + bu_y^2 - au^2]\psi'\,dxdy$$
$$- \iint [5u_x^2 + bu^2]\psi^{(3)}\,dxdy + \iint u^2\psi^{(5)}\,dxdy = 2\iint f_0 u\psi\,dxdy - \iint 2f_1(u\psi)_x\,dxdy, \tag{43}$$

if $f_1 \equiv 0$, then in equality (43) one can put $\psi \equiv 1$.

Proof. Multiplying (23) by $2u(x,y,t)\psi(x)$ and integrating over Σ_+, thus we obtain (43) with $\mu_2 \equiv u_{xx}\big|_{x=0}$. According to (20) for arbitrary $\varepsilon > 0$

$$|\iint f_1(u\psi)_x\,dxdy| \le c\|(|u_x| + |u|)(\psi')^{1/4}\psi^{1/4}\|_{L_{2,+}}\|f_1\psi^{3/4}(\psi')^{-1/4}\|_{L_{2,+}}$$
$$\le c_1\big[\|(|u_{xx} + u_{yy}|)(\psi')^{1/2}\|_{L_{2,+}}^{1/2}\|u\psi^{1/2}\|_{L_{2,+}}^{1/2} + \|u\psi^{1/2}\|_{L_{2,+}}\big]\|f_1\psi^{3/4}(\psi')^{-1/4}\|_{L_{2,+}}$$
$$\le \varepsilon\iint (u_{xx}^2 + u_{yy}^2)\psi'\,dxdy + c(\varepsilon)\|f_1\|^{4/3}_{L_{2,+}^{\psi^{3/2}(x)(\psi'(x))^{-1/2}}}\Big(\iint u^2\psi\,dxdy\Big)^{1/3}$$
$$+c_1\|f_1\|_{L_{2,+}^{\psi^{3/2}(x)(\psi'(x))^{-1/2}}}\Big(\iint u^2\psi\,dxdy\Big)^{1/2}, \tag{44}$$

and according to (19)

$$\iint u_x^2(\psi' + |\psi'''|)\,dxdy \le \varepsilon\iint u_{xx}^2\psi'\,dxdy + c(\varepsilon)\iint u^2\psi\,dxdy. \tag{45}$$

Moreover,

$$\iint u_y^2\psi'\,dxdy = -\iint uu_{yy}\psi'\,dxdy \le \varepsilon\iint u_{yy}^2\psi'\,dxdy + c(\varepsilon)\iint u^2\psi\,dxdy. \tag{46}$$

It follows from (43) – (45), that for smooth solutions

$$\|u\|_{X^{\psi(x)}(\Pi_T^+)} + \|u_{xx}\big|_{x=0}\|_{L_2(B_T)} \le c. \tag{47}$$

The end of the proof is standard. □

Lemma 2.9. *Let $\psi(x)$ be admissible weight function, such that $\psi'(x)$ is also an admissible weight function, $u_0 \in \widetilde{H}_+^{2,\psi(x)}$, $u_0(0,y) \equiv u_{0x}(0,y) \equiv 0$, $f \equiv f_0 + f_1$, where $f_0 \in \widetilde{H}_+^{2,\psi(x)}$, $f_1 \in L_2(0,T;L_{2,+}^{\psi^2(x)/\psi'(x)})$. Then there exist a strong solution $u \in X^{2,\psi(x)}(\Pi_+^T)$ to problem (23), (2) – (4) and a function $\mu_4 \in L_2(B_T)$ such that for any $t \in (0,T)$*

$$\|u\|_{X^{2,\psi(x)}(\Pi_t^+)} + \|\mu_4\|_{L_2(B_t)} \le c(T)\big(\|u_0\|_{\widetilde{H}_+^{2,\psi(x)}} + \|f_0\|_{L_2(0,t;\widetilde{H}_+^{2,\psi(x)})} + \|f_1\|_{L_2(0,t;L_{2,+}^{\psi^2(x)/\psi'(x)})}\big),$$

and for a. e. $t \in (0,T)$

$$\frac{d}{dt}\iint (u_{xx}^2 + u_{yy}^2 + bu_x^2 + bu_y^2)\psi\,dxdy + \int (u_{xxxx}^2\psi + 4u_{xxxx}u_{xxx}\psi' + 2u_{xxxx}u_{xx}\psi'' - 2bu_{xxxx}u_{xx}\psi$$
$$-3u_{xxx}^2\psi'' - 2u_{xxx}u_{xx}\psi''' + 4bu_{xxx}u_{xx}\psi' + u_{xx}^2\psi^{(4)} - 4bu_{xx}^2\psi'' + (b^2+a)u_{xx}^2\psi)\big|_{x=0}dy$$
$$+\iint (5u_{xxxx}^2 + 6u_{xxyy}^2 + 8bu_{xxx}^2 + 6bu_{xxy}^2 + u_{yyyy}^2 + 4bu_{xyy}^2 + 2bu_{yyy}^2 +$$
$$+(3b^2 - a)u_{xx}^2 + 4b^2u_{xy}^2 - abu_x^2 + (b^2 - a)u_{yy}^2)\psi'\,dxdy$$
$$+\iint (-5u_{xxx}^2 - 6bu_{xx}^2 - 5u_{xyy}^2 - bu_{yy}^2 - b^2u_x^2 - 5bu_{xy}^2 - b^2u_y^2)\psi'''\,dxdy$$
$$+\iint (u_{xx}^2 + u_{yy}^2 + bu_x^2 + bu_y^2)\psi^{(5)}\,dxdy$$
$$= 2\iint (f_{0xx}u_{xx} + f_{0yy}u_{yy} + bf_{0x}u_x + bf_{0y}u_y)\psi\,dxdy$$
$$-2\int (f_0(u_{xx}\psi)_x - f_{0x}u_{xx}\psi)\big|_{x=0}dy$$
$$+2\iint (f_1[(u_{xx}\psi)_{xx} + u_{yyyy}\psi - b(u_x\psi)_x - bu_{yy}\psi])\,dxdy, \tag{48}$$

if $f_1 \equiv 0$ then in equality (48) one can put $\psi(x) = 1$.

Proof. Multiplying (23) by $2(u_{xx}\rho(x))_{xx} + 2u_{yyyy}\rho(x) - 2b(u_x\rho(x))_x - 2bu_{yy}\rho(x)$ where either $\rho \equiv \psi(x)$ or $\rho(x) \equiv 1$ and integrating over Σ_+ we get equality (48) for $\mu_4 \equiv u_{xxxx}|_{x=0}$, where ψ is substituted by ρ. Here according to (20) for an arbitrary $\varepsilon > 0$

$$\int_0^L u_{xxx}^2\big|_{x=0} dy \le \varepsilon \iint u_{xxxx}^2 \psi' dxdy + c(\varepsilon) \iint u_{xx}^2 \psi dxdy, \tag{49}$$

similarly to (45) and (46)

$$\iint (u_{xxx}^2 + u_{yyy}^2 + u_{xyy}^2 + u_{xxy}^2)\psi' dxdy \le \varepsilon \iint (u_{xxxx}^2 + u_{yyyy}^2 + u_{xxyy}^2)\psi' dxdy + c(\varepsilon) \iint (u_{xx}^2 + u_{yy}^2)\psi dxdy, \tag{50}$$

and

$$\Big| \iint f_1 [(u_{xx}\psi)_{xx} + u_{yyyy}\psi] dxdy \Big| \le \varepsilon \iint (u_{xxxx}^2 + u_{yyyy}^2 + u_{xx}^2)\psi' dxdy + c(\varepsilon) \iint f_1^2 \psi^2 (\psi')^{-1} dxdy. \tag{51}$$

Inequalities (47), (49) – (51) and equality (48) imply that for smooth solutions

$$\|u\|_{X^{2,\psi(x)}(\Pi_T^+)} + \|u_{xxxx}\big|_{x=0}\|_{L_2(B_T)} \le c(T)\big(\|u_0\|_{\widetilde{H}_+^{2,\psi(x)}} + \|f_0\|_{L_2(0,t;\widetilde{H}_+^{2,\psi(x)})} + \|f_1\|_{L_2(0,t;L_{2,+}^{\psi^2(x)/\psi'(x)})}\big). \tag{52}$$

□

Lemma 2.10. *Let the hypothesis of Lemma 2.9 be satisfied for $\psi(x) \equiv e^{2\alpha x}$ for certain $\alpha > 0$. Let $g \in C^1(\mathbb{R})$, $g(0) = 0$. Consider the strong solution $u \in X^{2,\psi(x)}(\Pi_T^+)$ to problem (23), (2) – (4). Then for a.e. $t \in (0,T)$*

$$\begin{gathered}\frac{d}{dt}\iint g^*(u)\rho dxdy + \iint g'(u)u_x(u_{xxxx} - bu_{xx} + u_{yyyy} - bu_{yy})\rho dxdy + \\ \iint g(u)(u_{xxxx} - bu_{xx} + u_{yyyy} - bu_{yy})\rho' dxdy - a\iint g^*(u)\rho' dxdy = \iint g(u) f\rho dxdy.\end{gathered} \tag{53}$$

where either $\rho(x) = 1$ or $\rho(x)$ is an admissible weight function such that $\rho(x) \le c\psi(x)$ $\forall x \ge 0$.

Proof. In the smooth case equality (53) is obtained via multiplication of (23) by $g(u(t,x,y))\rho(x)$ and subsequent integration and in the general case via closure, which here is easily justified since $X^{2,\psi(x)}(\Pi_T^+) \in L_\infty(\Pi_T^+)$ and $\psi \sim \psi'$. □

3. Existence of solutions. The following is the appropriate text. In this section we proof of the existence of the solutions in the first two theorems.

Lemma 3.1. *Let $g \in C^1(\mathbb{R})$, $g(0) = 0$. $|g'(u)| \le c$ $\forall u \in \mathbb{R}$. $\psi(x) \equiv e^{2\alpha x}$ for certain $\alpha > 0$, $u_0 \in L_{2,+}^{\psi(x)}$, $f \in L_1(0,T; L_{2,+}^{\psi(x)})$. Then problem (1) – (4) has a unique weak solution $u \in X^{\psi(x)}(\Pi_T^+)$.*

Proof . We apply the contraction principle. For $t_0 \in (0,T]$ define a mapping Λ on $X^{\psi(x)}(\Pi_T^+)$ as follows: $u = \Lambda v \in X^{\psi(x)}(\Pi_{t_0}^+)$ is a weak solution to a linear problem:

$$u_t - (u_{xxxx} + u_{yyyy})_x + b(u_{xx} + u_{yy})_x + au_x = f(t,x,y) - (g(v))_x,$$

in $\Pi_{t_0}^+$ and boundary conditions (2) – (4).

Note that $\psi^{3/2}(\psi')^{-1/2} \le c\psi$, $|g(v)| \le c|v|$ thus, Lemma 3.1 provides that the mapping Λ exists. Moreover, for functions $v, \widetilde{v} \in X^{\psi(x)}(\Pi_{t_0}^+)$ according to inequality (42)

$$\begin{gathered}\|\Lambda v\|_{X^{\psi(x)}(\Pi_{t_0}^+)} \le c(T)(\|u_0\|_{L_{2,+}^{\psi(x)}} + \|f\|_{L_1(0,T;L_{2,+}^{\psi(x)})} + t_0^{3/4}\|v\|_{X^{\psi(x)}(\Pi_{t_0}^+)}), \\ \|\Lambda v - \Lambda\widetilde{v}\|_{X^{\psi(x)}(\Pi_{t_0}^+)} \le c(T)t_0^{3/4}\|v - \widetilde{v}\|_{X^{\psi(x)}(\Pi_{t_0}^+)}.\end{gathered} \tag{54}$$

whence first the local result succeeds. Next, since the constant in the right-hand side in the above inequalities is uniform with respect to u_0 and f, one can extend the solution to the whole time segment $[0,T]$ by the standard argument.□

Proof of Existence Part of Theorem 1.1. For $h \in (0;1]$ consider a set of initial-boundary value problems:

$$u_t - (u_{xxxx} + u_{yyyy})_x + b(u_{xx} + u_{yy})_x + au_x + g_h'(u)u_x = f(t,x,y), \tag{55}$$

with an initial condition:

$$u\big|_{t=0} = u_{0h}(x), \tag{56}$$

with boundary conditions (3) and (4), where

$$f_h(t,x,y) \equiv f(t,x,y)\eta(1/h - x), \quad u_{0h}(x,y) \equiv u_0(x)\eta(1/h - x),$$

$$g'_h(u) \equiv g'(u)\eta(2 - h|u|), \quad g_h(u) \equiv \int_0^u g'_h(\theta)d\theta.$$

Note, that $g_h(u) = g(u)$ if $|u| \le 1/h$, $g'_h(u) = 0$ if $|u| \ge 2/h$, $|g'_h(u)| \le c(h)$ $\forall u$ and the function g_h satisfy inequality (7) uniformly with respect to h.

Lemma 3.1 implies that there exists a unique solution to this problem $u_h \in X^{e^{2\alpha x}}(\Pi_T^+)$ for any $\alpha > 0$.

Next, establish appropriate estimates for functions u_h uniform with respect to h (we drop the subscript h in intermediate steps for simplicity). First, note that $g'(u)u_x \in L_1(0,T;L_{2,+}^{\psi(x)})$ and so the hypothesis of Lemma 3.1 is satisfied (for $f_1 = f_2 \equiv 0$). Then equality (43) provides that for both for $\rho(x) \equiv 1$ and $\rho(x) \equiv \psi$:

$$\begin{gathered}\frac{d}{dt}\iint u^2\rho dxdy + \rho(0)\int_0^L \mu_2^2\big|_{x=0}dy + \iint [5u_{xx}^2 + u_{yy}^2 + 3bu_x^2 + bu_y^2 - au^2]\rho' dxdy \\ - \iint [5u_x^2 + bu^2]\rho^{(3)}dxdy + \iint u^2\rho^{(5)}dxdy = 2\iint fu\rho dxdy + \iint (g'(u)u)^*\rho' dxdy.\end{gathered} \tag{57}$$

Choosing $\rho \equiv 1$ with respect to h and to L we get

$$\|u_h\|_{C([0,T];L_{2,+})} \le c. \tag{58}$$

Let $\rho \equiv \psi$. Note that uniformly with respect to h

$$|(g'_h(u)u)^*| \le c|u|^{p+2}. \tag{59}$$

Let $q = p+2$, $s = s_0(q)$ from (17), $\psi_1(x) \equiv \psi'(x)$, $\psi_2(x) \equiv (\psi'(x))^{\frac{2(1-qs)}{q(1-2s)}}$ $(qs = p/4 < 1)$. Applying (18), we obtain that

$$\begin{aligned}\iint |u|^{p+2}\psi' dxdy &= \iint |u|^{p+2}\psi_1^{qs}\psi_2^{q(1/2-s)}dxdy \\ &\le c\Big(\iint (u_{xx}^2 + u_{yy}^2 + u^2)\psi_1 dxdy\Big)^{qs}\Big(\iint u^2\psi_2 dxdy\Big)^{q(1/2-s)} \\ &= c\Big(\iint (u_{xx}^2 + u_{yy}^2 + u^2)\psi_1 dxdy\Big)^{qs}\Big(\iint (u^2\psi')^{\frac{2(1-qs)}{q(1-2s)}} u^{\frac{2(q-2)}{q(1-2s)}} dxdy\Big)^{q(1/2-s)} \\ &\le c\Big(\iint (u_{xx}^2 + u_{yy}^2 + u^2)\psi_1 dxdy\Big)^{p/4}\Big(\iint u^2\psi' dxdy\Big)^{(4-p)/4}\Big(\iint u^2 dxdy\Big)^{p/2}.\end{aligned} \tag{60}$$

Since the norm of the functions u_h in the space $L_{2,+}$ is already estimated in (58) it follows from (57), (59) and (60), that uniformly with respect to h

$$\|u_h\|_{X^{\psi(x)}(\Pi_T^+)} \le c. \tag{61}$$

Write the analogue of (55) where ρ is substituted by $\rho_0(x - x_0)$ for any $x_0 \ge 0$ Then it easily follows (5), that

$$\lambda^+(u_{hxx};T) + \lambda^+(u_{hyy}) \le c. \tag{62}$$

Let $\Sigma_n = (0,n) \times (0,L)$. It follows from (62) and interpolating inequality from [1] (where $Q_n = (n, n+1) \times (0,L)$):

$$\|f\|_{L_\infty(Q_n)} \le c(L)\Big(\iint_{Q_n} (f_{xx}^2 + f_{yy}^2 + f^2)dxdy\Big)^{1/4}\Big(\iint_{Q_n} f^2 dxdy\Big)^{1/4},$$

that uniformly with respect to h

$$\|u_h\|_{L_4(0,T;L_\infty(\Sigma_n))} \le c,$$

and

$$\|g_h(u_h)\|_{L_{4/p}(0,T;L_2(\Sigma_n))} \le c.$$

Then from equation (1) itself it follows, that uniformly with respect to h

$$\|u_{ht}\|_{L_1(0,T;H^{-5}(\Sigma_n))} \le c. \tag{63}$$

Since the embedding $H^1(\Sigma_n) \subset L_2(\Sigma_n)$ is compact, it follows from [15] that the set u_h is relatively compact in $L_q(0,T;L_2(\Sigma_n))$ for $q < +\infty$.

Extract a sub-sequence of the functions u_h, again denoted as u_h, such that as $h \to +0$

$u_h \to u$ *-weakly in $L_\infty(0,T;L_{2,+}^{\psi(x)})$;

$u_{hxx}, u_{hyy} \to u_{xx}, u_{yy}$ weakly in $L_2(0,T;L_{2,+}^{\psi'(x)})$;

$u_h \to u$ strongly in $L_{max(2,4/(4-p))}(0,T;L_2(\Sigma_n))$ $\forall n$.

Let ϕ is a test function from Definition 1.2 with $supp\phi \in \overline{\Sigma}_n$. Then, since

$$|g_h(u_h) - g_h(u)| \le c(|u_h|^p + |u|^p)|u_h - u|,$$

with the use of (63), we obtain, that the limit function u verifies (6).

Now, note that $g(u)\phi_x \in L_\infty(0,T;L_{1,+})$ if $p \le 1$. In case $p > 1$

$$\begin{aligned} \|g(u)\phi\|_{L_1(\Pi_T^+)} &\le c\int_0^T \|u(\psi')^{1/4}\psi^{1/4}\|^p_{L_{\infty,+}} \iint |u\phi_x|(\psi')^{-p/4}\psi^{-p/4}dxdydt \\ &\le c_1\int_0^T \Big[\Big(\iint (u_{xx}^2+u_{yy}^2+u^2)\psi' dxdy\Big)^{p/4}\Big(\iint u^2\psi dxdy\Big)^{(p+2)/4} \\ &\quad \Big(\iint \phi_x^2(\psi')^{-p/2}\psi^{-(1+p)/2}dxdy\Big)^{1/2}\Big]dt < \infty. \end{aligned} \tag{64}$$

since $(\psi')^{-p/2}\psi^{-(1+p)/2} \le c(1+x)^{pn/4}$ by virtue of the additional property of the function ψ. Approximating any test function from Definition 1.2 by the compactly supported ones and passing to the limit we obtain equality (1) in the general case. □

Lemma 3.2. *Let $g \in C^2(\mathbb{R})$, $g(0)=0$, $|g'(u)|, |g''(u)| \le c$ $\forall u \in \mathbb{R}$. $\psi(x) \equiv e^{2\alpha x}$ for certain $\alpha > 0$, $u_0 \in \widetilde{H}_+^{2,\psi(x)}$, $u_0(0,y) = u_{0y}(0,y) = 0$, $f \in L_2(0,T;\widetilde{H}_+^{2,\psi(x)})$. Then there exists $t_0 \in (0,T)$ such that the problem (1) – (4) has a unique strong solution $u \in X^{2,\psi(x)}(\Pi_{t_0}^+)$.*

Proof. Similarly to the proof of Lemma 3.1 we construct the desired solution as a fixed point of the map Λ but defined on the space $X^{2,\psi(x)}(\Pi_{t_0}^+)$. Here $\psi^2/\psi' \sim \psi$ and Lemma 2.9 where $f_0 \equiv f$, $f_1 \equiv g'(v)v_x$ ensures that such a map exists. Moreover, for functions $v, \in X^{2,\psi(x)}(\Pi_{t_0}^+)$ according to inequality (48)

$$\|\Lambda v\|_{X^{2,\psi(x)}} \le c(T)(\|u_0\|_{\widetilde{H}_+^{2,\psi(x)}} + \|f\|_{L_2(0,T;\widetilde{H}_+^{2,\psi(x)})} + t_0^{1/2}\|v\|_{X^{2,\psi(x)}}),$$

and, since $|g'(v)v_x - g'(\widetilde{v})\widetilde{v_x}| \le c(|v_x| + |\widetilde{v_x}|)|v-\widetilde{v}| + c|v_x - \widetilde{v_x}|$,

$$\|\Lambda v - \Lambda\widetilde{v}\|_{X^{2,\psi(x)}} \le c(T)t_0^{1/2}(\|v\|_{X^{2,\psi(x)}} + \|\widetilde{v}\|_{X^{2,\psi(x)}})\|v-\widetilde{v}\|_{X^{2,\psi(x)}},$$

whence the assertion of the lemma succeeds. Here for convenience we denoted $X^{2,\psi(x)}(\Pi_{t_0}^+)$ as $X^{2,\psi(x)}$. □

Proof of Existence Part of Theorem 1.2. We will proof, that if $X^{2,e^{2\alpha x}}(\Pi_T^+)$, $\alpha > 0$ is a solution to problem (1) – (4) for some $T' \in (0,T]$, where the function $g \in C^2(\mathbb{R})$ verifies (7), then for any admissible function $\psi(x)$, such that ψ' is also admissible and $\psi(x) \le ce^{2\alpha x}$, $\forall x \ge 0$,

$$\|u\|_{X^{2,\psi(x)}(\Pi_{T'}^+)} \le c(T, \|u_0\|_{\widetilde{H}_+^{2,\psi(x)}}, \|f\|_{L_2(0,T;\widetilde{H}_+^{2,\psi(x)})}). \tag{65}$$

Using (57), where $\mu_2 = u_{xx}|_{x=0}$ we obtain

$$\|u\|_{X^{\psi(x)}(\Pi_{T'}^+)} + \|u_{xx}|_{x=0}\|_{L_2(B_{T'})} \le c. \tag{66}$$

Next, since the hypotheses of Lemma 2.9 and Lemma 2.10 are satisfied, write down the corresponding analogues of equalities (48) and (53) and subtract from the first one the doubled second one, then with the use of (49) and (50) for sufficiently small ε we get

$$\begin{aligned} \frac{d}{dt}\iint (u_{xx}^2 + u_{yy}^2 + bu_x^2 + bu_y^2 - 2g^*(u))\rho dxdy + \int (u_{xxxx}^2\rho)\big|_{x=0}dy + \iint (5u_{xxxx}^2 + 6u_{xxyy}^2 + u_{yyyy}^2)\rho' dxdy \\ \le \iint 2g(u)(u_{xxxx} - bu_{xx} + u_{yyyy} - bu_{yy})\rho' dxdy - 2a\iint g^*(u)\rho' dxdy \\ +\varepsilon\int u_{xxxx}^2\big|_{x=0}dy + c(\varepsilon)\int u_{xx}^2\big|_{x=0}dy + \varepsilon\iint (u_{xxxx}^2 + u_{xxyy}^2 + u_{yyyy}^2)\rho' dxdy \\ +c(\varepsilon)\iint (u_{xx}^2 + u_{yy}^2)\rho dxdy + c\iint (f_{xx}^2 + f_{yy}^2 + f^2)\rho dxdy \\ +2\iint (g'(u)u_x[2u_{xxx}\rho' + u_{xx}\rho'' - bu_x\rho'])dxdy - 2\iint g(u)f\rho dxdy - 2\iint (g'(u)g(u))^*\rho' dxdy. \end{aligned} \tag{67}$$

Choose $\rho \equiv 1$. Note, that (7) with (66) imply that

$$\iint |g^*(u)|dxdy \le c\|u\|^p_{L_{\infty,+}}\|u\|_{L_{2,+}} \le c_1\Big(\iint (u_{xx}^2 + u_{yy}^2 + u^2)dxdy\Big)^{p/4}, \tag{68}$$

$$\iint g(u)f dxdy \le c\|u\|^p_{L_{\infty,+}}\|u\|_{L_{2,+}}\|f\|_{L_{2,+}}.$$

Thus, from (67) we get

$$\|u_{xx}\|_{L_\infty(0,T';L_{2,+})} + \|u_{yy}\|_{L_\infty(0,T';L_{2,+})} \le c.$$

In particular

$$\|u_{xx}\|_{L_\infty(\Pi^+_{T'})} \le c. \tag{69}$$

Now, in (67) chose $\rho(x) \equiv \psi(x)$. By virtue of (69) $|g(u)| \le c|u|$ and then estimate (65) easily follows.

Note, that from (67) (where $\rho(x) \equiv \rho_0(x - x_0)$ for any $x_0 \ge 0$) follows

$$\lambda^+(u_{xxxx}; T') + \lambda^+(u_{xxyy}; T') + \lambda^+(u_{yyyy}; T') \le c.$$

To finish the proof consider the set of initial-boundary value problems (55), (56), (3), (4). Lemma 3.2 imply that for any $h \in (0, 1]$ there exists a solution to such a problem $u_h \in X^{2,\psi(x)}(\Pi^+_{t_0(h)})$. Then with the use of estimate (65) we first extend this solution to the whole time segment $[0, T]$ and then similarly to the end of the proof of the previous theorem pass to the limit as $h \to +\infty$ and construct the desired solution. Note, that here due to (69) $g(u)\phi_x \in L_1(\Pi^+_T)$ $\forall p$ without any additional assumptions on the weight function ψ.□

4. Uniqueness of solutions. The following is the appropriate text. In the following section we give proof of the uniqueness of the solutions in the first two theorems.

Theorem 4.1. *Let $p \in [0, 3]$ in (7), $\psi(x)$ be an admissible weight function, such that $\psi'(x)$ is also an admissible weight function and inequality (8) be verified. Then for any $T > 0$ and $M > 0$ there exists a constant $c = c(T, M)$, such that for any two weak solutions $u(t, x, y)$ and $\widetilde{u}(t, x, y)$ to problem (1) – (4), satisfying $\|u\|_{X^{\psi(x)}_\omega}, \|\widetilde{u}\|_{X^{\psi(x)}_\omega} \le M$ with corresponding data $u_0, \widetilde{u}_0 \in L^{\psi(x)}_{2,+}$, $f, \widetilde{f} \in L_1(0, T; L^{\psi(x)}_{2,+})$ the following inequality holds:*

$$\|u - \widetilde{u}\|_{X^{\psi(x)}_\omega} \le c(\|u_0 - \widetilde{u}_0\|_{L^{\psi(x)}_{2,+}} + \|f - \widetilde{f}\|_{L_1(0,T;L^{\psi(x)}_{2,+})}). \tag{70}$$

Proof . Let $\omega \equiv u - \widetilde{u}$, $\omega_0 \equiv u_0 - \widetilde{u}_0$, $F \equiv f - \widetilde{f}$. For the function ω apply Lemma 3.1, where $f_1 \equiv 0$. Note that inequality (8) implies that $(\psi/\psi')^{1/4} \le c(\psi')^{p/4}\psi^{p/4}$, thus

$$\begin{aligned}
(\iint |u|^{2p} u_x^2 \psi\, dxdy)^{1/2} &\le \||u|^p(\psi/\psi')^{1/4}\|_{L_{\infty,+}} [\iint u_x^2 (\psi'\psi)^{1/2} dxdy]^{1/2} \\
&\le c\|u(\psi')^{1/4}\psi^{1/4}\|^p_{L_{\infty,+}} \|u(\psi')^{1/4}\psi^{1/4}\|^2_{L_{2,+}} \\
&\le c_1(\iint (u_{xx}^2 + u_{yy}^2 + u^2)\psi' dxdy)^{p/4+1/4} (\iint u^2\psi\, dxdy)^{p/4+1/4},
\end{aligned} \tag{71}$$

so $g'(u)u_x \in L_1(0, T; L^{\psi(x)}_{2,+})$, since $p \le 3$.

As a result, we derive from (43) that for $t \in (0, T]$

$$\begin{aligned}
&\iint \omega^2\psi\, dxdy + \psi(0)\int_0^L \mu_2^2\big|_{x=0} dy + \int_0^t \iint [5\omega_{xx}^2 + \omega_{yy}^2 + 3b\omega_x^2 + \omega_y^2 - a\omega^2]\psi' dxdyd\tau \\
&\le \iint \omega_0^2 \psi\, dxdy + c\int_0^t \iint \omega^2\psi\, dxdyd\tau + 2\int_0^t \iint (F - (g'(u)u_x - g'(\widetilde{u})\widetilde{u}_x))\omega\psi\, dxdtd\tau.
\end{aligned} \tag{72}$$

Where

$$2|\int_0^t \iint (g'(u) - g'(\widetilde{u})\widetilde{u}_x)\omega\psi\, dxdt| = 2|\int_0^t \iint (g(u) - g(\widetilde{u})(\omega\psi)_x dxdt| \le c\iint (|u|^p + |\widetilde{u}|^p)|\omega(\omega\psi)_x| dxdy, \tag{73}$$

where similarly to (71)

$$\begin{aligned}
&\iint |u|^p |\omega\omega_x| \psi' dxdy \le \||u|^p(\psi/\psi')^{1/4}\|_{L_{\infty,+}} (\iint \omega_x^2 (\psi')^{1/2}\psi^{1/2} dxdy)^{1/2} (\iint \omega^2\psi\, dxdy)^{1/2} \\
&\le c(\iint (u_{xx}^2 + u_{yy}^2 + u^2)\psi' dxdy)^{p/4} (\iint u^2\psi\, dxdy)^{p/4} (\iint (\omega_{xx}^2 + \omega_{yy}^2 + \omega^2)\psi' dxdy)^{1/4} (\iint \omega^2\psi\, dxdy)^{3/4} \\
&\le \varepsilon \iint (\omega_{xx}^2 + \omega_{yy}^2 + \omega^2)\psi' dxdy \\
&+ c(\varepsilon)(\iint (u_{xx}^2 + u_{yy}^2 + u^2)\psi' dxdy)^{p/3} \iint \omega^2\psi\, dxdy,
\end{aligned} \tag{74}$$

where $\varepsilon > 0$ can be chosen arbitrarily small. Then inequalities (72), (74) provide the desired result.□

The next theorem provides the uniqueness part of Theorem 1.2.

Theorem 4.2. *Let the function $g \in C^2(\mathbb{R})$ verifies condition (9). Let $\psi(x)$ be an admissible weight function, such that $\psi'(x)$ is also an admissible weight function and condition (10) holds. Then for any $T > 0$ and $M > 0$ there exists a constant $c = c(T, M)$, such that for any two strong solutions $u(t, x, y)$ and $\widetilde{u}(t, x, y)$ to problem (1) – (4), satisfying*

$\|u\|_{X_\omega^{2,\psi(x)}(\Pi_T^+)}$, $\|\widetilde{u}\|_{X_\omega^{2,\psi(x)}(\Pi_T^+)} \le M$, *with the corresponding data* $u_0, \widetilde{u_0} \in L_{2,+}^{\psi(x)}$, $f, \widetilde{f} \in L_1(0,T; L_{2,+}^{\psi(x)})$ *inequality (70) holds.*

Proof. The proof mostly repeats the proof of Theorem 4.1. Note that here obviously $g'(\widetilde{u})u_x$, $g'(\widetilde{u})\widetilde{u}_x \in L_\infty(0,T; L_{2,+}^{\psi(x)})$, thus equality (72) holds. The difference is related only to the nonlinear term. In comparison with (73) we estimate it in the following way: since

$$g'(u)u_x - g'(\widetilde{u})\widetilde{u}_x = (g'(u) - g(\widetilde{u}))u_x + g'(\widetilde{u})\omega_x, 2|\iint (g'(u)u_x - g'(\widetilde{u})\widetilde{u}_x)\omega\psi dxdy|$$
$$= |2\iint (g'(u) - g'(\widetilde{u}))u_x\omega\psi dxdy - \iint g''(\widetilde{u})\widetilde{u}_x\omega^2\psi dxdy - \iint g'(\widetilde{u})\omega^2\psi' dxd| \tag{75}$$
$$\le c\iint (|u|^q + |\widetilde{u}|^q)(|u_x|^q + |\widetilde{u}_x|^q)\omega^2\psi dxdy + c\iint \omega^2\psi dxdy.$$

By virtue of (10) $\psi \le c\psi^{(q+1)/2}(\psi')^{(r-2)/(2r)}\psi^{(r+2)/(2r)}$

$$\iint |u|^q|u_x|\omega^2\psi dxdy \le c\iint |u|^q\psi^{q/2}|u_x|\psi^{1/2}\omega^2(\psi')^{2s_0}\psi^{1-2s_0}dxdy$$
$$\le \|u\psi^{1/2}\|^q_{L_{\infty,+}}\|u_x\psi^{1/2}\|_{L_{\frac{r}{r-2},+}}\|\omega(\psi')^{s_0}\psi^{1/2-s_0}\|^2_{L_{r,+}}$$
$$\le c\|u\|^{q+1}_{\widetilde{H}_+^{2,\psi(x)}}(\iint (\omega_{xx}^2 + \omega_{yy}^2 + \omega^2)\psi' dxdy)^{2s_0}(\iint \omega^2\psi dxdy)^{1-2s_0}$$
$$\le \varepsilon\iint (\omega_{xx}^2 + \omega_{yy}^2 + \omega^2)\psi' dxdy + c(\varepsilon)\iint \omega^2\psi dxdy,$$

where $s_0(r) = \frac{1}{4} - \frac{1}{2r} \le \frac{1}{2}$ and $2 \le \frac{r}{r-2} < +\infty$. The desired result obtained from (72) and (75). □

Theorem 4.3. *Let the function* $g \in C^2(\mathbb{R})$ *verifies condition (9). Let* $\psi(x)$ *be an admissible weight function, such that* $\psi'(x)$ *is also an admissible weight function and for certain positive constant*

$$\psi'(x)\psi^q(x) \ge c_0 \quad \forall x \ge 0. \tag{76}$$

Then for any $T > 0$ *and* $M > 0$ *there exists constant* $c = c(T,M)$ *such that for any two strong solutions* $u(t,x,y)$ *and* $\widetilde{u}(t,x,y)$ *to problem (1) – (4), satisfying* $\|u\|_{X_\omega^{2,\psi(x)}}, \|\widetilde{u}\|_{X_\omega^{2,\psi(x)}} \le M$, *with corresponding data* $u_0, \widetilde{u}_0 \in H_+^{1,\psi(x)}$, $f, \widetilde{f} \in L_2(0,T; \widetilde{H}_+^{2\psi(x)})$, $u_0(0,y) = \widetilde{u}_0(0,y) \equiv 0$, *the following inequality holds:*

$$\|u - \widetilde{u}\|_{X_\omega^{2\psi(x)}(\Pi_T^+)} \le c(\|u_0 - \widetilde{u}_0\|_{H_+^{2,\psi(x)}} + \|f - \widetilde{f}\|_{L_2(0,T;H_+^{2,\psi(x)})}).$$

Proof. First of all note that the hypothesis of Theorem 4.2 is satisfied and, consequently, inequality (70) holds.

Let $g_1'(u) \equiv g'(u) - g'(0)$, then according to (9)

$$|g_1'(u)| \le c|u|^{q+1}. \tag{77}$$

Adjoin the term $g'(0)u_x$ to the linear term au_x and consider an equation of (1) type, where g' is substituted by g_1'. Condition (76) implies that

$$\frac{\psi^2(x)}{\psi'(x)} \le c\psi^{q+2}(x). \tag{78}$$

In particular it means that $g_1'(u)u_x, g_1'(\widetilde{u})\widetilde{u}_x \in L_\infty(0,T; L_{2,+}^{\psi^2/\psi'(x)})$. Write corresponding analog of (48) for $\omega \equiv u - \widetilde{u}$ and $f_1 \equiv g_1'(u)u_x - g_1'(\widetilde{u})\widetilde{u}_x$, then

$$\iint (\omega_{xx}^2 + \omega_{yy}^2 + b\omega_x^2 + b\omega_y^2)\psi dxdy + \int_0^t \iint (5u_{xxxx}^2 + 6u_{xxyy}^2 + u_{yyyy}^2)\psi' dxdyd\tau$$
$$\le \iint (\omega_{0xx}^2 + \omega_{0yy}^2 + b\omega_{0x}^2 + b\omega_{0y}^2)\psi dxdy + c\int_0^t \iint (g_1'(u)u_x - g_1'(\widetilde{u})\widetilde{u}_x)^2\frac{\psi^2}{\psi'}dxdyd\tau$$
$$\varepsilon\int_0^t \iint (\omega_{xxxx}^2 + \omega_{xxyy}^2 + \omega_{yyyy}^2)\psi' dxdyd\tau + c(\varepsilon)\int_0^t \iint (\omega_{xx}^2 + \omega_{yy}^2 + \omega^2)\psi dxdyd\tau$$
$$+c\int_0^t \iint (F_{xx}^2 + F_{yy}^2 + F^2)\psi dxdyd\tau,$$

To estimate the integral with the nonlinear term apply (77), (78) and the corresponding analogue of (75)

$$\iint (g_1'(u)u_x - g_1'(\widetilde{u})\widetilde{u}_x)^2\frac{\psi^2}{\psi'})dxdyd \le c\iint (|u|^{2q} + |\widetilde{u}|^{2q})u_x^2\omega^2\psi^{q+2}dxdy + c\iint |\widetilde{u}|^{2q+2}\omega_x^2\psi^{q+2}dxdy, \tag{79}$$

where

$$\iint |u|^{2q}u_x^2\omega^2\psi^{q+2}dxdy \le \|u\psi^{1/2}\|_{L_{\infty,+}}\|u_x\psi^{1/2}\|_{L_{6,+}}\|\omega\psi^{1/2}\|_{L_{3,+}}$$

$$\le c\|u\|^{2q+2}_{\widetilde{H}_+^{2,\psi(x)}} \iint (\omega_{xx}^2+\omega_{yy}^2+b\omega_x^2+b\omega_y^2)dxdy, \iint |\widetilde{u}|^{2q+2}\omega_x^2\omega^2\psi^{q+2}dxdy$$

$$\le \|u\psi^{1/2}\|^{2q+2}_{L_{\infty,+}} \iint \omega_x^2\psi dxdy.$$

The statement of the theorem follows from inequality (79). □

5. Large-time decay of solutions. Now, we proof last two theorems and establish large-time decay of solutions.

Proof of Theorem 1.3. Let $\psi(x) \equiv e^{2\alpha x}$ for $\alpha \in (0, \alpha_0]$, will be specified later, $u_0 \in L_{2,+}^{\psi(x)}$, $f \equiv 0$. Consider the unique solution to problem (1) – (4) from the space $X_\omega^{\psi(x)}(\Pi_T^+)$ $\forall T$.

Note that according to (71) $g'(u)u_x \in L_1(0,T;L_{2,+}^{\psi(x)})$.

Apply Lemma 3.1, where $f_0 \equiv g'(u)u_x$, $f_1 \equiv 0$, then equality (57) for $\rho \equiv 1$ provides, that

$$\|u(t,\cdot,\cdot)\|_{L_{2,+}} \le \|u_0\|_{L_{2,+}} \quad \forall t \ge 0.$$

Equality (57) for $\rho = \psi$ implies that

$$\frac{d}{dt}\iint u^2\psi dxdy + \int_0^L \mu_2^2 dy + 2\alpha \iint [5u_{xx}^2 + u_{yy}^2 + (3b+4\alpha^2)u_x^2 + bu_y^2 + (4\alpha^2 b + 16\alpha^4 - a)u^2]\psi dxdy$$
$$= 2\alpha \iint (g'(u)u)^*\psi dxdy \tag{80}$$

With the use of inequalities (59) and (60) we derive that uniformly with respect to L for certain constant c^* depending on the properties of the function g,

$$2\iint (g'(u)u)^*\psi dxd \le c\Big(\iint (u_{xx}^2+u_{yy}^2+u^2)\psi dxdy\Big)^{p/4}\Big(\iint u^2\psi dxdy\Big)^{(4-p)/4}\|u_0\|^p_{L_{2,+}}$$
$$\le \frac14 \iint (u_{xx}^2+u_{yy}^2)\psi dxdy + c^*(\|u_0\|^{(4p)/(4-p)}_{L_{2,+}} + \|u_0\|^p_{L_{2,+}})\iint u^2\psi dxdy. \tag{81}$$

It follows from (22) that

$$\iint u^2\psi dxdy \le \frac{L^2}{\pi^2}\iint u_y^2\psi dxdy \le \frac{L^2}{\pi^2}\Big(\iint u^2\psi dxdy\Big)^{1/2}\Big(\iint u_{yy}^2\psi dxdy\Big)^{1/2},$$

and, so

$$\frac{\pi^4}{L^4}\iint u^2\psi dxdy \le \iint u_{yy}^2\psi dxdy. \tag{82}$$

In particular

$$2\alpha\iint u_{yy}^2\psi dxdy \ge \frac{\pi^4\alpha}{4L^4}\iint u^2\psi dxdy + \frac{7\alpha}{4}\iint u_{yy}^2 dxdy. \tag{83}$$

Moreover,

$$|3b+4\alpha^2|\iint u_x^2\psi dxdy \le \iint u_{xx}^2\psi dxdy + c(b,\alpha_0)\iint u^2\psi dxdy, \tag{84}$$

$$2|b|\iint u_y^2\psi dxdy \le \frac14\iint u_{yy}^2\psi dxdy + c(b)\iint u^2\psi dxdt \tag{85}$$

Combining (80) – (85) we find that

$$\frac{d}{dt}\iint u^2\psi dxdy + \int_0^L \mu_2^2 dy$$
$$+\alpha\iint (u_{xx}^2+u_{yy}^2)\psi dxdy + \alpha\Big[\frac{\pi^4}{4L^4} - c(b,a,\alpha_0) - c^*(\|u_0\|^{4p/(4-p)}_{L_{2,+}} + \|u_0\|^p_{L_{2,+}})\Big]\iint u^2\psi dxdy \le 0. \tag{86}$$

Choose L_0, α_0 and ϵ_0, such that $\frac{\pi^4}{16L_0^4} \ge c^*(\epsilon_0^{4p/(4-p)} + \epsilon_0^p)$, $\frac{\pi^4}{16L_0^4} \ge c(b,a,\alpha_0)$. Then it follows from (86) that

$$\frac{d}{dt}\iint u^2\psi dxdy + \int_0^L \mu_2^2 dy + \alpha\iint (u_{xx}^2+u_{yy}^2)dxdy + \alpha\beta\iint u^2 dxdy \le 0. \tag{87}$$

where $\beta = \frac{\pi^4}{8L^4}$. □

Proof of Theorem 1.4. Let the values L_0, α_0, ϵ_0, β be the same as as in the proof of the previous theorem, $\psi(x) \equiv e^{2\alpha x}$ for certain $\alpha \in (0;\alpha_0]$, $u_0 \in \widetilde{H}_+^{2,\psi(x)}$, $u_0(0,y) \equiv u_{0x}(0,y) \equiv 0$, $\|u_0\|_{L_{2,+}} \leq \epsilon_0$. Consider the unique solution to problem (1) – (4) $u \in X_\omega^{2,\psi(x)}(\Pi_T^+)$, $\forall T$. Since $g'(u)u_x \in L_\infty(0,T;L_{2,+}^{\psi(x)})$. Repeat the proof of Theorem 1.3 and obtain (86). Besides (11), it follows from (87) that

$$\int_0^{+\infty} e^{\alpha\beta\tau}\Big[\int_0^L u_{xx}^2\big|_{x=0}dy + \alpha\iint [u_{xx}^2 + \beta u_{yy}^2]\psi dxdy\Big]d\tau \leq \|u_0\|_{L_{2,+}^{\psi(x)}}. \tag{88}$$

Similarly to (67), from (48) and (53) we get (for $\rho \equiv 0$)

$$\frac{d}{dt}\iint (u_{xx}^2 + u_{yy}^2 + bu_x^2 + bu_y^2 - 2g^*(u))\rho dxdy \leq c\int_0^L u_{xx}^2\big|_{x=0}dy,$$

whence with the use of (68) and (88) follows that uniformly with respect to $t \geq 0$

$$\|u_{xx}\|_{L_{2,+}} + \|u_{yy}\|_{L_{2,+}} \leq c,$$

and

$$\|u\|_{L_\infty(\Pi_\infty^+)} \leq c. \tag{89}$$

In (67) let $\rho \equiv \psi$

$$\begin{aligned}
&\frac{d}{dt}\iint (u_{xx}^2 + u_{yy}^2 + bu_x^2 + bu_y^2 - 2g^*(u))\psi dxdy + \int (u_{xxxx}^2)\big|_{x=0}dy \\
&+2\alpha\iint (5u_{xxxx}^2 + 6u_{xxyy}^2 + u_{yyyy}^2)\psi dxdy \\
\leq\ & 2\alpha\iint 2g(u)(u_{xxxx} - bu_{xx} + u_{yyyy} - bu_{yy})\psi dxdy - 4a\alpha\iint g^*(u)\psi dxdy \\
&+\varepsilon\int u_{xxxx}^2\big|_{x=0}dy + c(\varepsilon)\int u_{xx}^2\big|_{x=0}dy + 2\varepsilon\alpha\iint (u_{xxxx}^2 + u_{xxyy}^2 + u_{yyyy}^2)\psi dxdy \\
&+\alpha c(\varepsilon)\iint (u_{xx}^2 + u_{yy}^2)\psi dxdy + 2\iint (g'(u)u_x[4\alpha u_{xxx} + 4\alpha^2 u_{xx} - 2\alpha bu_x]\psi)dxdy \\
&-4\alpha\iint (g'(u)g(u))^*\psi dxdy.
\end{aligned}$$

Inequality (12) follows from (88) and (89). □

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