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FOURIER-BESSEL'S TRANSFORM OF A GENERALIZED FUNCTION VANISHING OUTSIDE A BOUNDED SURFACE

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Let $\mathbb{R}_n^+$ denote an Euclidean space of points $x=(x_1,...,x_n)$, $x_1>0,...,x_n>0$ and the multiindex $\gamma=(\gamma_1,...,\gamma_n)$ runs through fixed positive numbers. The space $S_{ev}(\mathbb{R}_n^+)=S_{ev}$ is the subspace of the Schwartz function space that consists of functions $\varphi(x)$ even in each variable $x_1,...,x_n$. The space of linear continuous functionals, whose regular representatives are generated by the linear weighted form

$$(f,\varphi)_\gamma=\int\limits_{\mathbb{R}_n^+} f(x)\varphi(x)x^\gamma dx, \qquad x^\gamma=\prod_{i=1}^n x_i^{\gamma_i},$$

is called the distribution space over S_{ev} and is denoted by $S'_{ev}(\mathbb{R}_n^+)=S'_{ev}$.

The Fourier-Bessel transform is denoted by formula

$$F_B[f](\xi)=\widehat{f}(\xi)=\int\limits_{\mathbb{R}_n^+} f(x)\mathbf{j}_\gamma(x,\xi)x^\gamma\,dx\,,$$

where $\mathbf{j}_\gamma(x,\xi)=\prod\limits_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\xi_i)$, $j_\nu(t)=\Gamma(\nu+1)\left(\frac{2}{t}\right)^\nu J_\nu(t)$, $t\in\mathbb{R}_1$, $J_\nu(t)$ is Bessel functions of the first kind. Spaces S_{ev} and S'_{ev} are invariant to Fourier-Bessel transform (see [1]).

For the generalized function $f\in S'_{ev}$, vanishing outside a bounded surface $\Omega\subset\mathbb{R}_n^+$ the Fourier-Bessel transform is functional

$$(f(x),\mathbf{j}_\gamma(x,\xi))_\gamma=\int\limits_\Omega f(\sigma)\mathbf{j}_\gamma(x,\xi)x^\gamma dx,$$

which acts as follows: function $\mathbf{j}_\gamma(x,\xi)$ is replaced by a test function $\varphi_0(x,\xi)=\mathbf{j}_\gamma(x,\xi)$ for $x\in\Omega$ and $\varphi_0(x,\xi)=0$ for $x\notin\Omega$, then functional f applies to $\varphi_0(x,\xi)$. The number obtained is independent of choice of this function $\varphi_0(x,\sigma)$.

The Fourier-Bessel transform of any generalized function $f\in S'_{ev}$ vanishing outside a bounded surface for any test function $\psi(x)\in S_{ev}$ is denoted by formula

$$\int (f(x),\mathbf{j}_\gamma(x,\sigma))_\gamma\,\widehat{\psi}(\sigma)\sigma^\gamma d\sigma=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}(f,\psi)_\gamma.$$

We shall introduce a singular generalized weighted function (compare with construction in [2] page 247)

$$(\delta_\gamma(r-a),\varphi)_\gamma=\int\limits_{S_n^+(a)}\varphi(x)x^\sigma dS,\qquad \varphi(x)\in S_{ev}.$$

The Fourier-Bessel transform of $\delta_\gamma(r-R)$ is calculated according to the formula:

$$F_B[\delta_\gamma(r-R)](\xi)=\int\limits_{S_n^+(R)}\mathbf{j}_\gamma(x,\xi)x^\gamma\,dS_R=R^{n+|\gamma|-1}|S_n^+(1)|_\gamma j_{\frac{n+|\gamma|-2}{2}}(R|\xi|). \tag{1}$$

Following [3] we introduce the operator Δ_γ:

$$\Delta_\gamma=\sum_{i=1}^n B_{\gamma_i},$$

where B_{γ_i} is Bessel operator:

$$B_{\gamma_i}=\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i},\quad i=1,...,n.$$

The formula (1) can be used for solving a problem

$$\frac{\partial^2 u}{\partial t^2}+\frac{\alpha}{t}\frac{\partial u}{\partial t}=\triangle_\gamma u(x,t), \tag{2}$$

$$u(x,0)=0,\quad u_t(x,0)=\delta_\gamma(x). \tag{3}$$

For $0<n+|\gamma|-\alpha<3$, $|\gamma|=\gamma_1+\ldots+\gamma_n$ the solution of (2)-(3) is

$$u(x,t)=C_{\alpha,\gamma}(n)t^{1-n-|\gamma|}\,{}_2F_1\left(\frac{n+|\gamma|-\alpha}{2},\frac{n+|\gamma|-1}{2},\frac{n+|\gamma|}{2};\frac{|x|^2}{t^2}\right),$$

where

$$C_{\alpha,\gamma}(n)=2^{n+|\gamma|-\alpha-2}|S_1^+(n)|_\gamma\frac{\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\frac{n+|\gamma|-\alpha}{2}\right)}{\Gamma\left(\frac{3-n-|\gamma|}{2}\right)}.$$

References

1. Lyakhov L.N. Multipliers of the mixed Fourier–Bessel transform // Tr. Mat. Inst. Steklova. – 1997. – 17, V.214. – P.234–249.

2. Gel'fand, I.M., Shilov, G.E. Generalized functions. Vol. I: Properties and operations / Boston: Academic Press, 1964.

3. Kipriyanov I.A. Singular Elliptic Boundary Value Problems / Moscow: Nauka, 1996.

ПРЕОБРАЗОВАНИЕ ФУРЬЕ-БЕССЕЛЯ ОБОБЩЕННОЙ ФУНКЦИИ ИСЧЕЗАЮЩЕЙ ВНЕ ОГРАНИЧЕННОЙ ПОВЕРХНОСТИ

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