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DETERMINATION OF SOURCE AND VARIABLE COEFFICIENT IN THE INVERSE PROBLEM FOR THE WAVE'S EQUATION

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Resume. Investigated the one-dimensional inverse problem with an unknown source and an unknown variable coefficient on two known solutions at fixed points in the plane. Inverse problem using the method of integral transforms is reduced to inverse problem of Sturm-Liouville problem. The uniqueness theorems are proved.

Keywords: integral transforms, operator, bilinear system of integral equations, uniqueness theorems.

1. Introduction

Inverse problems of mathematical physics are of interest to many disciplines: geophysics, seismic, acoustic, radar, and medicine. Variously statements and methods of solution of inverse problems. The actual problem is the determining the variable coefficients of the partial differential equations. A wide class of inverse problems are problems of interpretation of observational data, in which the results of the field measurements required to determine the sources of the fields or elements of the distribution environment [1].

The investigated problem is reduced to inverse problem of Sturm-Liouville problem. The original formulation is reduced to spectral by method of integral transformations, and solve the system of nonlinear integral equations containing unknown functions. The properties of the required functions are assumed set. Solution of inverse problems closely connected with the spectral theory of differential operators studied by M. M. Lavrentiev, V. G. Yakhno, K. G. Reznitskaya [2], A.S.Alekseev [3], M. M. Lavretiev and K.G. Reznitskaya [4].

The most similar problem formulation published in [2]. Closest to the topic of the research tasks are solved by reseach-workers V.G. Romanov [5], M. M.Lavrentiev et al.[2].

The purpose of the work is the building constructive solution of the problem; description of classes of functions, in which the recovered solution; the proof of the uniqueness theorems.

The investigated problem is of practical importance as a model for interpretation of seismic data and exploration. Algorithms can be useful in the numerical solution of inverse problems in special classes of functions. The obtained results can be used in further studies on the theory of inverse problems.

2. Statement of the problem

Let in the domain $-\infty < x, y < \infty,\ z \geq 0,\ t > 0$ is the wave equation

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - q(z)U(x,y,z,t). \tag{1}$$

Generalized function $U(x,y,z,t)$ is equal to zero when $t < 0$ and satisfies the equation (1), initial data

$$U(x,y,z,t)\Big|_{t=+0} = 0, \tag{2}$$

$$U_t'(x,y,z,t)\Big|_{t=+0} = 0, \tag{3}$$

and boundary condition

$$U_z'(x,y,z,t)\Big|_{z=0} = \delta(t)\cdot f\left(\sqrt{x^2+y^2}\right) = \delta(t)\cdot f(\rho). \tag{4}$$

$\delta(t)$ − the Delta function of Dirac with the carrier at the point $t = t_0$,

$\delta(t) = \lim_{t_0 \to +0} \delta(t - t_0)$. $f\left(\sqrt{x^2 + y^2}\right) = f(\rho)$ is an unknown real function: finite, positively defined and n times differentiable in the interval $(0, \varepsilon) \ni \rho$. $f(\rho) = 0$ when $\rho \geq \varepsilon$. $f(\rho)$ satisfies the Dirichlet conditions. We assume that the set of functions $f(\rho)$ with these properties is the class $\boldsymbol{\Phi}$.

$U(x, y, z, t)$ – is a generalized solution of the boundary problem (1)-(4) belongs to the class $\mathcal{U}$ [2].

For information about the solution of the direct problem in two fixed points in the plane $z = 0$, such that $\sqrt{(x^j)^2 + (y^j)^2} = \rho^j$, $j = 0,1$,

$$U(x^j, y^j, 0, t) = \varphi^j(t), \ j = 0,1, \tag{5}$$

need to identify unknown potential $q(z)$ in the class of functions $\boldsymbol{Q_M^a}$ and the unknown source $f(\rho)$ in the class of functions $\boldsymbol{\Phi}$.

The class of functions $\boldsymbol{Q_M^a}$ [9] contains all functions $q(z)$, having the properties:

1. $q(z) \in C^1[0, \infty) \cap L_1[0, \infty)$, $\|q(z)\|_{L_1[0,\infty)} \leq M$.

2. $q(z)$ has an absolute minimum $q_{\min\, abs} = q(b^*) = m < 0$.

3. For the large values of the argument $z \geq b^*$ $q(z)$ takes negative values and monotonically tends to zero: $q(z) = o\left(-\frac{1}{z^2}\right)$, $z \to \infty$.

4. A sequence of elements of a linear normed space $L_1[0, \infty)$

$$q_n(z) = \begin{cases} q(z), & if \ z \in [0, b_n], \\ 0, & if \ z \in (b_n, \infty) \end{cases}$$

is converged in the space $L_1[0, \infty)$ to an element of this space $q(z)$ by the norm:

$$\lim_{n \to \infty} \|q_n(z) - q(y)\|_{L_1[0,\infty)} = 0.$$

5. $q(z)$ – entire function such that $q(0) = A > 0$.

3. Solution of the direct problem

The solution of the direct problem will be conduct by method of integral transformations, described in [2]. Apply to problem (1)-(4) two-dimensional Fourier transform in the variables x and y. Let's introduce the designation

$$W_1(p, s, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ipx + isy}\, U(x, y, z, t) dx dy. \tag{6}$$

Get the auxiliary problem in the form

$$\frac{\partial^2 W_1}{\partial t^2} = -(p^2 + s^2) W_1(p, s, z, t) + \frac{\partial^2 W_1}{\partial z^2} - q(z) W_1(p, s, z, t), \tag{7}$$

$$W_1(p, s, z, t)\Big|_{t = +0} = 0, \tag{8}$$

$$[W_1]'_t(p, s, z, t)\Big|_{t = +0} = 0, \tag{9}$$

$$[W_1]'_z(p, s, z, t)\Big|_{z = 0} = \frac{1}{2\pi} \delta(t) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ipx + isy} f(x, y) dx dy = \delta(t) A(p, s), \tag{10}$$

where
$$A(p, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ipx + isy} f(x, y) dx dy. \tag{11}$$

To problem (7)-(10) we apply the Laplace transform on the variable t

$$W_2(p, s, z, r) = \int_0^{\infty} e^{-rt}\, W_1(p, s, z, t) dt. \tag{12}$$

Get the ordinary differential equation

$$(p^2 + s^2 + r^2) W_2(p, s, z, r) = \frac{\partial^2 W_2}{\partial z^2} - q(z) W_2(p, s, z, r) \tag{13}$$

with the initial condition

$$[W_2]'_z(p,s,z,r)\Big|_{z=0} = A(p,s). \tag{14}$$

Problem (13)-(14) will be reduced to a functional equation, if we use the generalized Fourier transform

$$W_3(p,s,\lambda,r) = \int_0^\infty \varphi(z,\lambda)\, W_2(p,s,z,r)dz$$

on the system of eigenfunctions of Sturm-Liouville operator.

$$W_3(p,s,\lambda,r) = -\frac{1}{p^2+s^2+r^2+\lambda}\cdot A(p,s).$$

Perform the inverse transform

$$W_2(p,s,z,r) = -A(p,s)\int_0^\infty \frac{1}{p^2+s^2+r^2+\lambda}\varphi(z,\lambda)d\sigma(\lambda) =$$

$$= \int_0^\infty e^{-rt}\{-A(p,s)\int_0^\infty \varphi(z,\lambda)\frac{1}{\sqrt{p^2+s^2+\lambda)}}\sin\left(t\sqrt{p^2+s^2+\lambda)}\right)d\sigma(\lambda)\}\,dt. \tag{15}$$

Compare different kind of image $W_2(p,s,z,r)$ – (12) and (15). According to a theorem of Lerch, two functions with the same Laplace transform coincide for all $t>0$, where both functions are continuous. The Laplace transform is unique for each function $W_1(p,s,z,t)$ having this conversion. So

$$W_1(p,s,z,t) = -A(p,s)\int_0^\infty \varphi(z,\lambda)\frac{\sin\left(t\sqrt{p^2+s^2+\lambda)}\right)}{\sqrt{p^2+s^2+\lambda)}}d\sigma(\lambda).$$

The treatment of the two-dimensional Fourier transform (4) and using the integral representation (11) $A(p,s)$ gives the solution of the direct problem:

$$U(x,y,z,t) =$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ipx-isy}A(p,s)\left\{\int_0^\infty \varphi(z,\lambda)\frac{\sin\left(t\sqrt{p^2+s^2+\lambda)}\right)}{\sqrt{p^2+s^2+\lambda)}}d\sigma(\lambda)\right\}dpds$$

$$= -\frac{1}{4\pi^2}\cdot\int_0^\infty \varphi(z,\lambda)\cdot[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tilde{x},\tilde{y})\cdot\{\int_{-\infty}^{\infty} e^{ip(\tilde{x}-x)}\cdot\pi J_0(\sqrt{p^2+\lambda}\sqrt{t^2-(y-\tilde{y})^2})\cdot$$

$$\theta(t^2-(y-\tilde{y})^2)dp\}d\tilde{x}d\tilde{y}]d\sigma(\lambda) = -\frac{1}{4\pi^2}\int_0^\infty \varphi(z,\lambda)\cdot[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tilde{x},\tilde{y})\cdot 2\pi$$

$$\frac{\cos\left(\sqrt{\lambda}\sqrt{t^2-(x-\tilde{x})^2-(y-\tilde{y})^2}\right)}{\sqrt{t^2-(x-\tilde{x})^2-(y-\tilde{y})^2}}\theta(t^2-(x-\tilde{x})^2-(y-\tilde{y})^2)d\tilde{x}d\tilde{y}]d\sigma(\lambda).$$

After the change of the variables $x-\tilde{x}=X$, $y-\tilde{y}=Y$ we write the solution of the direct problem $U(x,y,z,t)$ as convolution of an unknown source $f(x,y)$ and the fundamental solution of the problem for equation (1) with the known centered source

$$\delta(t)\delta(x,y).$$

$$U(x,y,z,t) = \frac{1}{2\pi}\int_0^\infty \varphi(z,\lambda)\left\{f(x,y) * \frac{\cos\left(\sqrt{\lambda}\sqrt{t^2-x^2-y^2}\right)}{\sqrt{t^2-x^2-y^2}}\theta(t^2-x^2-y^2)\right\}\cdot d\sigma(\lambda) =$$

$$= \frac{1}{2\pi}\int_0^\infty \varphi(z,\lambda)[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-X,y-Y)\frac{\cos\left(\sqrt{\lambda}\sqrt{t^2-X^2-Y^2}\right)}{\sqrt{t^2-X^2-Y^2}}\theta\left(t^2-X^2-Y^2\right)dXdY]d\sigma(\lambda).$$

4. Solution of the inverse problem. Derivation of the recurrence formula. Uniqueness theorems

On the plane $z = 0$ put $x^0 = 0,\ y^0 = 0$ и $(x^1)^2 + (y^1)^2 = (\rho^1)^2$.
Consider the solution of the direct problem (3)-(5) at two fixed points on the plane $z = 0$.

$$\varphi^j(t) = U(x^j, y^j, 0, t)$$

$$= \frac{1}{2\pi} \cdot \int_0^\infty d\sigma(\lambda) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x^j - X, y^j - Y) \frac{\cos\left(\sqrt{\lambda}\sqrt{t^2 - X^2 - Y^2}\right)}{\sqrt{t^2 - X^2 - Y^2}} \cdot$$

$$\theta(t^2 - X^2 - Y^2) dX dY,\ j = 0{,}1. \tag{16}$$

Let's move on to polar coordinates by the formulas $x = \rho cos\vartheta,\ y = \rho sin\vartheta$.
The system of integral equations (16) write in the new characters:

$$\begin{cases} \varphi^0(t) = \int_0^\infty d\sigma(\lambda) \int_{-\infty}^{\infty} f(\rho) \frac{\cos\left(\sqrt{\lambda}\sqrt{t^2-\rho^2}\right)}{\sqrt{t^2-\rho^2}} \theta(t^2 - \rho^2) \rho d\rho, \\ \varphi^1(t) = \int_0^\infty d\sigma(\lambda) \int_{-\infty}^{\infty} f(\rho^1 - \rho) \frac{\cos\left(\sqrt{\lambda}\sqrt{t^2-\rho^2}\right)}{\sqrt{t^2-\rho^2}} \theta(t^2 - \rho^2) \rho d\rho. \end{cases} \tag{17}$$

To given system apply the Fourier transform on the variable t with parameter α. In the transformations below we use the well-known formula

$$\int_0^\infty e^{it\alpha} \frac{\cos\left(\sqrt{\lambda}\sqrt{t^2-\rho^2}\right)}{\sqrt{t^2-\rho^2}} \theta(t^2 - \rho^2) dt = i\pi signa\alpha J_0(\rho\sqrt{\alpha^2 - \lambda}) \cdot \theta(\alpha^2 - \lambda);$$

and the formulas connecting the spherical and cylindrical functions [6]. Let's introduce the designation $\Phi^0(\alpha) = \int_0^\infty e^{it\alpha} \varphi^0(t) dt$. Then

$$\Phi^0(\alpha) = \int_0^\infty d\sigma(\lambda) \int_0^\infty \rho f(\rho) \{ \int_0^\infty e^{it\alpha} \frac{\cos(\sqrt{\lambda}\sqrt{t^2 - \rho^2})}{\sqrt{t^2 - \rho^2}} \theta(t^2 - \rho^2) dt \} d\rho =$$

$$\int_0^\infty \left[\frac{2}{\pi} d\sqrt{\lambda} + d\sigma_1(\lambda)\right] \int_0^\infty \rho f(\rho) \{ i\pi signa\alpha J_0(\rho\sqrt{\alpha^2 - \lambda}) \cdot \theta(\alpha^2 - \lambda) \} d\rho =$$

$$2isigna\alpha \int_0^\infty \rho f(\rho) \{ \int_0^\infty J_0\left(\rho\sqrt{\alpha^2 - \lambda}\right) \cdot \theta(\alpha^2 - \lambda) d\sqrt{\lambda} \} d\rho +$$

$$i\pi signa\alpha \int_0^\infty \rho f(\rho) \{ \int_0^\infty J_0(\rho\sqrt{\alpha^2 - \lambda}) \cdot \theta(\alpha^2 - \lambda) d\sigma_1(\lambda) \} d\rho =$$

$$2isigna\alpha \int_0^\infty f(\rho) \sin(\rho|\alpha|)\, d\rho$$

$$+ i\pi signa\alpha \int_0^\infty \rho f(\rho) \{ \int_0^\infty J_0(\rho\sqrt{\alpha^2 - \lambda}) \cdot \theta(\alpha^2 - \lambda) d\sigma_1(\lambda) \} d\rho.$$

Similarly, we write the Fourier transform of the second equation of system (17). Note that for all $\alpha > 0$ true equality $signa\alpha \sin(\rho|\alpha|) = \sin(\rho\alpha)$.

$$\Phi^1(\alpha) = \int_0^\infty e^{it\alpha} \varphi^1(t) dt = 2i \int_0^\infty f(\rho_1 - \rho) \sin(\rho\alpha)\, d\rho +$$

$$i\pi signa\alpha \int_0^\infty \rho f(\rho^1 - \rho) \{ \int_0^\infty J_0(\rho\sqrt{\alpha^2 - \lambda}) \cdot \theta(\alpha^2 - \lambda) d\sigma_1(\lambda) \} d\rho.$$

Denote the kernel of the integral equations as $K(\rho, \alpha)$ and make the change of variables $\sqrt{\alpha^2 - \lambda} = \nu,\ \alpha^2 - \nu^2 = \lambda,\ d\lambda = -2\nu d\nu$. Then

$$K(\rho, \alpha) = -\frac{1}{2} signa\alpha \int_0^\infty J_0\left(\rho\sqrt{\alpha^2 - \lambda}\right) \cdot \theta(\alpha^2 - \lambda) d\sigma_1(\lambda) signa\alpha \int_{-\alpha}^{\alpha} J_0(\rho\nu) \sigma_1'(\alpha^2 - \nu^2) \nu d\nu.$$

The solution of the inverse problem with an unknown source as a function of distance reduces to solving a bilinear system of integral equations

$$\begin{cases} \Phi^0(\alpha) = 2i\int_0^\infty f(\rho)\sin(\rho\alpha)\,d\rho - 2\pi i\int_0^\infty \rho f(\rho)K(\rho,\alpha)d\rho, \\ \Phi^1(\alpha) = 2i\int_0^\infty f(\rho^1-\rho)\sin(\rho\alpha)\,d\rho - 2\pi i\int_0^\infty \rho f(\rho^1-\rho)\,K(\rho,\alpha)\,d\rho, \end{cases} \tag{18}$$

containing the sought functions $f(\rho)$ and $\sigma_1'(\alpha^2-\nu^2)$. As it's known, [7-8], on the spectral functions $\sigma(\lambda)\in\boldsymbol{\sigma^a}$ unknown potential $q(z)$ is uniquely recovered in the class of functions $\boldsymbol{Q_M^a}$.

The class of functions $\boldsymbol{\sigma^a}$ [8] are spectral functions, answering to descriptions:

1. $\sigma(\lambda) = \lim_{n\to\infty}\sigma_{nn}(\lambda)$ mainly, i.e., at points of continuity $\sigma(\lambda)$.

2. $\sigma(\lambda)$ has the form $\sigma(\lambda) = \begin{cases}\frac{2}{\pi}\sqrt{\lambda}+\sigma_1(\lambda), \text{если } \lambda\ge 0,\\ 0, \text{если } \lambda<0.\end{cases}$

3. $\sigma(\lambda)\in C(\lambda\ge 0)\cap C^1(\lambda>0)$; $\sigma_1(s)$, $s=\sqrt{\lambda}$, is monotonically decreasing on the interval (0, ∞).

4. $\sigma_1(s)$ is absolutely continuous.

5. $\sigma_1'(\lambda)$ is an entire function in the interval $[0,\infty)$.

$\sigma_1'(0) = \lim_{s\to 0+}\sigma_1'(s) = -\frac{2}{\pi}$.

Note the properties of the kernel K (ρ,α).

1. $\lim_{\alpha\to 0}K(\rho,\alpha) = \lim_{\alpha\to 0} signa\int_{-\alpha}^{\alpha}J_0(\rho\nu)\sigma_1'(\alpha^2-\nu^2)\nu d\nu = 0.$

2. $K(\rho,\alpha)$ is continuously for the set of the arguments.

3. $K(\rho,\alpha)$ is continuously differentiable.

$$\lim_{\alpha\to 0}\frac{\partial K(\rho,\alpha)}{\partial\alpha} = \lim_{\alpha\to 0} signa\int_{-\alpha}^{\alpha}J_0(\rho\nu)\sigma_1''(\alpha^2-\nu^2)2\alpha\nu d\nu + 2\alpha J_0(\rho\alpha)\sigma_1'(0) = 0,$$

$$\lim_{\alpha\to 0}\frac{\partial^2 K(\rho,\alpha)}{\partial\alpha^2} = \lim_{\alpha\to 0}\{signa\int_{-\alpha}^{\alpha}J_0(\rho\nu)[\sigma_1'''(\alpha^2-\nu^2)(2\alpha)^2+\sigma_1''(\alpha^2-\nu^2)2]\nu d\nu$$

$$+(2\alpha)^2J_0(\rho\alpha)\sigma_1''(0)+2\left[J_0(\rho\alpha)+\alpha\frac{\partial J_0(\rho,\alpha)}{\partial\alpha}\right]\sigma_1'(0)\} = 2\sigma_1'(0).$$

.While differentiation of the kernel $K(\rho,\alpha)$ on the parameter α and calculating the limiting values of derivatives of even and odd order with respect to $\alpha\to 0$, there are the regularities:

$$\lim_{\alpha\to 0}\frac{\partial^{2n-1}K(\rho,\alpha)}{\partial\alpha^{2n-1}} = 0,\ n=1,2,\dots,\lim_{\alpha\to 0}\frac{\partial^{2n}K(\rho,\alpha)}{\partial\alpha^{2n}} =$$

$$(2n-1)!!\sum_{k=1}^{n-1}\frac{2^k}{[2(n-k)-1]!!}\frac{\partial^{2(n-k)}J_0(\rho\alpha)}{\partial\alpha^{2(n-k)}}\bigg|_{\alpha=0}\sigma_1^{(k)}(0)+2^{2n}(2n-1)!!\,\sigma_1^{(n)}(0)$$

$$= (2n-1)!!\sum_{k=1}^{n-1}(-1)^{n-k}\frac{[2(n-k)]!\,\rho^{2(n-k)}}{2^{2n-3k}[2(n-k)-1]!!\,(n-k)!\,\Gamma(n-k+1)}\sigma_1^{(k)}(0)$$

$$+2^{2n}(2n-1)!!\,\sigma_1^{(n)}(0),\ n\ge 2.$$

Because $\sigma_1'(\lambda)$ is entire in the half-line $\lambda \geq 0$, derive the recurrence formula for calculation of the coefficients of the Taylor series

$$\sigma_1'(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_1^{(n+1)}(0)\lambda^n.$$

To do this, we express $f(\rho)$ from the first equation of system (18), previously multiplying every term on $i\frac{1}{\sqrt{2\pi}}$.

$$-i\frac{1}{\sqrt{2\pi}}\Phi^0(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(\rho)\sin(\rho\alpha)\,d\rho - \sqrt{2\pi}\int_0^{\infty} \rho f(\rho)K(\rho,\alpha)d\rho.$$

Treatment of the sine-Fourier transform of the function $f(\rho)$ gives the representation $f(\rho)$ in the form

$$f(\rho) = -\frac{i}{\pi}\int_0^{\infty}\Phi^0(\alpha)\sin(\rho\alpha)d\alpha + 2\int_0^{\infty}\sin(\rho\alpha)\int_0^{\infty}\hat{\rho} f(\hat{\rho})K(\hat{\rho},\alpha)d\hat{\rho}d\alpha = -i\widehat{\Phi}^0(\rho) + \Omega(\rho), \quad (19)$$

where $\widehat{\Phi}^0(\rho) = \frac{1}{\pi}\int_0^{\infty}\Phi^0(\alpha)\sin(\rho\alpha)\,d\alpha = \frac{1}{\pi}\int_0^{\infty}\sin(\rho\alpha)\int_0^{\infty} e^{it\alpha}\varphi^0(t)dtd\alpha$ is known function, $\Omega(\rho) = \Omega_{[f,\sigma_1']} = 2\int_0^{\infty}\sin(\rho\alpha)\int_0^{\infty}\hat{\rho} f(\hat{\rho})K(\hat{\rho},\alpha)d\hat{\rho}d\alpha -$ sine-transform bilinear quadratic form, that contains the unknown functions $f(\rho)$ and $\sigma_1'(\lambda)$.

Similarly to (19) we express the function

$$f(\rho^1-\rho) = -i\widehat{\Phi}^0(\rho^1-\rho) + \Omega(\rho^1-\rho) \quad (20)$$

and substitute it into the second equation of system (18). Then $\Phi^1(\alpha) =$

$$2i\int_0^{\infty}\{-i\widehat{\Phi}^0(\rho^1-\rho)+\Omega(\rho^1-\rho)\}\sin(\rho\alpha)\,d\rho - 2\pi i\int_0^{\infty}\rho\left\{-i\widehat{\Phi}^0(\rho^1-\rho)+\Omega(\rho^1-\rho)\right\}$$
$$K(\rho,\alpha)\,d\rho = 2\int_0^{\infty}\widehat{\Phi}^0(\rho^1-\rho)\sin(\rho\alpha)\,d\rho + 2i\int_0^{\infty}\Omega(\rho^1-\rho)\sin(\rho\alpha)\,d\rho -$$
$$-2\pi i\int_0^{\infty}\rho\left\{-i\widehat{\Phi}^0(\rho^1-\rho)+\Omega(\rho^1-\rho)\right\}K(\rho,\alpha)\,d\rho \quad (21)$$

We introduce the notation $\overline{\Phi}^0(\rho^1,\alpha) = \int_0^{\infty}\widehat{\Phi}^0(\rho^1-\rho)\sin(\rho\alpha)\,d\rho$ and rewrite the equality (18) in the form $-\frac{i}{2}\Phi^1(\alpha) + i\overline{\Phi}^0(\rho^1,\alpha) =$

$$\int_0^{\infty}\Omega(\rho^1-\rho)\sin(\rho\alpha)\,d\rho + \pi\int_0^{\infty}\rho\left\{i\widehat{\Phi}^0(\rho^1-\rho)-\Omega(\rho^1-\rho)\right\}K(\rho,\alpha)\,d\rho.$$

Differentiate with respect to α the getting equal $2n$ times, and putting $\alpha = 0$, for $n = 1$ we get the moment of quadratic form $M_{\Omega}^{(1)}$ of the first order:

$$M_{\Omega}^{(1)} = \frac{\partial}{\partial\alpha}[-\frac{i}{2}\Phi^1(\alpha) + i\overline{\Phi}^0(\rho^1,\alpha)]\Big|_{\alpha=0} = \int_0^{\infty}\rho\,\Omega(\rho^1-\rho)d\rho. \quad (22)$$

When $n = 2$ we receive

$\sigma_1'(0) = \frac{1}{2\pi}\cdot\frac{\frac{\partial^2}{\partial\alpha^2}[-\frac{i}{2}\Phi^1(\alpha)+i\overline{\Phi}^0(\rho^1,\alpha)]|_{\alpha=0}}{\int_0^{\infty}\rho\left\{i\widehat{\Phi}^0(\rho^1-\rho)-\Omega(\rho^1-\rho)\right\}d\rho}$ or, taking into consideration the calculated moment (22), express $\sigma_1'(0)$ through the integral transforms of given information.

$$\sigma_1'(0) = \frac{1}{2\pi} \cdot \frac{\frac{\partial^2}{\partial\alpha^2}[-\frac{i}{2}\Phi^1(\alpha) + i\overline{\Phi}^0(\rho^1,\alpha)]\Big|_{\alpha=0}}{\int_0^\infty \rho\left\{i\widehat{\Phi}^0(\rho^1-\rho)\right\}d\rho - M_\Omega^{(1)}} =$$

$$\frac{1}{2\pi} \cdot \frac{\frac{\partial^2}{\partial\alpha^2}[-\frac{i}{2}\Phi^1(\alpha) + i\overline{\Phi}^0(\rho^1,\alpha)]\Big|_{\alpha=0}}{\int_0^\infty \rho\left\{i\widehat{\Phi}^0(\rho^1-\rho)\right\}d\rho - \frac{\partial}{\partial\alpha}[-\frac{i}{2}\Phi^1(\alpha) + i\overline{\Phi}^0(\rho^1,\alpha)]\Big|_{\alpha=0}}.$$

The derivative $(2n-1)$-order of the difference well-known integral transformations of information about the solution of direct problem

$[-\frac{i}{2}\Phi^1(\alpha) + i\overline{\Phi}^0(\rho^1,\alpha)]$, calculated with $\alpha = 0$, allows to get a moment of quadratic forms $M_\Omega^{(2n-1)}$ $(n = 1,2, ...)$.

$$M_\Omega^{(2n-1)} = \int_0^\infty \rho^{2n-1}\Omega(\rho^1-\rho)\,d\rho = (-1)^{n+1}\frac{\partial^{(2n-1)}}{\partial\alpha^{(2n-1)}}\left[-\frac{i}{2}\Phi^1(\alpha) + i\overline{\Phi}^0(\rho^1,\alpha)\right]\Bigg|_{\alpha=0}.$$

The derivative $2n$ - order of the difference $[-\frac{i}{2}\Phi^1(\alpha) + i\overline{\Phi}^0(\rho^1,\alpha)]$ with the zero value of the parameter alpha determines uniquely the coefficients $\sigma_1^{(n+1)}(0)$ of the spectral function $\sigma_1'(\lambda)$ in a Taylor series.

$$\frac{\partial^{(2n)}}{\partial\alpha^{(2n)}}[-\frac{i}{2}\Phi^1(\alpha) + i\overline{\Phi}^0(\rho^1,\alpha)] = \pi\int_0^\infty \rho\left\{i\widehat{\Phi}^0(\rho^1-\rho) - \Omega(\rho^1-\rho)\right\}[\frac{\partial^{(2n)}}{\partial\alpha^{(2n)}}K(\rho,\alpha)]\,d\rho.$$

$$\frac{\partial^{(2n)}}{\partial\alpha^{(2n)}}\left[-\frac{i}{2}\Phi^1(\alpha) + i\overline{\Phi}^0(\rho^1,\alpha)\right]\Bigg|_{\alpha=0} =$$

$$\pi\int_0^\infty \rho\left\{i\widehat{\Phi}^0(\rho^1-\rho) - \Omega(\rho^1-\rho)\right\}[(2n-1)!!\sum_{k=1}^{n-1}(-1)^{n-k}\frac{[2(n-k)]!}{2^{2n-3k}[2(n-k)-1]!!}\cdot$$

$$\cdot\frac{\rho^{2(n-k)}}{(n-k)!\,\Gamma(n-k+1)}\sigma_1^{(k)}(0) + 2^{2n}(2n-1)!!\,\sigma_1^{(n)}(0)]d\rho, \qquad n \geq 2.$$

$$\sigma_1^{(n)}(0) = \frac{\frac{\partial^{(2n)}}{\partial\alpha^{(2n)}}\left[-\frac{i}{2}\Phi^1(\alpha) + i\overline{\Phi}^0(\rho^1,\alpha)\right]\Big|_{\alpha=0}}{\pi 2^{2n}(2n-1)!!\int_0^\infty \rho\left\{i\widehat{\Phi}^0(\rho^1-\rho) - \Omega(\rho^1-\rho)\right\}d\rho} -$$

$$\frac{\pi(2n-1)!!\sum_{k=1}^{n-1}(-1)^{n-k}\cdot\frac{[2(n-k)]!}{2^{2n-3k}[2(n-k)-1]!!}\cdot\frac{\sigma_1^{(k)}(0)}{(n-k)!\,\Gamma(n-k+1)}}{\pi 2^{2n}(2n-1)!!\int_0^\infty \rho\left\{i\widehat{\Phi}^0(\rho^1-\rho) - \Omega(\rho^1-\rho)\right\}d\rho}.$$

$$\cdot\int_0^\infty \rho^{2(n-k)+1}\cdot\left\{i\widehat{\Phi}^0(\rho^1-\rho) - \Omega(\rho^1-\rho)\right\}d\rho.$$

Theorem 1. The spectral function of the operator Sturm-Liouville $\sigma(\lambda)$ is unique in the class of functions $\boldsymbol{\sigma^a}$ and is expressed through the integral transforms of the given information (5) on the solution of the direct problem (1)-(4).

Theorem 2. Unknown potential q(z) of one-dimensional inverse problem (1)-(5) is unambiguously reconstructed in the class of functions $\boldsymbol{Q_M^a}$.

Theorem 3. Moments of the function $f(\rho)$ uniquely determine the unknown source $f(\rho) = f\left(\sqrt{x^2+y^2}\right)$ in the boundary condition (4) of the inverse problem (1)-(5) in the class of functions **Φ**.

While differentiation of the first equation of system (18), assuming $\alpha = 0$, we obtain the odd moments of the function $f(\rho)$:

$$M_f^{(2n-1)} = \int_0^\infty \rho^{2n-1} f(\rho)\, d\rho = (-1)^{n+1} \cdot \frac{1}{2i} \cdot \frac{\partial^{(2n-1)}}{\partial \alpha^{(2n-1)}} \left[\Phi^0(\alpha)\right]\Big|_{\alpha = 0}, \quad n = 1,2,\dots.$$

5. Summary

The novelty of the work lies in the fact that it was decided a new, not previously studied problem; tested recovery method of a source and potential in the special classes of functions, first devised by author and used in the work [9]; also uniqueness theorems are proved. The proofs of theorems are constructive, and can be base for creating numerical methods for solution of inverse problems, which are reduced to inverse problem of Sturm-Liouville problem.

Studied classes of functions that can be restored by this method. Between the classes of functions $\boldsymbol{Q_M^a}$ and $\boldsymbol{\sigma^a}$ established one-to-one correspondence [8].

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