On the Solvability of Initial Problems for Abstract Singular Equations Containing Fractional Derivatives

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Abstract With the help of integral representations of the Poisson type, it is established that the Cauchy problem for a number of abstract singular equations with fractional derivatives reduces to a simpler problem for a non-singular equation.

Keywords Abstract singular equations · Fractional derivatives · Transformation operator · Cauchy problem

1 Introduction

One of the methods for studying differential equations is the method of transformation operators. Using conversion operators, many important results are established for various classes differential equations, including those for singular differential equations containing the Bessel differential expression

$$\frac{d^2}{dt^2} + \frac{k}{t}\frac{d}{dt}, \quad k \in R.$$

So in the monograph [1] the singular equation of Euler–Poisson–Darboux in partial derivatives

$$\frac{\partial^2 u(t,x)}{\partial t^2} + \frac{k}{t}\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x), \quad k > 0, \ \ x \in R^n,$$

where Δ is the Laplace operator in space variables, investigated by reduction with the help of a suitable transformation operator to a simpler wave equation when $k = 0$. In this case, the formulas for the solution are written using spherical averages over spatial variables.

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The review paper [2] presents the results of studies in which transformation operators are used in more general situation, when in the Euler–Poisson–Darboux equation the Laplace operator in space variables is replaced by some abstract operator A acting in a Banach space, as well as for some other singular equations of integer order. In these studies, a class of operators A is described for which the corresponding initial value problem is well-posed and an explicit representation is established for the enabling operator.

In this paper, the method of transformation operators is applied to abstract singular differential equations, containing fractional derivatives (see [3, Sect. 5], [4, Chap. 2]).

2 Generalized Euler–Poisson–Darboux Differential Equation

Let A be a closed operator in a Banach space E with dense in E domain $D(A)$. For $k \geq 0,\ 0 < \alpha < 1$, consider abstract singular equation with fractional derivatives

$$B_{k,\alpha}u(t) \equiv \frac{d}{dt}\partial^{\alpha}_{0,t}u(t) + \frac{k}{t}\partial^{\alpha}_{0,t}u(t) = Au(t), \quad t > 0, \tag{1}$$

where $\partial^{\alpha}_{0,t}u(t)$ is the fractional Caputo derivative defined by the equality

$$\partial^{\alpha}_{0,t}u(t) = D^{\alpha}_{0,t}\left(u(t) - u(0)\right), \quad \partial^{\alpha}_{0,t}u(0) = \lim_{t\to 0}\ \partial^{\alpha}_{0,t}u(t),$$

wherein

$$D^{\alpha}_{0,t}\left(u(t) - u(0)\right) = \frac{d}{dt}I^{1-\alpha}_{0,t}\left(u(t) - u(0)\right),\ I^{1-\alpha}_{0,t}u(t) = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^t \frac{u(\tau)}{(t-\tau)^{\alpha}}d\tau$$

respectively, the left-hand fractional derivative and the fractional Riemann–Liouville integral, $\Gamma(\cdot)$ is the gamma function.

If $\alpha = 1$, then the Eq. (1) becomes the Euler–Poisson–Darboux equation

$$u''(t) + \frac{k}{t}u'(t) = Au(t), \quad t > 0, \tag{2}$$

for which the abstract Cauchy problem with conditions

$$u(0) = u_0, \quad u'(0) = 0 \tag{3}$$

previously explored in detail in [5–7] (see also [2]). In these papers there is a review of the studies of the Euler–Poisson–Darboux equation, the class G_k of operators A is described, with which the problem (2), (3) is uniformly well-posed, the construction of the resolving operator of the problem (2), (3), which is called the operator Bessel function and which we denote by $Y_{k,1}(t)$.

In this paper, we present the setting of initial conditions for an equation with fractional derivatives (1), let us describe the class of operators A with which the corresponding initial problems are solvable and establish a number of properties of the solutions.

We will look for a solution to the Eq. (1) that satisfies the initial conditions

$$u(0) = u_0, \quad \partial_{0,t}^{\alpha} u(0) = 0. \tag{4}$$

Definition 1 A solution of the problem (1), (4) is a function continuous for $t > 0$ $u(t)$ such that for $t > 0$ the functions $I^{1-\alpha}u(t)$ are twice continuously differentiable, the function $u(t)$ takes values from the domain $D(A)$ of the operator A and satisfies the equalities (1), (4).

We begin the study of the solvability of the problem (1), (4) from the case when the parameter $k = 0$ in the Eq. (1) and describe the class considered operators A.

Condition 1 If $\mathrm{Re}\,\lambda > \omega \geq 0$ and $0 < \alpha \leq 1$, then $\lambda^{\alpha+1}$ belongs to the resolvent set $\rho(A)$ of the operator A and for all integers $n \geq 0$ the resolution $R(\lambda) = (\lambda I - A)^{-1}$ satisfies the inequalities

$$\left\| \frac{d^n}{d\lambda^n} \left(\lambda^{\alpha} R \left(\lambda^{\alpha+1} \right) \right) \right\| \leq \frac{Mn!}{(\mathrm{Re}\,\lambda - \omega)^{n+1}}. \tag{5}$$

Theorem 1 *Let $k = 0$, $0 < \alpha \leq 1$, $u_0 \in D(A)$ and the operator A satisfies Condition 1. Then the problem (1), (4) uniquely resolvable.*

Proof After applying to the Eq. (1) the integration operator $I_{0,t}^{1}$ and fractional differentiation $D_{0,t}^{1-\alpha}$ the problem (1), (4) reduces to the next initial problem

$$u'(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)\, ds, \quad t \geq 0, \tag{6}$$

$$u(0) = u_0. \tag{7}$$

Problem (6), (7) is a special case of the problem studied in [8]. In Theorem 3 of [8], it is established that Condition 1 is necessary and sufficient condition on the operator A, which, under the assumptions made in the theorem being proved, ensures the unique solvability problem (6), (7), and thus the equivalent problem (1), (4). The resolving operator of the problem (6), (7) will be denoted by $Y_{0,\alpha}(t)$, while $u(t) = Y_{0,\alpha}(t)u_0$. For $Y_{0,\alpha}(t)$ in [8] the representation and estimate are set respectively

$$Y_{0,\alpha}(t)u_0 = \frac{1}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\, \lambda^{\alpha} R\left(\lambda^{\alpha+1}\right) u_0\, d\lambda, \quad u_0 \in D\left(A^2\right), \tag{8}$$

$$\left\| Y_{0,\alpha}(t) \right\| \le M\, e^{\sigma t}, \quad \sigma > \omega.$$

□

Let us proceed to consider the case $k > 0$ and introduce the Poisson-type transformation operator

$$P_{k,\alpha}u(t) = c_{k,\alpha} \int\limits_0^1 \left(1 - s^{\alpha+1}\right)^{k/(\alpha+1)-1} u(ts)\, ds, \tag{9}$$

where $B(\cdot, \cdot)$ is the beta function,

$$c_{k,\alpha} = \frac{\alpha + 1}{B(k/(\alpha+1), 1/(\alpha+1))}.$$

The Poisson-type transformation operator is expressed in terms of the Erdelyi–Kober fractional integral $I^{\gamma}_{0+,\sigma,\eta}$ (see [3, Sect. 18]) as follows

$$P_{k,\alpha}u(t) = \frac{\Gamma\left((k+1)/(\alpha+1)\right)}{\Gamma\left(1/(\alpha+1)\right)}\, I^{k/(\alpha+1)}_{0+,\alpha+1,-\alpha/(\alpha+1)} u(t),$$

and the constant $c_{k,\alpha}$ is chosen so that

$$\lim_{t\to 0} P_{k,\alpha}u(t) = u(0).$$

Theorem 2 *Let $k > 0$, $0 < \alpha \le 1$ and the function $u(t)$ be that there is a fractional derivative of the form $\left(\partial^{\alpha}_{0,t}u(t)\right)'$. Then the equality*

$$B_{k,\alpha}\; P_{k,\alpha}u(t) = P_{k,\alpha}\left(\partial^{\alpha}_{0,t}u(t)\right)' + \frac{c_{k,\alpha}}{t}\; \partial^{\alpha}_{0,t}u(0). \tag{10}$$

Proof Applying the operator $B_{k,\alpha}$ to (9), after integrating by parts we get

$$B_{k,\alpha}\; P_{k,\alpha}u(t) = c_{k,\alpha} \int\limits_0^1 \left(1 - s^{\alpha+1}\right)^{k/(\alpha+1)-1}\; s^{\alpha+1}\, \frac{d}{d(ts)}\; \partial^{\alpha}_{0,ts}u(t)\, ds +$$

$$+ \frac{k\; c_{k,\alpha}}{t} \int\limits_0^1 \left(1 - s^{\alpha+1}\right)^{k/(\alpha+1)-1}\; s^{\alpha}\; \partial^{\alpha}_{0,ts}u(t)\, ds =$$

$$= c_{k,\alpha} \int\limits_0^1 \left(1 - s^{\alpha+1}\right)^{k/(\alpha+1)-1} \; s^{\alpha+1} \; \frac{d}{d(ts)} \; \partial^{\alpha}_{0,ts} u(t) \; ds +$$

$$+ \frac{c_{k,\alpha}}{t} \; \partial^{\alpha}_{0,t} u(0) + c_{k,\alpha} \int\limits_0^1 \left(1 - s^{\alpha+1}\right)^{k/(\alpha+1)} \; \frac{d}{d(ts)} \; \partial^{\alpha}_{0,ts} u(t) \; ds =$$

$$= c_{k,\alpha} \int\limits_0^1 \left(1 - s^{\alpha+1}\right)^{k/(\alpha+1)-1} \; \left(s^{\alpha+1} + 1 - s^{\alpha+1}\right) \; \frac{d}{d(ts)} \; \partial^{\alpha}_{0,ts} u(t) \; ds +$$

$$+ \frac{c_{k,\alpha}}{t} \; \partial^{\alpha}_{0,t} u(0) = P_{k,\alpha} \left(\partial^{\alpha}_{0,t} u(t)\right)' + \frac{c_{k,\alpha}}{t} \; \partial^{\alpha}_{0,t} u(0).$$

□

An immediate consequence of Theorem 2 is a theorem that establishes the solvability of the problem (1), (4) for $k > 0$.

Theorem 3 *Let* $k > 0$, $0 < \alpha \leq 1$, $u_0 \in D(A)$ *and operator* A *satisfy Condition 1. Then the function*

$$u(t) = P_{k,\alpha} Y_{0,\alpha}(t) u_0 = c_{k,\alpha} \int\limits_0^1 \left(1 - s^{\alpha+1}\right)^{k/(\alpha+1)-1} Y_{0,\alpha}(ts) u_0 \; ds \qquad (11)$$

is a solution to the problem (1), (4).

In what follows, for $k > 0$, $0 < \alpha \leq 1$ we will use the notation

$$Y_{k,\alpha}(t) = P_{k,\alpha} Y_{0,\alpha}(t).$$

Example 1 If the operator A is bounded and $0 < \alpha \leq 1$, then it is easy to verify directly that the function

$$Y_{0,\alpha}(t) u_0 = E_{\alpha+1,1}(t^{\alpha+1} A) u_0 = \sum_{j=0}^{\infty} \frac{t^{(\alpha+1)j} A^j u_0}{\Gamma((\alpha+1)j+1)},$$

where $E_{\alpha,\beta}(\cdot)$ is the Mittag–Leffler function, is the solution to the problem

$$\frac{d}{dt} \partial^{\alpha} u(t) = A u(t), \quad u(0) = u_0 \in E, \;\; \partial^{\alpha} u(0) = 0.$$

By virtue of Theorem 3, the function

$$u(t) = Y_{k,\alpha}(t)u_0 = P_{k,\alpha}Y_{0,\alpha}(t)u_0 =$$

$$= \frac{\Gamma((k+1)/(\alpha+1))}{\Gamma(1/(\alpha+1))} \sum_{j=0}^{\infty} \frac{\Gamma(j+1/(\alpha+1))\, t^{(\alpha+1)j} A^j w_0}{\Gamma((\alpha+1)j+1)\, \Gamma(j+(k+1)/(\alpha+1))} =$$

$$= \frac{\Gamma((k+1)/(\alpha+1))}{\Gamma(1/(\alpha+1))}\, {}_2\Psi_2 \left[\begin{matrix} (1/(\alpha+1), 1)\,,\ (1,1) \\ (1, \alpha+1)\,,\ ((k+1)/(\alpha+1), 1) \end{matrix} \middle| t^\alpha A \right] w_0, \quad (12)$$

where ${}_p\Psi_q(\cdot)$ is the Fox–Wright function (see [9, 10]) is the solution to the problem (1), (4).

Note that for $\alpha = 1$ the series in the formula (12) turns into the operator Bessel function (see [2, 5–7])

$$Y_{k,1}(t) = \Gamma(k/2+1/2) \sum_{j=0}^{\infty} \frac{\left(t\sqrt{A}/2\right)^{2j}}{j!\, \Gamma(j+k/2+1/2)} =$$

$$= \Gamma(k/2+1/2) \left(t\sqrt{A}/2\right)^{1/2-k/2} I_{k/2-1/2}\left(t\sqrt{A}\right),$$

where $I_\nu(\cdot)$ is the modified Bessel function.

Example 2 The operator function $Y_{0,\alpha}(t)$ satisfies the principle of subordination, which for the Eq. (1) with $k = 0$ was actually established in Chap. 3 of [11]. Let $0 \le \beta < \alpha \le 1$, then the following shift formula with respect to the second parameter is valid

$$Y_{0,\beta}(t) = \frac{1}{t^{(1+\beta)/(1+\alpha)}} \int_0^{\infty} \phi\left(-\frac{1+\beta}{1+\alpha}, \frac{\alpha-\beta}{1+\alpha}; -\frac{\tau}{t^{(1+\beta)/(1+\alpha)}}\right) Y_{0,\alpha}(\tau)\, d\tau,$$

in which the Wright function is used

$$\phi(\mu, \nu; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\, \Gamma(\mu n + \nu)}.$$

In particular, if the operator A is the generator of the operator cosine function $C(t; A)$, then for $\alpha = 1$ we obtain

$$Y_{0,\beta}(t) = \frac{1}{t^{(1+\beta)/2}} \int_0^{\infty} \phi\left(-\frac{1+\beta}{2}, \frac{1-\beta}{2}; -\frac{\tau}{t^{(1+\beta)/2}}\right) C(\tau; A)\, d\tau, \quad (13)$$

$$Y_{k,\beta}(t) = P_{k,\beta} Y_{0,\beta}(t).$$

In the limiting case, when $\beta = 0$, $\alpha = 1$, the equality (13) becomes the well-known semigroup connection formula $T(t; A)$ and cosine of the operator-function $C(t; A)$ generated by the operator A, which has the form

$$T(t; A) = \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{\tau^2}{4t}\right) C(\tau; A)\, d\tau. \tag{14}$$

The operator function $Y_{k,\alpha}(t)$ also satisfies the shift formula with respect to the first parameter.

Theorem 4 *Let $m > k \geq 0$, $0 < \alpha \leq 1$ and operator A satisfy Condition 1. Then there is an equality*

$$Y_{m,\alpha}(t) = \frac{\alpha + 1}{B((m-k)/(\alpha+1), (k+1)/(\alpha+1))} \times$$

$$\times \int_0^1 s^k \left(1 - s^{\alpha+1}\right)^{(m-k)/(\alpha+1)-1} Y_{k,\alpha}(ts)\, ds. \tag{15}$$

Proof After a series of obvious transformations, using the integral 2.2.5.1 [12], we obtain

$$\int_0^1 s^k \left(1 - s^{\alpha+1}\right)^{(m-k)/(\alpha+1)-1} Y_{k,\alpha}(ts)\, ds =$$

$$= \int_0^t \tau^k \left(t^{\alpha+1} - \tau^{\alpha+1}\right)^{(m-k)/(\alpha+1)-1} Y_{k,\alpha}(\tau)\, d\tau =$$

$$= c_{k,\alpha} \int_0^t \tau^\alpha \left(t^{\alpha+1} - \tau^{\alpha+1}\right)^{(m-k)/(\alpha+1)-1} \times$$

$$\times \int_0^\tau \left(\tau^{\alpha+1} - \xi^{\alpha+1}\right)^{k/(\alpha+1)-1} Y_{0,\alpha}(\xi)\, d\xi d\tau = c_{k,\alpha} \times$$

$$\times \int_0^t Y_{0,\alpha}(\xi) \int_\xi^t \tau^\alpha \left(t^{\alpha+1} - \tau^{\alpha+1}\right)^{(m-k)/(\alpha+1)-1} \left(\tau^{\alpha+1} - \xi^{\alpha+1}\right)^{k/(\alpha+1)-1} d\tau d\xi =$$

$$= \frac{c_{k,\alpha}}{\alpha+1} \int\limits_0^t Y_{0,\alpha}(\xi) \int\limits_{\xi^{\alpha+1}}^{t^{\alpha+1}} (t^{\alpha+1} - \eta)^{(m-k)/(\alpha+1)-1} \left(\eta - \xi^{\alpha+1}\right)^{k/(\alpha+1)-1} \, d\eta d\xi =$$

$$= \frac{c_{k,\alpha}\, B((m-k)/(\alpha+1), k/(\alpha+1))}{\alpha+1} \int\limits_0^t (t^{\alpha+1} - \xi)^{m/(\alpha+1)-1} Y_{0,\alpha}(\xi)\, d\xi =$$

$$= \frac{\Gamma\left((m-k)/(\alpha+1)\right)\ \Gamma\left((k+1)/(\alpha+1)\right)}{(\alpha+1)\ \Gamma\left((m+1)/(\alpha+1)\right)}\, t^{m-\alpha} Y_{m,\alpha}(t).$$

Consequently,

$$Y_{m,\alpha}(t) = \frac{(\alpha+1)\Gamma\left((m+1)/(\alpha+1)\right) t^{\alpha-m}}{\Gamma\left((k+1)/(\alpha+1)\right) \Gamma\left((m-k)/(\alpha+1)\right)} \times$$

$$\times \int\limits_0^t \tau^k \, (t^{\alpha+1} - \tau^{\alpha+1})^{(m-k)/(\alpha+1)-1} \, Y_{k,\alpha}(\tau)\, d\tau =$$

$$= \frac{\alpha+1}{B((m-k)/(\alpha+1), (k+1)/(\alpha+1))} \int\limits_0^1 s^k \, (1 - s^{\alpha+1})^{(m-k)/(\alpha+1)-1} \, Y_{k,\alpha}(ts)\, ds,$$

and the required equality (15) is established. □

3 Generalized Functional-Differential Bessel–Struve Equation

Let us proceed to the study of the case of a nonzero second initial condition $\partial^{\alpha}_{0,t} u(0) \neq 0$ and we will study the following initial problem for the functional differential equation

$$\frac{d}{dt}\partial^{\alpha}_{0,t} u(t) + \frac{k}{t}\left(\partial^{\alpha}_{0,t} u(t) - \partial^{\alpha}_{0,t} u(0)\right) = Au(t), \quad t > 0, \tag{16}$$

$$u(0) = 0, \quad \partial^{\alpha}_{0,t} u(0) = u_1. \tag{17}$$

For $\alpha = 1$ the problem (16), (17) becomes the initial problem for the Bessel–Struve equation, which was previously investigated by the author in [13].

Let us first consider the case when the parameter $k = 0$ in the Eq. (16).

Theorem 5 *Let* $k = 0$, $0 < \alpha \le 1$, $u_1 \in D(A)$ *and the operator* A *satisfies Condition 1. Then the problem (16), (17) is uniquely solvable.*

Proof After applying to the Eq. (1) the integration operator $I^1_{0,t}$ and fractional differentiation $D^{1-\alpha}_{0,t}$ problem (16), (17) reduces to the following initial problem for the inhomogeneous equation

$$u'(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} Au(s)\, ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} u_1, \quad t \ge 0, \tag{18}$$

$$u(0) = 0. \tag{19}$$

Just like task (6), (7), task (18), (19) is is a special case of the problem investigated in [8] and is uniquely solvable. The resolving operator of the problem (18), (19) will be denoted by $L_{0,\alpha}(t)$, and $u(t) = L_{0,\alpha}(t)u_1$, and $L_{0,\alpha}(t)$ in [8] is set to the representation

$$L_{0,\alpha}(t) = I^{\alpha}_{0,t} Y_{0,\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} Y_{0,\alpha}(s)\, ds. \tag{20}$$

□

An immediate consequence of Theorems 5 and 2 is the solvability of the problem (16), (17) for $k > 0$. For $0 < \alpha \le 1$ we introduce the following notation:

$$d_{k,\alpha} = \frac{k}{\alpha+1} B\left(\frac{k}{\alpha+1}, \frac{1}{\alpha+1}\right), \quad L_{k,\alpha}(t) = d_{k,\alpha} P_{k,\alpha} L_{0,\alpha}(t).$$

Theorem 6 *Let* $k > 0$, $0 < \alpha \le 1$, $u_1 \in D(A)$ *and operator* A *satisfy Condition 1. Then the function*

$$u(t) = L_{k,\alpha}(t)u_1 = d_{k,\alpha} P_{k,\alpha} L_{0,\alpha}(t) u_1 \tag{21}$$

is a solution to the problem (16), (17).

Example 3 If $0 < \alpha \le 1$ and A is a bounded operator, then

$$L_{k,\alpha}(t) = \Gamma\left(k/(\alpha+1)+1\right) \sum_{j=0}^{\infty} \frac{\Gamma(j+1)\, t^{(\alpha+1)j+\alpha} A^j}{\Gamma\left((\alpha+1)j+\alpha+1\right)\ \Gamma\left(j+k/(\alpha+1)+1\right)}. \tag{22}$$

For $\alpha = 1$, the series on the right-hand side (22) is expressed in terms of the Struve function

$$L_k(t) = \frac{\sqrt{\pi}}{2}\,\Gamma\,(k/2+1)\sum_{j=0}^{\infty}\frac{\left(t\sqrt{A}/2\right)^{2j}}{\Gamma\,(j+3/2)\,\Gamma\,(j+k/2+1)} =$$

$$= \frac{2^{k/2-1/2}\sqrt{\pi}\,\Gamma(k/2+1)}{A^{k/4+1/4}\,t^{k/2-1/2}}\,\mathbf{L}_{k/2-1/2}\left(t\sqrt{A}\right),$$

where $\mathbf{L}_{\nu}(\cdot)$ is the modified Struve function ([14], p. 655).

Example 4 If $0<\beta<1$ and the operator A is the generator of the operator cosine function $C(t;A)$, then

$$L_{k,\beta}(t) = d_{k,\beta}P_{k,\beta}I^{\beta}_{0,t}Y_{0,\beta}(t),$$

where the operator function $Y_{0,\beta}(t)$ is defined by the equality (13).

The operator function $L_{k,\alpha}(t)$ satisfies the shift formula with respect to the first parameter, whose proof is carried out in the same way as in Theorem 4.

Theorem 7 *Let $m>k\geq 0$, $0<\alpha\leq 1$ and operator A satisfy Condition 1. Then*

$$L_{m,\alpha}(t) = \frac{\alpha+1}{B((m-k)/(\alpha+1),k/(\alpha+1)+1)}\ \times$$

$$\times\int_0^1 s^k\,(1-s^{\alpha+1})^{(mk)/(\alpha+1)-1}\,L_{k,\alpha}(ts)\,ds.$$

The constructed operator functions $Y_{k,\alpha}(t)$, $L_{k,\alpha}(t)$, as well as Theorems 3 and 6 allow us to establish the following statement about the solvability of the general initial problem for the Eq. (16).

Theorem 8 *Let $k\geq 0$, $0<\alpha\leq 1$, $u_0,u_1\in D(A)$ and the operator A satisfies Condition 1. Then the function $u(t)=Y_{k,\alpha}(t)u_0+L_{k,\alpha}(t)u_1$ is a solution to the Eq. (16) satisfying the conditions*

$$u(0)=u_0,\quad \partial^{\alpha}_{0,t}u(0)=u_1. \tag{23}$$

Theorems 3, 6, 8 do not contain a statement about the uniqueness of the solution. To prove the uniqueness of the solution of these problems, we make an additional assumption. We assume that $A\in G_k$, i.e., with the operator A, the Cauchy problem (2), (3) is uniformly well-posed for the Euler–Poisson–Darboux equation, and the resolving operator of this problem, as indicated earlier, is denoted by $Y_{k,1}(t)$.

Theorem 9 *Let $k\geq 0$, $0<\alpha\leq 1$ and operator $A\in G_k$. Then the solutions of problems (1), (4) and (16), (23) are unique.*

Proof Proof of the uniqueness of the solution to the problem (16), (23) we will lead from the contrary. If $u_1(t)$ and $u_2(t)$ are two solutions to the problem (16), (23), then consider a function of two variables $w(t,s) = f\,(Y_k(s)\,(u_1(t) - u_2(t)))$, where $f \in E^*$ (E^* is the dual space), $t, s \geq 0$. She, obviously satisfies the equation

$$B_{k,\alpha} w(t,s) = \frac{\partial^2 w(t,s)}{\partial s^2} + \frac{k}{s}\,\frac{\partial w(t,s)}{\partial s}, \quad t, s > 0 \tag{24}$$

and conditions

$$\lim_{t\to 0}\, w(t,s) = \lim_{t\to 0}\, \partial^{\alpha}_{0,t} w(t,s) = \lim_{s\to 0} \frac{\partial w(t,s)}{\partial s} = 0. \tag{25}$$

As was done in [15], we interpret $w(t,s)$ as a generalized function of moderate growth and on the variable s we apply the Fourier–Bessel transformation

$$\hat{w}(t,\lambda) = \int\limits_0^\infty s^{2p+1}\; j_p(\lambda s)\; w(t,s)\; ds, \quad w(t,s) = \gamma_p \int\limits_0^\infty \lambda^{2p+1}\; j_p(\lambda s)\; \hat{w}(t,\lambda)\; d\lambda,$$

$$p = \frac{1-k}{2}, \quad \gamma_p = \frac{1}{2^{2p}\;\Gamma^2(p+1)}, \quad j_p(s) = \frac{2^p\;\Gamma(p+1)}{s^p}\; J_p(s),$$

where $J_p(\cdot)$ is the Bessel function.

From (24), (25) for the image $\hat{w}(t,\lambda)$ we get the following problem

$$B_{k,\alpha}\hat{w}(t,\lambda) = -\lambda^2\;\hat{w}(t,\lambda), \quad t > 0, \tag{26}$$

$$\lim_{t\to 0}\;\hat{w}(t,\lambda) = \lim_{t\to 0}\;\partial^{\alpha}_{0,t}\hat{w}(t,\lambda) = 0. \tag{27}$$

By virtue of Examples 1 and 3, the general solution of the Eq. (26) has the form

$$\hat{w}(t,\lambda) = \frac{d_1(\lambda)\Gamma\,((k+1)/(\alpha+1))}{\Gamma\,(1/(\alpha+1))} \sum_{j=0}^{\infty} \frac{\Gamma\,(j+1/(\alpha+1))\,t^{(\alpha+1)j}(-\lambda^2)^j}{\Gamma\,((\alpha+1)j+1)\,\Gamma\,(j+(k+1)/(\alpha+1))} +$$

$$+ d_2(\lambda)\;\Gamma\,(k/(\alpha+1)+1) \sum_{j=0}^{\infty} \frac{\Gamma(j+1)\;t^{(\alpha+1)j+\alpha}(-\lambda^2)^j}{\Gamma\,((\alpha+1)j+\alpha+1)\;\;\Gamma\,(j+k/(\alpha+1)+1)},$$

and the initial conditions (27) imply the equalities $d_1(\lambda) = d_2(\lambda) = 0$. Hence $\hat{w}(t,\lambda) = w(t,s) = 0$ for any $s \geq 0$. Since the functional $f \in E^*$ is arbitrary, for $s = 0$ we obtain the equality $u_1(t) \equiv u_2(t)$, and the uniqueness of the solution of the considered problems is established. □

As an application of Theorem 8 consider the problem (16), (23) with the operator which is a fractional power of the operator A. Let A be the generator of a uniformly bounded cosine-operator function. Then one can define a positive fractional power of the operator $-A$ (see, for example, [16, p. 358])

$$-\left(-A\right)^{\gamma} x=\frac{\sin \gamma\pi}{\pi}\int\limits_{0}^{\infty} \lambda^{\gamma-1}\left(\lambda I-A\right)^{-1} Ax\, d\lambda, \tag{28}$$

where $\gamma\in(0,1)\,,\ x\in D(A)$.

Moreover, if $y\in E,\ \ \mu>0$, then the resolvent of the operator $A_{\gamma}=-(-A)^{\gamma}$ satisfies the representation

$$\left(\mu I-A_{\gamma}\right)^{-1} y=\frac{\sin \gamma\pi}{\pi}\int\limits_{0}^{\infty} \frac{\lambda^{\gamma}\left(\lambda I-A\right)^{-1} y\, d\lambda}{\mu^{2}-2\mu\lambda^{\gamma}\ \cos\gamma\pi+\lambda^{2\gamma}}. \tag{29}$$

Next, we establish the solvability of the initial problem (16), (23) with the operator A_{γ}, where the exponent is $\gamma=(\alpha+1)/2$.

Theorem 10 *Let $\gamma=(\alpha+1)/2,\ \ 0<\alpha<1,\ \ u_0,u_1\in D(A)$, the operator A — generator of uniformly bounded cosine-operator function $C(t;A)$ and operator A_{γ} defined by (28). Then the solution of the initial problem*

$$\frac{d}{dt}\partial^{\alpha}_{0,t}u(t)+\frac{k}{t}\left(\partial^{\alpha}_{0,t}u(t)-\partial^{\alpha}_{0,t}u(0)\right)=A_{\gamma}u(t),\quad t>0, \tag{30}$$

$$u(0)=u_0,\quad \partial^{\alpha}_{0,t}u(0)=u_1. \tag{31}$$

is the function $u(t)=Y_{k,\alpha}(t;A_{\gamma})u_0+L_{k,\alpha}(t;A_{\gamma})u_1$, where

$$Y_{0,\alpha}(t;A_{\gamma})=\frac{\sin\gamma\pi}{\gamma\pi}\int\limits_{0}^{\infty}\frac{C\left(ts^{-1/(2\gamma)};A\right)\,ds}{s^{2}-2s\cos\gamma\pi+1}, \tag{32}$$

while the operator functions $Y_{k,\alpha}(t;A_{\gamma}),\ \ L_{0,\alpha}(t;A_{\gamma}),\ \ L_{k,\alpha}(t;A_{\gamma})$ are defined respectively by the formulas (11), (20), (21).

Proof The operator A is the generator of a uniformly bounded cosine operator function, and in order to to use Theorem 8, one should check the fulfillment of Condition 1 for the operator A_{γ}. In the In our case, this condition is that for Re $\mu>0$ the resolvent $\left(\mu^{\alpha+1}I-A_{\gamma}\right)^{-1}$ satisfied the inequality

$$\left\|\frac{d^{n}\left(\mu^{\alpha}\left(\mu^{\alpha+1}I-A_{\gamma}\right)^{-1}\right)}{d\mu^{n}}\right\|\le\frac{Mn!}{(\mathrm{Re}\ \mu)^{n+1}}. \tag{33}$$

Given the representation (29), after the change of variables, we get

$$\mu^{\alpha}\left(\mu^{\alpha+1}I-A_{\gamma}\right)^{-1}y=\frac{\mu\sin\gamma\pi}{\gamma\pi}\int\limits_{0}^{\infty}\frac{s^{1/\gamma}\left(\mu^{2}s^{1/\gamma}I-A\right)^{-1}y\,ds}{s^{2}-2s\;\cos\gamma\pi+1}=$$

$$=\frac{\sin\gamma\pi}{\gamma\pi}\int\limits_{0}^{\infty}\frac{s^{1/(2\gamma)}\xi\left(\xi^{2}I-A\right)^{-1}y\,ds}{s^{2}-2s\;\cos\gamma\pi+1},$$

where $\xi=\mu s^{1/(2\gamma)}$ and hence

$$\frac{d^{n}\left(\mu^{\alpha}\left(\mu^{\alpha+1}I-A_{\gamma}\right)^{-1}y\right)}{d\mu^{n}}=$$

$$=\frac{\sin\gamma\pi}{\gamma\pi}\int\limits_{0}^{\infty}\frac{s^{(1+n)/(2\gamma)}}{s^{2}-2s\;\cos\gamma\pi+1}\frac{d^{n}}{d\xi^{n}}\left(\xi\left(\xi^{2}I-A\right)^{-1}y\right)\,ds. \tag{34}$$

Since for the resolvent of the generator of a uniformly bounded cosine-operator function for Re $\xi>0$ there is an estimate

$$\left\|\frac{d^{n}}{d\xi^{n}}\left(\xi\left(\xi^{2}I-A\right)^{-1}y\right)\right\|\le\frac{M_{1}\;n!}{(\mathrm{Re}\;\xi)^{n+1}}, \tag{35}$$

then (34), (35) implies the validity of the inequality

$$\left\|\frac{d^{n}}{d\mu^{n}}\left(\mu^{\alpha}\left(\mu^{\alpha+1}I-A_{\gamma}\right)^{-1}\right)\right\|\le\frac{M_{1}\;n!}{(\mathrm{Re}\;\mu)^{n+1}}\int\limits_{0}^{\infty}\frac{ds}{s^{2}-2s\;\cos\gamma\pi+1}\le\frac{Mn!}{(\mathrm{Re}\;\mu)^{n+1}},$$

and thus the inequality (33) is proved, and with it the solvability of the problem (30), (31).

It remains for us to obtain the representation (32) for the operator function $Y_{0,\alpha}(t;A_{\gamma})$. Using (8), (29), for $u_0\in D(A^2)$ we get

$$Y_{0,\alpha}(t;A_{\gamma})u_{0}=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}e^{\lambda t}\,\lambda^{\alpha}\left(\lambda^{\alpha+1}I-A_{\gamma}\right)^{-1}u_{0}\,d\lambda=$$

$$=\frac{\sin\gamma\pi}{\gamma\pi}\int\limits_{0}^{\infty}\frac{s^{1/\gamma}}{s^{2}-2s\;\cos\gamma\pi+1}\,\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\lambda e^{\lambda t}\left(\lambda^{2}s^{1/\gamma}I-A\right)^{-1}u_{0}\,d\lambda ds=$$

$$=\frac{\sin\gamma\pi}{\gamma\pi}\int\limits_{0}^{\infty}\frac{C\left(ts^{-1/(2\gamma)};A\right)u_{0}\,ds}{s^{2}-2s\cos\gamma\pi+1}.$$

The representation established on the dense set $D(A^2) \subset E$ (32) for the operator function $Y_{0,\alpha}(t;\, A_\gamma)$ extends by continuity to all E.

Operator functions $Y_{k,\alpha}(t;\, A_\gamma),\ L_{0,\alpha}(t;\, A_\gamma),\ L_{k,\alpha}(t;\, A_\gamma)$ are defined respectively by the formulas (11), (20), (21). In particular,

$$L_{0,\alpha}(t;\, A_\gamma) = I^{\alpha}_{0,t} Y_{0,\alpha}(t;\, A_\gamma) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} Y_{0,\alpha}(s;\, A_\gamma)\, ds =$$

$$= \frac{\sin\gamma\pi}{\gamma\pi\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} \int\limits_0^\infty \frac{C\left(s\eta^{-1/(2\gamma)};\, A\right)\, d\eta}{\eta^2 - 2\eta\cos\gamma\pi + 1} ds =$$

$$= \frac{2\gamma\sin\gamma\pi}{\gamma\pi\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} \int\limits_0^\infty \frac{s^{2\gamma}\xi^{2\gamma-1} C(\xi;\, A)\, d\xi}{s^{4\gamma} - 2(s\xi)^{2\gamma}\cos\gamma\pi + \xi^{4\gamma}} ds =$$

$$= \frac{2\gamma\ \sin\gamma\pi}{\gamma\pi\Gamma(\alpha)} \int\limits_0^\infty \xi^{2\gamma-1} C(\xi;\, A) \int\limits_0^t \frac{s^{2\gamma}(t-s)^{\alpha-1}\, ds}{s^{4\gamma} - 2(s\xi)^{2\gamma}\cos\gamma\pi + \xi^{4\gamma}}\, d\xi.$$

$\square$

4 Appendix

If A is the generator of an exponentially bounded β times integrated cosine operator of the function $C_\beta(t;\, A)$, then for

$$0 < \alpha < 1,\ \ \beta \le \frac{1-\alpha}{1+\alpha},\quad \gamma = \frac{(\beta-1)(1+\alpha)}{2} + \alpha \le 0$$

performance for $Y_{0,\alpha}(t;\, A)$ in Theorem 1 can be written as

$$Y_{0,\alpha}(t;\, A) = \frac{1}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\, \lambda^{\alpha} \left(\lambda^{\alpha+1} I - A\right)^{-1}\, d\lambda =$$

$$= \frac{1}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} \lambda^{\gamma} \int\limits_0^\infty e^{-\tau\lambda^{(\alpha+1)/2}} C_\beta(\tau;\, A) d\tau d\lambda =$$

$$= \int_0^\infty C_\beta(\tau; A) \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \lambda^\gamma e^{\lambda t - \tau\lambda^{(\alpha+1)/2}} d\lambda d\tau = \int_0^\infty C_\beta(\tau; A) I_{0,t}^{-\gamma} f_{\tau,(\alpha+1)/2}(t)\, d\tau, \tag{36}$$

in doing so, we used the introduced in [16, p. 357] function

$$f_{\tau,\gamma}(t) = \begin{cases} \frac{1}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \exp\left(tz - \tau z^\gamma\right)\, dz, & t \geq 0, \\ 0, \quad t < 0, \end{cases}$$

where $\sigma > 0,\ \tau > 0,\ 0 < \gamma < 1$.

The function $f_{\tau,\gamma}(t)$ for $t > 0$ is expressed in terms of a Wright-type function ([17, Chap. 1]) $f_{\tau,\gamma}(t) = t^{-1} e_{1,\gamma}^{1,0}\left(-\tau t^{-\gamma}\right)$, where is the function

$$e_{\alpha,\beta}^{\mu,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \mu)\, \Gamma(\delta - \beta k)}, \quad \alpha > \max\{0; \beta\}, \quad \mu, z \in C$$

satisfies the assessment

$$e_{1,\beta}^{1,\delta}(-\tau) \leq M_n(\tau) \exp\left(-(1-\beta)\beta^{\beta/(1-\beta)} \tau^{1/(1-\beta)}\right), \tag{37}$$

$$M_n(\tau) = \sum_{m=0}^{n} \frac{(\beta\tau)^m}{\Gamma(\delta + m(1-\beta))},$$

and the number n is chosen from the condition $\delta + n(1-\beta) \geq 1$.

In the equality (36), the fractional integral $I_{0,t}^{-\gamma} f_{\tau,(\alpha+1)/2}(t)$ is calculated (see formula (1.2.12) in [17]) and we arrive at the equality

$$Y_{0,\alpha}(t; A) = \frac{1}{t^{\gamma+1}} \int_0^\infty C_\beta(\tau; A) e_{1,(\alpha+1)/2}^{1,-\gamma}\left(-\tau t^{-(\alpha+1)/2}\right)\, d\tau. \tag{38}$$

Note that the convergence of the integral in the representation (38) is ensured by the estimate (37).

In the limiting case, when $\alpha = 0$, $\beta = 0$, $\gamma = -1/2$, the formula (36) becomes (14). Indeed, in this particular case we have (see [16, p. 369, formula (32)])

$$f_{\tau,1/2}(t) = \frac{\tau}{2t\sqrt{\pi t}} \exp\left(-\frac{\tau^2}{4t}\right)$$

and, taking into account the integral 2.3.4.1 [12]), we obtain

$$Y_{0,0}(t;A) = \int_0^\infty C(\tau;A) I_{0,t}^{1/2} f_{\tau,(1/2)}(t)d\tau =$$

$$= \int_0^\infty C(\tau;A) I_{0,t}^{1/2} \left(\frac{\tau}{2t\sqrt{\pi t}} \exp\left(-\frac{\tau^2}{4t}\right)\right) d\tau =$$

$$= \frac{1}{2\pi} \int_0^\infty \tau C(\tau;A) \int_0^t \frac{1}{s\sqrt{s(t-s)}} \exp\left(-\frac{\tau^2}{4s}\right) ds d\tau =$$

$$= \frac{1}{2\pi\sqrt{t}} \int_0^\infty \tau C(\tau;A) \int_{1/t}^\infty \left(\xi - \frac{1}{t}\right)^{-1/2} \exp\left(-\frac{\xi\tau^2}{4}\right) d\xi d\tau =$$

$$= \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{\tau^2}{4t}\right) C(\tau;A)d\tau,$$

which coincides with the representation (14), while, naturally, one should assume that $Y_{0,0}(t;A) = T(t;A)$, $Y_{0,1}(t;A) = C_0(t;A) = C(t;A)$.

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