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ORIGINAL ARTICLE

On a general discrete boundary value problem for an elliptic pseudo-differential equation in a quadrant

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Abstract
We study a general discrete boundary value problem in Sobolev–Slobodetskii spaces in a plane quadrant and reduce it to a system of integral equations. We show a solvability of the system for a small size of discreteness starting from a solvability of its continuous analogue.

Keywords Elliptic symbol · Invertibility · Digital pseudo-differential operator · Discrete equation · Periodic wave factorization · System of integral equation

Mathematics Subject Classification Primary: 35S15 · Secondary: 65T50

1 Introduction

The theory of pseudo-differential operators [1, 2] and related equations and boundary value problems [3] exists more than a half-century, and up to now it takes attention of scholars. Also there are some discrete theories of boundary value problems for partial differential equations [6, 7], but these studies are not applicable for pseudo-differential equations. According to this fact the first author has initiated a studying discrete theory of pseudo-differential equations [4, 12–14] having in mind forthcoming studies their approximation properties and applications to computational algorithms [8, 11].

Since model equations in [3] were studied in a half-space, it is a canonical domain for manifold with smooth boundary, next step was done with a cone, it is a canonical domain for manifold with conical points at boundary [10]. At this step one needs a

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special factorization for an elliptic symbol, it was transferred to the discrete case [15] using analogies with Fourier series [5].

This paper is devoted to a model discrete pseudo-differential equation and discrete boundary value problem in a quadrant in the plane and its solvability starting from solvability their continuous analogues under small parameter of discreteness.

2 Digital pseudo-differential operators and discrete equations

Here we introduce some starting concepts and results which will help us moving to statement of a general boundary value problem.

Let $\mathbb{Z}^2$ be the integer lattice in a plane, $K = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_1 > 0, x_2 > 0\}$ be the first quadrant, $K_d = h\mathbb{Z}^2 \cap K$, $h > 0$, $\mathbb{T}^2 = [-\pi, \pi]^2$, $\hbar = h^{-1}$. We consider functions of a discrete variable $u_d(\tilde{x})$, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in h\mathbb{Z}^2$.

We use also notations $\zeta^2 = h^{-2}((e^{ih\cdot\xi_1} - 1)^2 + (e^{ih\cdot\xi_2} - 1)^2)$ and $S(h\mathbb{Z}^2)$ for the discrete analogue of the Schwartz space of infinitely differentiable rapidly decreasing at infinity functions.

Definition 1 The space $H^s(h\mathbb{Z}^2)$ consists of discrete functions and it is a closure of the space $S(h\mathbb{Z}^2)$ with respect to the norm

$$||u_d||_s = \left(\int\limits_{\hbar\mathbb{T}^2} (1 + |\zeta^2|)^s |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}, \tag{1}$$

where $\tilde{u}_d(\xi)$ denotes the discrete Fourier transform

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^2} e^{i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^2, \quad \xi \in \hbar\mathbb{T}^2.$$

Let $A_d(\xi)$ be a measurable periodic function defined in $\mathbb{R}^2$ with the basic cube of periods $\hbar\mathbb{T}^2$.

Definition 2 A digital pseudo-differential operator A_d with the symbol $A_d(\xi)$ in discrete quadrant K_d is called the following operator

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbb{Z}^2} h^2 \int\limits_{\hbar\mathbb{T}^2} A_d(\xi) e^{i(\tilde{y}-\tilde{x})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in K_d,$$

Here we will consider symbols satisfying the condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \leq |A_d(\xi)| \leq c_2(1 + |\zeta^2|)^{\alpha/2}$$

with positive constants c_1, c_2 non-depending on h. The class of symbols satisfying this condition will be denoted by E_α. The number $\alpha \in \mathbb{R}$ is called an order of the digital pseudo-differential operator A_d.

We study solvability of the discrete equation

$$(A_d u_d)(\tilde{x}) = 0, \quad \tilde{x} \in K_d, \tag{2}$$

in the space $H^s(K_d)$, and for this purpose we need certain specific domains of two-dimensional complex space $\mathbb{C}^2$. A domain of the type $\mathcal{T}_h(K) = \hbar\mathbb{T}^2 + iK$ is called a tube domain over the quadrant K. We will work with holomorphic functions $f(x+i\tau)$ in such domains $\mathcal{T}_h(K)$.

Definition 3 Periodic wave factorization of the symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form

$$A_d(\xi) = A_{d,\neq}(\xi) A_{d,=}(\xi),$$

where the factors $A_{d,\neq}(\xi), A_{d,=}(\xi)$ admit holomorphic continuation into tube domains $\mathcal{T}_h(K), \mathcal{T}_h(-K)$ respectively satisfying the estimates

$$c_1(1+|\hat{\zeta}^2|)^{\frac{æ}{2}} \leq |A_{d,\neq}(\xi+i\tau)| \leq c'_1(1+|\hat{\zeta}^2|)^{\frac{æ}{2}},$$
$$c_2(1+|\hat{\zeta}^2|)^{\frac{\alpha-æ}{2}} \leq |A_{d,=}(\xi-i\tau)| \leq c'_2(1+|\hat{\zeta}^2|)^{\frac{\alpha-æ}{2}},$$

with positive constants c_1, c'_1, c_2, c'_2 non-depending on h;

$$\hat{\zeta}^2 \equiv \hbar^2\left((e^{ih(\xi_1+i\tau_1)}-1)^2+(e^{ih(\xi_2+i\tau_2)}-1)^2\right), \quad \xi = (\xi_1,\xi_2) \in \hbar\mathbb{T}^2,$$
$$\tau = (\tau_1,\tau_2) \in K.$$

The number $æ \in \mathbb{R}$ is called an index of periodic wave factorization.

Everywhere below we assume that we have this periodic wave factorization of the symbol $A_d(\xi)$ with the index æ.

Using methods developed in [12] we can prove the following result.

Theorem 1 *Let $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$. Then a general solution of the equation (2) has the following form*

$$\tilde{u}_d(\xi) = A^{-1}_{d,\neq}(\xi)\left(\sum_{k=0}^{n-1}\left(\tilde{c}_k(\xi_1)\zeta_2^k + \tilde{d}_k(\xi_2)\zeta_1^k\right)\right), \tag{3}$$

where $\tilde{c}_k(\xi_1), \tilde{d}_k(\xi_2), k = 0, 1, \cdots, n-1$, are arbitrary functions from– $\widetilde{H}^{s_k}(h\mathbb{T})$, $s_k = s - æ + k - 1/2$.

The a priori estimate

$$||u_d||_s \leq const \sum_{k=0}^{n-1}([c_k]_{s_k} + [d_k]_{s_k}),$$

holds, where $[\cdot]_{s_k}$ denotes a norm in the space $H^{s_k}(h\mathbb{T})$, and const doesn't depend on h.

3 Discrete boundary value problem

3.1 Statement and solvability

Starting from Theorem 1 we introduce the following boundary conditions:

$$(B_{d,j}u_d)(\tilde{x}_1, 0) = b_{d,j}(\tilde{x}_1),$$
$$begineqnarray*3pt](G_{d,j}u_d)(0, \tilde{x}_2) = g_{d,j}(\tilde{x}_2), \quad j = 0, 1. \cdots, n-1, \tag{4}$$

where $B_{d,j}, G_{d,j}$ are digital pseudo-differential operators of order $\beta_j, \gamma_j \in \mathbb{R}$ with symbols $\widetilde{B}_{d,j}(\xi) \in E_{\beta_j}, \widetilde{G}_{d,j}(\xi) \in E_{\gamma_j}$

$$(B_{d,j}u_d)(\tilde{x}) = \frac{1}{(2\pi)^2} \int\limits_{\hbar\mathbf{T}^2} \sum_{\tilde{y} \in h\mathbf{Z}^2} e^{i\xi\cdot(\tilde{x}-\tilde{y})} \widetilde{B}_{d,j}(\xi)\tilde{u}_d(\xi)d\xi,$$
$$(G_{d,j}u_d)(\tilde{x}) = \frac{1}{(2\pi)^2} \int\limits_{\hbar\mathbf{T}^2} \sum_{\tilde{y} \in h\mathbf{Z}^2} e^{i\xi\cdot(\tilde{x}-\tilde{y})} \widetilde{G}_{d,j}(\xi)\tilde{u}_d(\xi)d\xi.$$

One can rewrite boundary conditions (4) in Fourier images

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \widetilde{B}_{d,j}(\xi_1, \xi_2)\tilde{u}_d(\xi_1, \xi_2)d\xi_2 = \tilde{b}_{d,j}(\xi_1),$$
$$\int\limits_{-\hbar\pi}^{\hbar\pi} \widetilde{G}_{d,j}(\xi_1, \xi_2)\tilde{u}_d(\xi_1, \xi_2)d\xi_1 = \tilde{g}_{d,j}(\xi_2), \quad j = 0, 1. \cdots, n-1, \tag{5}$$

so that according to properties of digital pseudo-differential operators and trace properties we need to require $b_{d,j}(\tilde{x}_1) \in H^{s-\beta_j-1/2}(h\mathbb{Z})$, $g_{d,j}(\tilde{x}_2) \in H^{s-\gamma_j-1/2}(h\mathbb{Z})$.

Multiplying the equality (3) by $\widetilde{B}_{d,j}(\xi_1, \xi_2)$ and $\widetilde{G}_{d,j}(\xi_1, \xi_2)$, integrating over $[-\hbar\pi, \hbar\pi]$ on ξ_2 and ξ_1, taking into account the conditions (5) we obtain the following $(2n \times 2n)$-system of linear integral equations

$$\sum_{k=0}^{n-1} \left(r_{jk}(\xi_1)\tilde{c}_k(\xi_1) + \int\limits_{-\hbar\pi}^{\hbar\pi} l_{jk}(\xi_1, \xi_2)\tilde{d}_k(\xi_2)d\xi_2 \right) = \tilde{b}_{d,j}(\xi_1)$$
$$\sum_{k=0}^{n-1} \left(\int\limits_{-\hbar\pi}^{\hbar\pi} m_{jk}(\xi_1, \xi_2)\tilde{c}_k(\xi_1)d\xi_1 + p_{jk}(\xi_2)\tilde{d}_k(\xi_2) \right) = \tilde{g}_{d,j}(\xi_2), \tag{6}$$
$$j = 0, 1, \ldots, n-1,$$

with unknown functions $\tilde{c}_k, \tilde{d}_k, k = 0, 1, \ldots, n-1$. We have used the following notations:

$$r_{jk}(\xi_1) = \int_{-\hbar\pi}^{\hbar\pi} \tilde{B}_{d,j}(\xi) A_{d,\neq}^{-1}(\xi)\zeta_2^k d\xi_2, \quad p_{jk}(\xi_2) = \int_{-\hbar\pi}^{\hbar\pi} \tilde{G}_{d,j}(\xi) A_{d,\neq}^{-1}(\xi)\zeta_1^k d\xi_1,$$
$$l_{jk}(\xi_1,\xi_2) = \tilde{B}_{d,j}(\xi) A_{d,\neq}^{-1}(\xi)\zeta_1^k, \quad m_{jk}(\xi_1,\xi_2) = \tilde{G}_{d,j}(\xi) A_{d,\neq}^{-1}(\xi)\zeta_2^k,$$

$j, k = 0, 1, \ldots, n-1$.

Thus, we can formulate the following assertion:

Theorem 2 *The boundary value problem (2),(4) is uniquely solvable in the space $H^s(K_d)$ with data $b_{d,j} \in H^{s-\beta_j-1/2}(h\mathbb{Z}_+)$, $g_{d,j} \in H^{s-\gamma_j-1/2}(h\mathbb{Z}_+)$ if and only if the system (6) has the unique solution $\tilde{c}_k, \tilde{d}_k \in \tilde{H}^{s_k}(\hbar\mathbb{T})$, $j, k = 0, 1, \ldots, n-1$.*

3.2 Continuous case

Here we will describe continuous boundary value problem which is related to considered discrete boundary value problem (2),(4).

Let A be a pseudo-differential operator

$$(Au)(x) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{A}(\xi) e^{i\xi(y-x)} u(y) dy d\xi$$

with symbol $\tilde{A}(\xi)$ satisfying the condition

$$|\tilde{A}(\xi)| \sim (1+|\xi|)^{\alpha} \tag{7}$$

and admitting the wave factorization with respect to K

$$\tilde{A}(\xi) = A_{\neq}(\xi) \cdot A_{=}(\xi).$$

with index æ such that æ $- s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$.

Further, let $B_j, G_j, j = 0, 1, \ldots, n-1$ be pseudo-differential operators with symbols $\tilde{B}_j(\xi), \tilde{G}_j(\xi)$ satisfying the condition (7) with β_j, γ_j instead of α.

The following boundary value problem:

$$\begin{aligned} (Au)(x) &= 0, \quad x \in K, \\ beginegnarray*3pt](B_j u)(x_1, 0) &= b_j(x_1), \\ beginegnarray*3pt](G_j u)(0, x_2) &= g_j(x_2), \quad j = 0, 1, \ldots, n-1 \end{aligned} \tag{8}$$

is a continuous analogue of the discrete boundary value problem (2),(4). It was shown in [10] the problem (8) is equivalent to the following system of integral equations

$$\begin{aligned} &\sum_{k=0}^{n-1}\left(R_{jk}(\xi_1)\tilde{C}_k(\xi_1)+\int_{-\infty}^{+\infty}L_{jk}(\xi_1,\xi_2)\tilde{D}_k(\xi_2)d\xi_2\right)=\tilde{b}_j(\xi_1)\\ &\sum_{k=0}^{n-1}\left(\int_{-\infty}^{+\infty}M_{jk}(\xi_1,\xi_2)\tilde{C}_k(\xi_1)d\xi_1+P_{jk}(\xi_2)\tilde{D}_k(\xi_2)\right)=\tilde{g}_j(\xi_2)\\ &j=0,1,\ldots,n-1 \end{aligned} \tag{9}$$

with unknown functions $\tilde{C}_k, \tilde{D}_k, k = 0, 1, \ldots, n-1$. The following notations are used:

$$R_{jk}(\xi_1)=\int_{-\infty}^{+\infty}\tilde{B}_j(\xi)A_{\neq}^{-1}(\xi)(i\xi_2)^k d\xi_2,\quad P_{jk}(\xi_2)=\int_{-\infty}^{+\infty}\tilde{G}_j(\xi)A_{\neq}^{-1}(\xi)(i\xi_1)^k d\xi_1,$$
$$L_{jk}(\xi_1,\xi_2)=\tilde{B}_j(\xi)A_{\neq}^{-1}(\xi)(i\xi_1)^k,\quad M_{jk}(\xi_1,\xi_2)=\tilde{G}_j(\xi)A_{\neq}^{-1}(\xi)(i\xi_2)^k,$$

$j, k = 0, 1, \ldots, n-1$. If we can solve the system (9) and find $\tilde{C}_k, \tilde{D}_k, k = 0, 1, \ldots, n-1$ the solution of the boundary value problem (9) can be constructed by the formula [10]

$$\tilde{u}(\xi)=A_{\neq}^{-1}(\xi)\left(\sum_{k=0}^{n-1}\left(\tilde{C}_k(\xi_1)(i\xi_2)^k+\tilde{D}_k(\xi_2)(i\xi_1)^k\right)\right), \tag{10}$$

where $\tilde{C}_k(\xi_1), \tilde{D}_k(\xi_2), k = 0, 1, \cdots, n-1$, are arbitrary functions from $\widetilde{H}^{s_k}(\mathbb{R})$, $s_k = s - æ + k - 1/2$.

Our next problems are the following. Given operator A and boundary operators B_j, G_j how to choose the digital operators A_d and $B_{d,j}, G_{d,j}$ to obtain the implication: the unique solvability of the system (9) gives the unique solvability of the system (6) for enough small h. This question will be discussed in the next section.

4 Comparison theorems

4.1 Projection method

Let us introduce the following space of vector-functions:

$$\tilde{\mathbf{H}}^{\Lambda}(\mathbb{R})=\tilde{\mathbf{H}}^{S}(\mathbb{R})\oplus\tilde{\mathbf{H}}^{S}(\mathbb{R}),\quad \tilde{\mathbf{H}}^{S}(\mathbb{R})=\oplus\sum_{k=0}^{n-1}\tilde{H}^{s_k}(\mathbb{R}),$$

Norms in these spaces will be defined in the following way. For $f \in \tilde{\mathbf{H}}^S(\mathbb{R})$, $f = (f_0, \dots, f_{n-1})$, $f_k \in \tilde{H}^{s_k}(\mathbb{R})$, $g \in \tilde{\mathbf{H}}^S(\mathbb{R})$, $g = (g_0, \dots, g_{n-1})$, $g_k \in \tilde{H}^{s_k}(\mathbb{R})$ we put

$$||f||_S = \sum_{k=0}^{n-1} ||f_k||_{s_k}. \quad ||g||_S = \sum_{k=0}^{n-1} ||g_k||_{s_k},$$

and if $F \in \tilde{\mathbf{H}}^\Lambda(\mathbb{R})$, $F = (f, g)$, $f \in \tilde{\mathbf{H}}^S(\mathbb{R})$, $g \in \tilde{\mathbf{H}}^S(\mathbb{R})$ we put

$$||F||_\Lambda = ||f||_S + ||g||_S.$$

Let us introduce the following notations. We denote the system (9) in the following way:

$$\begin{pmatrix} R & L \\ M & P \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} B \\ G \end{pmatrix},$$

where $C = (\tilde{c}_0, \dots, \tilde{c}_{n-1})^T$, $D = (\tilde{d}_0, \dots, \tilde{d}_{n-1})^T$, $B = (\tilde{b}_0, \dots, \tilde{b}_{n-1})^T$, $G = (\tilde{g}_0, \dots, \tilde{g}_{n-1})^T$; operators R, L, M, P acting in the space $\tilde{\mathbf{H}}^S(\mathbb{R})$ are the following: R is multiplier by the matrix-function $(r_{jk})_{j,k=0}^{n-1}$, P is multiplier by the matrix-function $(p_{jk})_{j,k=0}^{n-1}$, L, M are matrix integral operators with kernels L_{jk}, M_{jk}, respectively.

Further, we will denote Ξ_h the restriction operator on the segment $\hbar\mathbb{T}$ so that for $f \in \tilde{\mathbf{H}}^S(\mathbb{R})$, $f = (f_0, \dots, f_{n-1})$ the notation $\Xi_h f$ means the following:

$$\Xi_h f = (\chi_h f_0, \dots, \chi_h f_{n-1}),$$

where χ_h is an indicator of $\hbar\mathbb{T}$.

We denote by Q the operator

$$Q = \begin{pmatrix} R & L \\ M & P \end{pmatrix}$$

Theorem 3 *Let $s - \beta_j > 1, s - \gamma_j > 2, j = 0, 1, \dots, n-1$. We have the following estimate:*

$$||\Xi_h Q - Q \Xi_h||_{\tilde{\mathbf{H}}^\Lambda(\mathbb{R}) \to \tilde{\mathbf{H}}^\Lambda(\mathbb{R})} \le \; const\ h^\varepsilon,$$

where

$$\varepsilon = \min_{0 \le j \le n-1} \{s - \beta_j - 1, s - \gamma_j - 1\},$$

const does not depend on h, $s_k = s - æ + k - 1/2, k = 0, 1, \dots, n-1$.

Proof 1 Obviously, the matrices R, P give vanishing result in the norm, and we need to work with integral operators only. Let us consider the operator L, and extract one its component L_{jk},

$$\int_{-\infty}^{+\infty} L_{jk}(\xi_1,\xi_2)\tilde{D}_k(\xi_2)d\xi_2, \quad L_{jk}(\xi_1,\xi_2) = \tilde{B}_j(\xi)A_{\neq}^{-1}(\xi)\xi_1^k.$$

We have

$$\chi_h(\xi_1)\int_{-\infty}^{+\infty} L_{jk}(\xi_1,\xi_2)\tilde{D}_k(\xi_2)d\xi_2 - \int_{-\bar{\pi}}^{+\hbar\pi} L_{jk}(\xi_1,\xi_2)\tilde{D}_k(\xi_2)d\xi_2$$
$$= \begin{cases} \left(\int\limits_{-\infty}^{-\hbar\pi} + \int\limits_{\hbar\pi}^{+\infty}\right) L_{jk}(\xi_1,\xi_2)\tilde{D}_k(\xi_2)d\xi_2, & \xi_1 \in \hbar\mathbb{T}, \\ -\int\limits_{-\hbar\pi}^{\hbar\pi} L_{jk}(\xi_1,\xi_2)\tilde{D}_k(\xi_2)d\xi_2, & \xi_1 \notin \hbar\mathbb{T}. \end{cases}$$

Let us consider the first case and estimate as follows:

$$\left|\int_{\hbar\pi}^{+\infty} L_{jk}(\xi_1,\xi_2)\tilde{D}_k(\xi_2)d\xi_2\right| \le \; const \int_{\hbar\pi}^{+\infty} (1+|\xi|)^{\beta_j - æ}|\xi_1|^k|\tilde{D}_k(\xi_2)|d\xi_2$$
$$\le \; const \int_{\hbar\pi}^{+\infty} (1+|\xi|)^{\beta_j - s + 1/2}|\tilde{D}_k(\xi_2)|(1+|\xi_2|)^{s_k}d\xi_2$$

(we have taken into account $s_k = s - æ + k - 1/2$ and now we apply the Cauchy–Schwartz inequality)

$$\le \; const(1+|\xi_1|+\hbar)^{\beta_j - s + 1}||\tilde{D}_k||_{s_k} \le \; const \; h^{s-\beta_j-1}||D_k||_{s_k}.$$

Squaring the latter inequality, multiplying by $(1+|\xi_|)^{2s_k}$ and integrating over $\hbar\mathbb{T}$ we obtain

$$\int_{-\hbar\pi}^{\hbar\pi} (1+|\xi_|)^{2s_k}\left|\int_{\hbar\pi}^{+\infty} L_{jk}(\xi_1,\xi_2)\tilde{D}_k(\xi_2)d\xi_2\right|^2 d\xi_1$$
$$\le \; const \; h^{2(s-\beta_j-1)}||D_k||_{s_k}^2 \int_0^{+\infty} (1+|\xi_|)^{2s_k}d\xi_1 \le \; const \; h^{2(s-\beta_j-1)}||D_k||_{s_k}^2$$

For the second case ($|\xi_1| > \hbar\pi$) we obtain

$$\left|\int_{-\hbar\pi}^{+\hbar\pi} L_{jk}(\xi_1,\xi_2)\tilde{D}_k(\xi_2)d\xi_2\right| \le const \int_{-\hbar\pi}^{+\hbar\pi} (1+|\xi|)^{\beta_j-æ}|\xi_1|^k|\tilde{D}_k(\xi_2)|d\xi_2$$

$$\le const \int_{-\hbar\pi}^{+\hbar\pi} (1+|\xi|)^{\beta_j-æ}|\xi_1|^{n-1}(1+|\xi_2|)^{-s_k}|\tilde{D}_k(\xi_2)|(1+|\xi_2|)^{s_k}d\xi_2$$

$$\le const\ |\xi_1|^{n-1}(1+|\xi_1|)^{-s_k}\int_{-\hbar\pi}^{+\hbar\pi} (1+|\xi|)^{\beta_j-æ}|\tilde{D}_k(\xi_2)|(1+|\xi_2|)^{s_k}d\xi_2$$

(we apply the Cauchy–Schwartz inequality in the integral)

$$\le const\ (1+|\xi_1|)^{n-s_k-1}(1+|\xi_1|)^{\beta_j-æ+1/2k}||D_k||_{s_k}$$

Squaring the latter inequality, multiplying by $(1+|\xi_|)^{2s_k}$ and integrating over $\mathbb{R}\setminus\hbar\mathbb{T}$ we obtain

$$\left(\int_{-\infty}^{-\hbar\pi}+\int_{\hbar\pi}^{+\infty}\right)(1+|\xi_|)^{2s_k}\left|\int_{\hbar\pi}^{+\infty} L_{jk}(\xi_1,\xi_2)\tilde{D}_k(\xi_2)d\xi_2\right|^2 d\xi_1$$

$$\le const||D_k||_{s_k}^2\int_{\hbar\pi}^{+\infty}(1+\xi_1)^{2n-2+2\beta_j+1-2æ}d\xi_1 \le const||D_k||_{s_k}^2 h^{2s-2\beta_j+2\delta},$$

since $2n+2\beta_j-2æ = 2n+2\beta_j-2(s+n+\delta) = 2\beta_j-2s-2\delta < 0$.

Thus, we have proved that

$$||\chi_h L_{jk} - L_{jk}\chi_h||_{H^{s_k}(\mathbb{R})\to H^{s_k}(\mathbb{R})} \le const\ h^{s-\beta_j-1},$$

since $s-\beta_j-1 < s-\beta_j+\delta$.

Almost the same inequality can be obtained for M_{jk}

$$||\chi_h M_{jk} - M_{jk}\chi_h||_{H^{s_k}(\mathbb{R})\to H^{s_k}(\mathbb{R})} \le const\ h^{s-\gamma_j-1},$$

These estimates complete the proof. □

Corollary 1 *Under conditions of Theorem 3 the invertibility of the operator Q in the space $\tilde{\mathbf{H}}^\Lambda(\mathbb{R})$ implies the invertibility of the operator $\Xi_h Q \Xi_h$ in the space $\tilde{\mathbf{H}}^\Lambda(\hbar\mathbb{T})$ for enough small h.*

Proof 2 We apply the results of the paper [9] which imply the following: If

$$||\Xi_h Q - Q\Xi_h||_{\tilde{\mathbf{H}}^\Lambda(\mathbb{R})\to\tilde{\mathbf{H}}^\Lambda(\mathbb{R})} \to 0, \quad h\to 0$$

then the equation in the space $\tilde{\mathbf{H}}^{\Lambda}(\mathbb{R})$

$$Qu = v \tag{11}$$

admits applying so called *projection method*. In other words it means that unique solvability of the equation (11) in the space $\tilde{\mathbf{H}}^{\Lambda}(\mathbb{R})$ implies unique solvability of the equation

$$\Xi_h Q \Xi_h u = \Xi_h v \tag{12}$$

in the space $\tilde{\mathbf{H}}^{\Lambda}(\hbar\mathbb{T})$ for enough small h. Moreover, if there is bounded operator Q^{-1} in the space $\tilde{\mathbf{H}}^{\Lambda}(\mathbb{R})$ then there is bounded operator $(\Xi_h Q \Xi_h)^{-1}$ for enough small h and

$$||(\Xi_h Q \Xi_h)^{-1}||_{\tilde{\mathbf{H}}^{\Lambda}(\hbar\mathbb{T}) \to \tilde{\mathbf{H}}^{\Lambda}(\hbar\mathbb{T})} \leq \; const,$$

where const does not depend on h.

Indeed, a reader can easily verify that

$$||(\Xi_h Q \Xi_h)^{-1} - \Xi_h Q^{-1} \Xi_h||_{\tilde{\mathbf{H}}^{\Lambda}(\hbar\mathbb{T}) \to \tilde{\mathbf{H}}^{\Lambda}(\hbar\mathbb{T})} \to 0, \quad h \to 0.$$

□

4.2 Discrete and continuous

To compare discrete and continuous operators we need a special choice of discrete operators. We will do it in the following way:

The symbol $A_d(\xi)$ of the discrete operator A_d will be constructed as follows. Given wave factorization for $\tilde{A}(\xi)$

$$\tilde{A}(\xi) = A_{\neq}(\xi) \cdot A_{=}(\xi)$$

we take restrictions of factors $A_{\neq}(\xi)$, $A_{=}(\xi)$ on $\hbar\mathbb{T}^2$ and periodically continue them into $\mathbb{R}^2$. We denote these elements by $A_{d,\neq}(\xi)$, $A_{d,=}(\xi)$ and construct the periodic symbol $A_d(\xi)$ which admits periodic wave factorization with respect to K

$$A_d(\xi) = A_{d,\neq}(\xi) \cdot A_{d,=}(\xi)$$

with the same index æ. We construct discrete pseudo-differential operators $B_{d,j}$, $G_{d,j}$ taking their symbol as restrictions of symbols $\tilde{B}_j(\xi)$, $\tilde{G}_j(\xi)$ on $\hbar\mathbb{T}^2$ with periodical continuations into $\mathbb{R}^2$, , $j = 0, 1, \ldots, n-1$. The discrete boundary functions $b_{d,j}$, $g_{d,j}$ are constructed in the same way. Thus, we have the corresponding discrete boundary value problem (2),(4).

Lemma 1 *The estimate*

$$|(i\xi_m)^k - \zeta_m^k| \leq \ const \ h|\xi_m|^{k+1}$$

holds for $\xi_m \in \hbar\mathbb{T}, m = 1, 2$, *const does not depend on* h.

Proof 3 First, we estimate

$$|\zeta_1| = \left|\sum_{\nu=1}^{\infty} \frac{(i\xi_1)^{\nu+1}h^{\nu}}{(\nu+1)!}\right| = |\xi_1| \left|\sum_{\nu=0}^{\infty} \frac{(i\xi_1)^{\nu}h^{m}}{(\nu+1)!}\right| \leq |\xi_1| \sum_{\nu=0}^{\infty} \frac{(|\xi_1|h)^{\nu}}{\nu!}$$
$$= |\xi_1|e^{|\xi_1|h} \leq |\xi_1|e^{\pi}.$$

Second,

$$|\zeta_1 - i\xi_1| = |\hbar(e^{i\xi_1 h} - 1) - i\xi_1| = \left|\sum_{\nu=1}^{\infty} \frac{(i\xi_1)^{\nu+1}h^{\nu}}{(\nu+1)!}\right|$$
$$\leq |\xi_1|^2 h \sum_{\nu=0}^{\infty} \frac{|\xi_1|^{\nu}h^{\nu}}{\nu!} = |\xi_1|^2 h e^{|\xi_1|h} \leq |\xi_1|^2 h e^{\pi}.$$

We have

$$\zeta_1^k - (i\xi_1)^k = (\zeta_1 - i\xi_1)\left(\sum_{\nu=0}^{k-1} \zeta_1^{\nu}(i\xi_1)^{k-1-\nu}\right),$$

and thus

$$|\zeta_1^k - (i\xi_1)^k| \leq |\zeta_1 - i\xi_1| \sum_{\nu=0}^{k-1} |\zeta_1|^{\nu}|\xi_1|^{k-1-\nu}$$

Applying above estimates we obtain required inequality.

□

Lemma 2 *Let* $s - \beta_j > 2, s - \gamma_j > 2, j = 0, 1, \dots, n-1$.*The following estimates*

$$|L_{jk}(\xi_1, \xi_2) - l_{jk}(\xi_1, \xi_2)| \leq \ const \ h(1 + |\xi|)^{\beta_j - æ + k + 1},$$
$$|M_{jk}(\xi_1, \xi_2) - m_{jk}(\xi_1, \xi_2)| \leq \ const \ h(1 + |\xi|)^{\gamma_j - æ + k + 1},$$
$$|R_{jk}(\xi_1) - r_{jk}(\xi_1)| \leq \ const \ h(1 + |\xi_1|)^{\beta_j - æ + k + 2},$$
$$|P_{jk}(\xi_2) - p_{jk}(\xi_2)| \leq \ const \ h(1 + |\xi_1|)^{\gamma_j - æ + k + 2}$$

hold for $\xi_1, \xi_2 \in \hbar\mathbb{T}$.

Proof 4 According to above conventions for $\xi \in \hbar\mathbb{T}^2$ and using Lemma 1 we have

$$|L_{jk}(\xi_1,\xi_2) - l_{jk}(\xi_1,\xi_2)| = |\tilde{B}_j(\xi)A^{-1}_{\neq}(\xi) - \tilde{B}_{d,j}\xi)A^{-1}_{d,\neq}(\xi)||B_j(\xi)||\xi_1^k - \zeta_1^k|$$
$$\leq \, const \, (1+|\xi|)^{\beta_j - æ} h|\xi_1|^{k+1} \leq \, const \, h(1+|\xi|)^{\beta_j - æ + k+1}.$$

Further,

$$|R_{jk}(\xi_1) - r_{jk}(\xi_1)| = \left| \int_{-\infty}^{+\infty} \tilde{B}_j(\xi)A^{-1}_{\neq}(\xi)\xi_2^k d\xi_2 - \int_{-\hbar\pi}^{+\hbar\pi} \tilde{B}_{d,j}(\xi)A^{-1}_{d,\neq}(\xi)\zeta_2^k d\xi_2 \right|$$
$$\leq \int_{-\hbar\pi}^{+\hbar\pi} |\tilde{B}_{d,j}(\xi)A^{-1}_{d,\neq}(\xi)||\xi_2^k - \zeta_2^k| d\xi_2 + \left(\int_{-\infty}^{-\hbar\pi} + \int_{\hbar\pi}^{+\infty} \right) |\tilde{B}_j(\xi)A^{-1}_{\neq}(\xi)\xi_2^k| d\xi_2.$$

For the first integral we have

$$\int_{-\hbar\pi}^{+\hbar\pi} |\tilde{B}_{d,j}(\xi)A^{-1}_{d,\neq}(\xi)||\xi_2^k - \zeta_2^k| d\xi_2 \leq \, const \, h \int_{-\hbar\pi}^{+\hbar\pi} (1+|\xi|)^{\beta_j - æ + k+1} d\xi_2$$
$$\leq \, const \, h(1+|\xi_1|)^{\beta_j - æ + k+2},$$

since $\beta_j - æ + k + 2 < 0, s - \beta_j > 2$.

The second summand

$$\left| \left(\int_{-\infty}^{-\hbar\pi} + \int_{\hbar\pi}^{+\infty} \right) |\tilde{B}_j(\xi)A^{-1}_{\neq}(\xi)\xi_2^k| d\xi_2 \right| \leq \, const \int_{\hbar\pi}^{+\infty} (1+|\xi_1|+|\xi_2|)^{\beta_j - æ + k} d\xi_2$$
$$\leq \, const (1+|\xi_1|+\hbar)^{\beta_j - æ + k+1} \leq \, const \, h(1+|\xi_1|)^{\beta_j - æ + k+2}.$$

The same estimates are valid for $M_{jk} - m_{jk}$ and P_{jk} with γ_j instead of β_j. □

We introduce similar notations for the system (6) so that this system takes the following form:

$$\begin{pmatrix} r & l \\ m & p \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} B_d \\ G_d \end{pmatrix},$$

where

$$q = \begin{pmatrix} r & l \\ m & p \end{pmatrix}$$

is linear bounded operator acting in the space $\tilde{\mathbf{H}}^{\Lambda}(\hbar\mathbb{T})$.

Theorem 4 *Let $s-\beta_j > 3, s-\gamma_j > 3, j = 0, 1, \dots, n-1$. A comparison between operators Q and q is given by the estimate*

$$||\Xi_h Q \Xi_h - q||_{\tilde{\mathbf{H}}^{\Lambda}(\hbar\mathbb{T}) \to \tilde{\mathbf{H}}^{\Lambda}(\hbar\mathbb{T})} \leq \ const \ h,$$

where const does not depend on h.

Proof 5 We need to estimate $H^{s_k}(\hbar\mathbb{T})$-norms of the following elements

$$(R_{jk}(\xi_1) - r_{jk}(\xi_1))f(\xi_1), \quad (R_{jk}(\xi_2) - p_{jk}(\xi_2))f(\xi_2),$$
$$\int\limits_{-\hbar\pi}^{\hbar\pi} (L_{jk}(\xi_1, \xi_2) - l_{jk}(\xi_1, \xi_2))f(\xi_2)d\xi_2,$$
$$\int\limits_{-\hbar\pi}^{\hbar\pi} (M_{jk}(\xi_1, \xi_2) - m_{jk}(\xi_1, \xi_2))f(\xi_1 d\xi_1.$$

We have according to Lemma 2

$$|(R_{jk}(\xi_1) - r_{jk}(\xi_1))f(\xi_1)| \leq \ const \ h(1 + |\xi_1|)^{\beta_j - æ + k + 2}|f(\xi_1)|.$$

Multiplying the latter inequality by $(1 + |\xi_1|)^{s_k}$, squaring, integrating over $\hbar\mathbb{T}$ and applying the Cauchy–Schwartz inequality we obtain

$$\int\limits_{-\hbar\pi}^{+\hbar\pi} (1 + |\xi_1|)^{2s_k}|R_{jk}(\xi_1) - r_{jk}(\xi_1)|^2|f(\xi_1)|^2 d\xi_1 \leq \ const \ h^2||f||_{s_k}^2$$

since $\beta_j - æ + k + 2 < 0$.

Let us consider

$$\int\limits_{-\hbar\pi}^{\hbar\pi} (L_{jk}(\xi_1, \xi_2) - l_{jk}(\xi_1, \xi_2))f(\xi_2)d\xi_2.$$

Using Lemma 2 we have

$$\left|\int_{-\hbar\pi}^{\hbar\pi}(L_{jk}(\xi_1,\xi_2)-l_{jk}(\xi_1,\xi_2))f(\xi_2)d\xi_2\right|$$
$$\le\ const\ h\int_{-\hbar\pi}^{\hbar\pi}(1+|\xi|)^{\beta_j-æ+k+1}|f(\xi_2)|d\xi_2$$
$$\le\ const\ h\int_{-\hbar\pi}^{\hbar\pi}(1+|\xi|)^{\beta_j-s+5/2}|f(\xi_2)|(1+|\xi_2|)^{s_k}d\xi_2$$

since $\beta_j-æ+k+2-s_k=-\beta_j-s+5/2$. Now applying Cauchy–Schwartz inequality we find

$$\left|\int_{-\hbar\pi}^{\hbar\pi}(L_{jk}(\xi_1,\xi_2)-l_{jk}(\xi_1,\xi_2))f(\xi_2)d\xi_2\right|$$
$$\le\ const\ h||f||_{s_k}\left(\int_{-\hbar\pi}^{\hbar\pi}(1+|\xi_1|+|\xi_2|)^{2\beta_j-2s+5}d\xi_2\right)^{1/2}$$
$$\le\ const\ h||f||_{s_k}(1+|\xi_1|)^{\beta_j-s+3}$$

according to the condition $s-\beta_j>3$. Squaring, multiplying by $(1+|\xi_1|)^{2s_k}$ and integrating over $\hbar\mathbb{T}$ we conclude

$$\int_{-\hbar\pi}^{\hbar\pi}(1+|\xi_1|)^{2s_k}\left|\int_{-\hbar\pi}^{\hbar\pi}(L_{jk}(\xi_1,\xi_2)-l_{jk}(\xi_1,\xi_2))f(\xi_2)d\xi_2\right|^2 d\xi_1$$
$$\le\ const\ h^2||f||^2_{s_k}\int_{-\hbar\pi}^{\hbar\pi}(1+|\xi_1|)^{2(\beta_j-s+3+s_k)}d\xi_1\le\ const\ h^2||f||^2_{s_k}$$

since $2(\beta_j-s+3+s_k)<-1$. Indeed, $2(\beta_j-s+3+s_k)=2(\beta_j-s+3+s-æ+k-1/2)=2(\beta_j-s-\delta)$. Obviously, the inequality $2(\beta_j-s-\delta)<-1$ is equivalent to $s-\beta_j>-1-\delta$.

□

Corollary 2 *Under conditions of Theorem 4 the invertibility of the operator Q in the space $\tilde{\mathbf{H}}^{\Lambda}(\mathbb{R})$ implies the invertibility of the operator q in the space $\tilde{\mathbf{H}}^{\Lambda}(\hbar\mathbb{T})$ for enough small h.*

Proof 6 Indeed, we have the invertibility of $\Xi_h Q \Xi_h$ by Corollary 1 and the invertibility of q is obtained by Theorem 4.

□

Conclusion

Main goal of the paper was to prove unique solvability of discrete boundary value problem for small h having in mind unique solvability of its continuous analogue. It was done by a special choice of a discrete operator and discrete boundary conditions. We hope that estimates of Theorem 3 and 4 will help us to obtain some estimates for discrete and continuous solutions.

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