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PARTIAL DIFFERENTIAL EQUATIONS

On Some Elliptic Boundary Value Problems in Conic Domains

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Abstract—A model elliptic pseudodifferential equation in a polyhedral cone is considered, and the situation when some of the parameters of the cone tend to their limiting values is investigated. In Sobolev–Slobodetskii spaces, a solution of the equation in the cone is constructed in the case of a special wave factorization of the elliptic symbol. It is shown that a limit solution of the boundary value problem with an additional integral condition can exist only under additional constraints on the boundary function.

Keywords: pseudodifferential equation, elliptic symbol, cone, wave factorization, domain with cut

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1. INTRODUCTION

In the theory of pseudodifferential equations in nonsmooth domains (domains with a nonsmooth boundary), model equations in canonical nonsmooth domains (cones) play a special role: the unique solvability of such an equation guarantees the Fredholm property of a general pseudodifferential equation in a domain with a corresponding conic point at the boundary. This fact is called the local principle and, to a varying degree of generality, is involved in all variants of theories of pseudodifferential equations and related boundary value problems in nonsmooth domains.

The theory of pseudodifferential equations is widely used in numerous branches of mathematics and physics. Specifically, such equations appear in problems of electromagnetic wave scattering (see, e.g., [1–3]), where the factorization method is widely applied. Below, the multidimensional variant of this method is used to derive integral representations of solutions to the considered boundary value problems.

This paper deals with some boundary value problems for model pseudodifferential equations in the case of a cone degenerating into a lower-dimensional one, in other words, when some of the parameters of the original cone tend to zero. The study is based on the theory of boundary value problems for elliptic pseudodifferential equations (see [4]), on the theory of one-dimensional singular integral equations (see [5–7]), on multidimensional complex analysis (see [8]), and on the wave factorization method developed by the author, which has led to numerous results on the solvability of boundary value problems for elliptic pseudodifferential equations in canonical nonsmooth domains of Euclidean space $\mathbb{R}^m$ (see [9–13]).

2. PSEUDODIFFERENTIAL EQUATIONS IN A CONE

2.1. Model Operators in a Canonical Domain

Let $D \subset \mathbb{R}^m$ be a domain in an m-dimensional space, and a function be defined on $\mathbb{R}^m$. A model pseudodifferential operator A in D is an operator of the form

$$(Au)(x) = \int_D \int_{\mathbb{R}^m} A(\xi)u(y)e^{i(y-x)\xi}d\xi dy, \quad x \in D,$$

where the function $A(\xi)$ is called its symbol. Here, we consider the class of symbols satisfying the condition

$$c_1 \le \left|A(\xi)(1+|\xi|)^{-\alpha}\right| \le c_2, \quad \xi \in \mathbb{R}^m.$$

The number $\alpha \in \mathbb{R}$ is the order of the pseudodifferential operator A.

A domain $D \subset \mathbb{R}^m$ is called canonical if D is a cone $C \subset \mathbb{R}^m$ that does not contain a whole straight line in the space $\mathbb{R}^m$.

Cones are associated with important domains in multidimensional complex space and with the concept of special factorization of an elliptic symbol, with the help of which the solvability of a pseudodifferential equation in a cone can be described.

Definition 1. A *radial tube domain over a cone C* is a subset of the multidimensional complex space $\mathbb{C}^m M$ of the form

$$T(C) \equiv \{z \in \mathbb{C}^m : z = x + iy,\, x \in \mathbb{R}^m,\, y \in C\}.$$

The *dual cone* $\overset{*}{C}$ is the one satisfying the condition

$$x \cdot y > 0 \quad \forall y \in C,$$

where $x \cdot y$ denotes the inner product of x and y.

Throughout this paper, we assume that the symbol $A(\xi)$ admits a wave factorization with respect to the cone C (see [8, 14]) with an index æ:

$$A(\xi) = A_{\neq}(\xi) \cdot A_{=}(\xi),$$

where $A_{\neq}(\xi)$ and $A_{=}(\xi)$ can be analytically continued to $T(-\overset{*}{C})$ and $T(\overset{*}{C})$, respectively. Below is the precise definition of such a factorization, since the value of the index has a large effect on the solvability of a pseudodifferential equation.

Definition 2. The *wave factorization of the symbol $A(\xi)$ with respect to the cone C* is its representation of the form

$$A(\xi) = A_{\neq}(\xi) A_{=}(\xi),$$

where the factors $A_{\neq}(\xi)$, $A_{=}(\xi)$ satisfy the following conditions:

(i) $A_{\neq}(\xi)$, $A_{=}(\xi)$ are defined for all $\xi \in \mathbb{R}^m$, except possibly the points of $\partial(\overset{*}{C} \cup (-\overset{*}{C}))$;

(ii) $A_{\neq}(\xi)$, $A_{=}(\xi)$ admit analytic continuations to the radial tube domains $T(\overset{*}{C})$, $T(-\overset{*}{C})$, respectively, and satisfy the estimates

$$c_1'(1 + |\xi| + |\tau|)^{æ} \le |A_{\neq}(\xi + i\tau)| \le c_1(1 + |\xi| + |\tau|)^{æ},$$

$$c_2'(1 + |\xi| + |\tau|)^{\alpha - æ} \le |A_{=}(\xi - i\tau)| \le c_2(1 + |\xi| + |\tau|)^{\alpha - æ} \quad \forall \tau \in \overset{*}{C}.$$

The number $æ \in \mathbb{R}$ is called the *index of the wave factorization*.

The class of symbols admitting a wave factorization is fairly rich. This issue is discussed in a whole chapter in [14], where numerous examples are given.

2.2. Construction of the Solution

We consider a polyhedral cone and the equation

$$(Au)(x) = 0, \quad x \in \mathbb{R}^3 \backslash \overline{C_+^{ab}}, \tag{1}$$

in the Sobolev–Slobodetskii space $H^s(\mathbb{R}^3 \backslash (\overline{C_+^{ab}}))$, where

$$C_+^{ab} = \left\{x \in \mathbb{R}^3 : x = (x_1, x_2, x_3),\, x_3 < a|x_1| + b|x_2|,\, a, b > 0\right\}.$$

Consider the special case $æ - s = 1 + \delta$, $|\delta| < 1/2$, for which results concerning the structure of the solution to Eq. (1) were obtained in [11–13]. We introduce the following one-dimensional singular integral operators (see [5–7]):

$$(S_1 u)(\xi_1, \xi_2, \xi_3) = \text{v.p.} \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{u(\tau, \xi_2, \xi_3) d\tau}{\xi_1 - \tau}, \quad (S_2 u)(\xi_1, \xi_2, \xi_3) = \text{v.p.} \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{u(\xi_1, \eta, \xi_3) d\eta}{\xi_2 - \eta}.$$

The following formula was derived in terms of these operators:

$$A_{\neq}(\xi)\tilde{u}(\xi) = \tilde{C}_1(\xi_1 - a\xi_3, \xi_2 - b\xi_3) + \tilde{C}_2(\xi_1 - a\xi_3, \xi_2 + b\xi_3) \\ + \tilde{C}_3(\xi_1 + a\xi_3, \xi_2 - b\xi_3) + \tilde{C}_4(\xi_1 + a\xi_3, \xi_2 + b\xi_3), \tag{2}$$

where

$$\tilde{C}_1(\xi_1 - a\xi_3, \xi_2 - b\xi_3) = \frac{1}{4}\tilde{c}_0(\xi_1 - a\xi_3, \xi_2 - b\xi_3) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 - b\xi_3) \\ - \frac{1}{2}(S_2\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 - b\xi_3) + (S_1S_2\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 - b\xi_3);$$

$$\tilde{C}_2(\xi_1 - a\xi_3, \xi_2 + b\xi_3) = \frac{1}{4}\tilde{c}_0(\xi_1 - a\xi_3, \xi_2 + b\xi_3) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 + b\xi_3) \\ + \frac{1}{2}(S_2\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 + b\xi_3) - (S_1S_2\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 + b\xi_3);$$

$$\tilde{C}_3(\xi_1 + a\xi_3, \xi_2 - b\xi_3) = \frac{1}{4}\tilde{c}_0(\xi_1 + a\xi_3, \xi_2 - b\xi_3) + \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 - b\xi_3) \\ - \frac{1}{2}(S_2\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 - b\xi_3) - (S_1S_2\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 - b\xi_3);$$

$$\tilde{C}_4(\xi_1 + a\xi_3, \xi_2 + b\xi_3) = \frac{1}{4}\tilde{c}_0(\xi_1 + a\xi_3, \xi_2 + b\xi_3) + \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 + b\xi_3) \\ + \frac{1}{2}(S_2\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 + b\xi_3) + (S_1S_2\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 + b\xi_3),$$

and $c_0(x_1, x_2)$ is an arbitrary function from the space $H^{s-æ+1/2}(\mathbb{R}^2)$. Thus, the kernel of the operator A is a one-dimensional subspace of $H^s(\mathbb{R}^3\setminus\overline{(C_+^{ab})})$.

To single out a unique solution (a uniquely determined function $c_0(\xi_1, \xi_2)$), we need an additional condition. Based on the form of the general solution, it seems the most convenient to specify the restriction $\tilde{u}(\xi_1, \xi_2, 0)$ or, in other words, to specify the integral condition

$$\int_{-\infty}^{+\infty} u(x_1, x_2, x_3)dx_3 \equiv g(x_1, x_2), \tag{3}$$

which, in terms of Fourier images, has the form

$$\tilde{u}(\xi_1, \xi_2, 0) = \tilde{g}(\xi_1, \xi_2), \tag{4}$$

where $g(x_1, x_2)$ is a given function.

Setting $\xi_3 = 0$ in formula (2) yields

$$\sum_{k=1}^{4}\tilde{C}_k(\xi_1, \xi_2) = \frac{1}{4}\tilde{c}_0(\xi_1, \xi_2) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1, \xi_2) - \frac{1}{2}(S_2\tilde{c}_0)(\xi_1, \xi_2) + (S_1S_2\tilde{c}_0)(\xi_1, \xi_2) \\ + \frac{1}{4}\tilde{c}_0(\xi_1, \xi_2) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1, \xi_2) + \frac{1}{2}(S_2\tilde{c}_0)(\xi_1, \xi_2) - (S_1S_2\tilde{c}_0)(\xi_1, \xi_2) \\ + \frac{1}{4}\tilde{c}_0(\xi_1, \xi_2) + \frac{1}{2}(S_1\tilde{c}_0)(\xi_1, \xi_2) - \frac{1}{2}(S_2\tilde{c}_0)(\xi_1, \xi_2) - (S_1S_2\tilde{c}_0)(\xi_1, \xi_2) \\ + \frac{1}{4}\tilde{c}_0(\xi_1, \xi_2) + \frac{1}{2}(S_1\tilde{c}_0)(\xi_1, \xi_2) + \frac{1}{2}(S_2\tilde{c}_0)(\xi_1, \xi_2) + (S_1S_2\tilde{c}_0)(\xi_1, \xi_2) = \tilde{c}_0(\xi_1, \xi_2).$$

In view of condition (4), we find

$$\tilde{c}_0(\xi') = \tilde{A}_{\neq}(\xi', 0)\,\tilde{g}(\xi'). \tag{5}$$

Below is the result of our calculations, which is the starting point for further research (see [11, 13]).

Theorem 1. *Suppose that* $æ - s = 1 + \delta$, $|\delta| < 1/2$, *and* $g \in H^{s+1/2}(\mathbb{R}^2)$. *Then the unique solution of problem* (1), (3) *is given by formula* (2), *where* $c_0(x_1, x_2)$ *is defined by* (5).

The cone has two parameters a, b, and the cases when they tend to their limiting values 0 or ∞ are of interest. The domain thus obtained is one with a cut, and the cut is a cone of dimension lower than the space dimension. We are interested in the behavior of the solution to problem (1), (3) in these limiting cases. It will be shown below that limit solutions can exist only under certain additional conditions on the function g.

It should be noted that, in the two-dimensional case, there is only one cone, and the limit solutions are considered in [15]. In multidimensional cases, cones are much more in number, and, in particular, the limit situations $a \to 0$, $b = \text{const}$ and $a = \text{const}$, $b \to 0$ are described in [13]. Note also that the case $a, b \to 0$ corresponds to a half-space and is completely treated in [4].

Below, relying on formula (2), we analyze some limit situations and determine the conditions to be imposed on g for limit solutions to exist.

Consider equality (2). Using the changes of variables $\xi_1 - a\xi_3 = t_1$, $\xi_1 + a\xi_3 = t_3$ and finding $\xi_1 = (t_3 + t_1)/2$, $\xi_3 = (t_3 - t_1)/(2a)$, we can define the function $\tilde{c}_0$ in the new variables t_1, ξ_2, t_3 with condition (4) taken into account. Rewriting formula (2) in the new variables t_1, ξ_2, t_3 yields

$$\begin{aligned} A_{\neq}\left(\frac{t_2 + t_1}{2}, \xi_2, \frac{t_3 - t_1}{2a}\right)\tilde{u}\left(\frac{t_2 + t_1}{2}, \xi_2, \frac{t_3 - t_1}{2a}\right) &= \tilde{C}_1\left(t_1, \xi_2 - b\frac{t_3 - t_1}{2a}\right) \\ + \tilde{C}_2\left(t_1, \xi_2 + b\frac{t_3 - t_1}{2a}\right) + \tilde{C}_3\left(t_3, \xi_2 - b\frac{t_3 - t_1}{2a}\right) &+ \tilde{C}_4\left(t_3, \xi_2 + b\frac{t_3 - t_1}{2a}\right). \end{aligned} \tag{6}$$

Sending $a \to +\infty$, we obtain

$$A_{\neq}\left(\frac{t_2 + t_1}{2}, \xi_2, 0\right)\tilde{u}\left(\frac{t_2 + t_1}{2}, \xi_2, 0\right) = \tilde{C}_1(t_1, \xi_2) + \tilde{C}_2(t_1, \xi_2) + \tilde{C}_3(t_3, \xi_2) + \tilde{C}_4(t_3, \xi_2).$$

Next, we do some calculations:

$$\tilde{C}_1(t_1, \xi_2) + \tilde{C}_2(t_1, \xi_2) + \tilde{C}_3(t_3, \xi_2) + \tilde{C}_4(t_3, \xi_2) = \frac{\tilde{c}_0(t_1, \xi_2) + \tilde{c}_0(t_3, \xi_2)}{2} - (S_1\tilde{c}_0)(t_1, \xi_2) + (S_1\tilde{c}_0)(t_3, \xi_2).$$

Using condition (4), taking into account formula (5) in (6), and introducing the new notation

$$\tilde{A}_{\neq}(\xi_1, \xi_2, 0)\tilde{g}(\xi_1, \xi_2) \equiv h(\xi_1, \xi_2),$$

we obtain the following equation with parameter ξ_2 for the function$g$:

$$h\left(\frac{t_2 + t_1}{2}, \xi_2\right) = \frac{h(t_1, \xi_2) + h(t_3, \xi_2)}{2} - (S_1 h)(t_1, \xi_2) + (S_1 h)(t_3, \xi_2). \tag{7}$$

Combining the above calculations yields the following result.

Theorem 2. *If the symbol* $A(\xi)$ *admits a wave factorization with respect to the cone* C_+^{ab} *with an index* æ *such that* $æ - s = 1 + \delta$, *where* $|\delta| < 1/2$, *for all sufficiently large* a, *then the unique solution of the boundary value problem* (1), (3) *has a limit as* $a \to +\infty$ *if and only if the function* $g \in H^{s+1/2}(R^2)$ *in the boundary condition is a solution of Eq.* (7).

2.3. Two-Dimensional Situation

Now we describe what happens in the plane. Equation (1) is written as

$$(Au)(x) = 0, \quad x \in \mathbb{R}^2 \backslash \overline{C_+^a}, \tag{8}$$

and is considered in the space $H^s(\mathbb{R}^2 \backslash \overline{C_+^a})$. The number æ denotes the index of the wave factorization with respect to the angle C_+^a [14] (as before, we assume that such a factorization for the symbol $A(\xi)$ exists). Assume that $1/2 < æ - s < 3/2$ and the angle C_+^a has the form

$$C_+^a = \left\{x \in \mathbb{R}^2 : x_2 > a|x_1|, a > 0\right\}.$$

The general solution of Eq. (8) in the Sobolev–Slobodetskii space $H^s(\mathbb{R}^2\backslash\overline{C_+^a})$ has the following form (see [11]):

$$\tilde{u}(\xi) = \frac{\tilde{c}_0(\xi_1 + a\xi_2) + \tilde{c}_0(\xi_1 - a\xi_2)}{2A_{\neq}(\xi_1,\xi_2)} + A_{\neq}^{-1}(\xi_1,\xi_2)\left(\text{v.p.}\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{c}_0(\eta)d\eta}{\xi_1 + a\xi_2 - \eta} - \text{v.p.}\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{c}_0(\eta)d\eta}{\xi_1 - a\xi_2 - \eta}\right),$$

where c_0 is an arbitrary function from $H^{s-æ+1/2}(\mathbb{R})$.

Introducing the notation

$$\text{v.p.}\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{\tilde{c}_0(\eta)d\eta}{\xi_1 + a\xi_2 - \eta} \equiv \tilde{d}_0(\xi_1 + a\xi_2), \quad \text{v.p.}\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{\tilde{c}_0(\eta)d\eta}{\xi_1 - a\xi_2 - \eta} \equiv \tilde{d}_0(\xi_1 - a\xi_2) \tag{9}$$

we obtain

$$\tilde{u}(\xi_1,\xi_2) = \frac{\tilde{c}_0(\xi_1 + a\xi_2) + \tilde{c}_0(\xi_1 - a\xi_2) + \tilde{d}_0(\xi_1 + a\xi_2) - \tilde{d}_0(\xi_1 - a\xi_2)}{2A_{\neq}(\xi_1,\xi_2)} \equiv \frac{\tilde{c}(\xi_1 + a\xi_2) + \tilde{d}(\xi_1 - a\xi_2)}{2A_{\neq}(\xi_1,\xi_2)}, \tag{10}$$

where $\tilde{c}(\xi_1 + a\xi_2) \equiv \tilde{c}_0(\xi_1 + a\xi_2) + \tilde{d}_0(\xi_1 + a\xi_2)$ and $\tilde{d}(\xi_1 - a\xi_2) \equiv \tilde{c}_0(\xi_1 - a\xi_2) - \tilde{d}_0(\xi_1 - a\xi_2)$.

Next, we add the integral condition

$$\int_{-\infty}^{+\infty} u(x_1,x_2)dx_2 \equiv g(x_1), \tag{11}$$

which, in terms of Fourier images, becomes $\tilde{u}(\xi_1,0) = \tilde{g}(\xi)$. Taking into account formula (10), we conclude that

$$\frac{\tilde{c}_0(\xi_1)}{A_{\neq}(\xi_1,0)} = \tilde{g}(\xi_1).$$

Therefore, we can define the function $\tilde{c}_0(\xi_1) = A_{\neq}(\xi_1,0)\tilde{g}(\xi_1)$. Then, using formula (9), we can define $\tilde{d}_0(\xi_1)$ such that formula (10) gives a solution of Eq. (8). Summing up, we conclude that the solution of Eq. (8) under condition (11) is given by the formula

$$\tilde{u}(\xi_1,\xi_2) = \frac{A_{\neq}(\xi_1 + a\xi_2,0)\tilde{g}(\xi_1 + a\xi_2) + A_{\neq}(\xi_1 - a\xi_2,0)\tilde{g}(\xi_1 - a\xi_2)}{2A_{\neq}(\xi_1,\xi_2)}$$
$$+ \frac{1}{2A_{\neq}(\xi_1,\xi_2)}\text{v.p.}\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{\xi_1 + a\xi_2 - \eta} - \frac{1}{2A_{\neq}(\xi_1,\xi_2)}\text{v.p.}\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{\xi_1 - a\xi_2 - \eta}$$

(for more details, see [12]).

Now, we try to find out what happens to the solution as $a \to \infty$; this situation corresponds to a plane with a cut along a ray.

Introducing the new variables

$$a_{\neq}(t_1,t_2) \equiv A_{\neq}\left(\frac{t_1 + t_2}{2}, \frac{t_1 - t_2}{2a}\right), \quad \tilde{U}(t_1,t_2) \equiv \tilde{u}\left(\frac{t_1 + t_2}{2}, \frac{t_1 - t_2}{2a}\right)$$

we write

$$\tilde{U}(t_1,t_2) = \frac{A_{\neq}(t_1,0)\tilde{g}(t_1) + A_{\neq}(t_2,0)\tilde{g}(t_2)}{2a_{\neq}(t_1,t_2)}$$
$$+ \frac{1}{2a_{\neq}(t_1,t_2)}\text{v.p.}\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{t_1 - \eta} - \frac{1}{2a_{\neq}(t_1,t_2)}\text{v.p.}\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{t_2 - \eta}.$$

Sending $a \to +\infty$ and introducing the new notation $A_{\neq}(t,0)\tilde{g}(t) \equiv G(t)$ and $\lim_{a\to+\infty} a_{\neq}(t_1,t_2) \equiv h(t_1,t_2)$, we can write

$$\tilde{u}\left(\frac{t_1+t_2}{2},0\right) = \tilde{U}(t_1,t_2) = \frac{A_{\neq}(t_1,0)\tilde{g}(t_1) + A_{\neq}(t_2,0)\tilde{g}(t_2)}{2h(t_1,t_2)}$$
$$+\frac{1}{2h(t_1,t_2)}\,\text{v.p.}\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{t_1-\eta} - \frac{1}{2h(t_1,t_2)}\,\text{v.p.}\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{t_2-\eta}. \tag{12}$$

Taking into account condition (11) gives

$$G\left(\frac{t_1+t_2}{2}\right) = \frac{G(t_1)+G(t_2)}{2} + (SG)(t_1) - (SG)(t_2), \tag{13}$$

where

$$(SG)(t) = \text{v.p.}\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{G(\eta)d\eta}{t-\eta}.$$

Our considerations are summarized in the following result.

Theorem 3. *If the symbol $A(\xi_1,\xi_2)$ admits a wave factorization with respect to the cone C_+^a for all sufficiently large a, then solution* (10) *to the boundary value problem* (8), (11) *has a limit as $a \to +\infty$ if and only if condition* (13) *is satisfied.*

3. CUTS IN A MULTIDIMENSIONAL SPACE

Relying on the material of the preceding sections, we describe several multidimensional situations. More precisely, we show multidimensional domains with cuts that can be obtained via a similar passage to the limit and formulate the corresponding boundary value problems.

The basic idea is as follows. Let C_1 and C_2 be cones in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, that do not contain a whole straight line. Obviously, $C_1 \times C_2$ is a cone in $\mathbb{R}^{m+n}$ that does not contain a whole straight line in $\mathbb{R}^{m+n}$. Then we can consider a boundary value problem similar to (1), (5) in the domain $\mathbb{R}^{m+n}\backslash(C_1\times C_2)$. Writing a formula for the solution of this problem (in the presence of a wave factorization with respect to the "large" cone), we can consider boundary value problems in domains with multidimensional cuts of various geometry, sending the parameters of C_1 and C_2 to limiting values.

The first variant of such a boundary value problem is

$$(Au)(x) = 0, \quad x \in \mathbb{R}^5\backslash\overline{\left(C_+^a \times C_+^{bd}\right)}, \quad \int_{\mathbb{R}^2} u(x_1,x_2,x_3,x_4,x_5)dx_2dx_5 = g(x_1,x_3,x_4), \tag{14}$$

where $C_+^a \subset \mathbb{R}^2$ and $C_+^{bd} \subset \mathbb{R}^3$.

The second variant is

$$(Au)(x) = 0, \quad x \in \mathbb{R}^4\backslash\overline{\left(C_+^a \times C_+^{b}\right)}, \quad \int_{\mathbb{R}^2} u(x_1,x_2,x_3,x_4)dx_2dx_4 = g(x_1,x_3), \tag{15}$$

where $C_+^a \subset \mathbb{R}^2$ and $C_+^b \subset \mathbb{R}^2$.

It is possible to consider a variant with two polyhedral angles in $\mathbb{R}^6$, more precisely, the boundary value problem

$$(Au)(x) = 0, \quad x \in \mathbb{R}^6\backslash\overline{(C_+^{ab} \times C_+^{dl})}, \quad \int_{\mathbb{R}^2} u(x_1,x_2,x_3,x_4,x_5,x_6)dx_3dx_6 = g(x_1,x_2,x_4,x_5), \tag{16}$$

where $C_+^{ab} \subset \mathbb{R}^3$ and $C_+^{dl} \subset \mathbb{R}^3$.

Finally, we can return to problem (14) and consider the same boundary value problem. Of course, its solution will be the same, but we consider a different limit variant: $a \to \infty$, $b \to \infty$, $d = \text{const}$, obtaining a cut of different geometry. In more detail, these situations will be addressed elsewhere.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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