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On Discrete Neumann Problem in a Quadrant

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Abstract—We study a discrete analogue on the Neumann boundary value problem for elliptic pseudo-differential equation in a quadrant. This approach is based on a special factorization of an elliptic symbol which permits to construct a general solution for a discrete pseudo-differential equation in discrete analogues of Sobolev–Slobodetskii spaces. The discrete Neumann boundary conditions are considered in the paper. Unique solvability of discrete Neumann boundary value problem is proved and a comparison between discrete and continuous solutions is given.

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1. INTRODUCTION

The theory of pseudo-differential operators and related equations has started its development from second half of last century [1, 2]. It was a convenient mathematical language to combine the theory of partial differential equations and related boundary value problems and the theory of convolution equations based on the distribution theory. At the same time there were certain discrete versions of boundary value problems for partial differential equations [3, 4] and convolution equations [8, 9] including one-dimensional singular integral equations [5–7]. It seems these discrete theories do not join although it is not so for the continuous situation. Taking into account this problem the third author has suggested to develop the discrete theory of pseudo-differential equations and boundary value problems and study a possibility to apply this theory to approximate solution of continuous boundary value problems.

First, studies in this direction were devoted to multidimensional singular integral equations with Calderon–Zygmund kernels [12]. We have worked with model symbols non-depending on a spatial variable in canonical domains of Euclidean space in view of the local principle. In the theory, it means that to obtain Fredholm property for a general equation in an arbitrary domain we need to describe invertibility conditions for model operators in special canonical domains. These canonical domains are cones, and we have started studying discrete pseudo-differential equations in different cones.

The case of whole discrete space was enough simple, but for a discrete half-space we need a periodic analogue of factorization for an elliptic symbol [2]. We have described solvability conditions for discrete pseudo-differential equations and some discrete boundary value problems [13–16] and have given a comparison between discrete and continuous solution under small values of a parameter [17].

The typical conical case is more complicated and it requires a periodic analogue of wave factorization for an elliptic symbol [10]. This concept uses elements of multidimensional complex analysis [11] in difference from the half-space case in which the classical theory of Riemann boundary value problem and one-dimensional singular integral equations could be applied [5–7].

In this paper, we consider a model discrete pseudo-differential equation in a quadrant, use a special factorization for a symbol with a certain index and describe solvability conditions for the discrete Neumann problem. Most principal point is a comparison between discrete and continuous solutions.

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2. AUXILIARIES: DISCRETE SPACES AND DIGITAL OPERATORS

In this section we introduce some notations and definitions which we will use in the paper. Basic details can be found in the paper [15].

Let $\mathbb{Z}^2$ be an integer lattice in a plane, and $K = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_1 > 0, x_2 > 0\}$, $K_d = h\mathbb{Z}^2 \cap K, h > 0$. We consider functions of a discrete variable $u_d(\tilde{x}), \tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in h\mathbb{Z}^2$.

We also use the following notations: $\mathbb{T}^2 = [-\pi, \pi]^2, \hbar = h^{-1}$, $\zeta^2 = \zeta_1^2 + \zeta_2^2 = h^{-2}((e^{-ih\cdot\xi_1} - 1)^2 + (e^{-ih\cdot\xi_2} - 1)^2)$, $S(h\mathbb{Z}^2)$ for discrete analogue of the Schwartz space of infinitely differentiable rapidly decreasing functions at infinity.

We introduce the discrete space $H^s(h\mathbb{Z}^2)$ which consists of discrete functions and it is a closure of the space $S(h\mathbb{Z}^2)$ with respect to the norm

$$||u_d||_s = \left(\int\limits_{\hbar\mathbb{T}^2} (1 + |\zeta^2|)^s |\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2}, \tag{1}$$

where $\tilde{u}_d(\xi)$ denotes the discrete Fourier transform

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^2} e^{i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^2, \quad \xi \in \hbar\mathbb{T}^2.$$

The space $H^s(K_d)$ consists of discrete functions from the space $H^s(h\mathbb{Z}^2)$, and their supports belong to $\overline{K}_d$. Norm in the space $H^s(K_d)$ is induced by norm of the space $H^s(h\mathbb{Z}^2)$.

The Fourier image of the space $H^s(K_d)$ is denoted by $\widetilde{H}^s(K_d)$.

Let $A_d(\xi)$ be a measurable periodic function defined in $\mathbb{R}^2$ with the basic cube of periods $\hbar\mathbb{T}^2$.

Definition 1. A *digital pseudo-differential operator* A_d with the symbol $A_d(\xi)$ in discrete quadrant K_d is called the following operator

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbb{Z}^2} h^2 \int\limits_{\hbar\mathbb{T}^2} A_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in K_d. \tag{2}$$

Here we will consider symbols, satisfying the condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \leq |A_d(\xi)| \leq c_2(1 + |\zeta^2|)^{\alpha/2} \tag{3}$$

with positive constants c_1, c_2, that do not depend on h. This class of symbols satisfying (3) will be denoted by E_α. The number $\alpha \in \mathbb{R}$ is called an order of the digital pseudo-differential operator A_d.

We study solvability of the discrete equation

$$(A_d u_d)(\tilde{x}) = 0, \quad \tilde{x} \in K_d, \tag{4}$$

in the space $H^s(K_d)$, and for this purpose we need certain specific domains of two-dimensional complex space $\mathbb{C}^2$.

Definition 2. A domain of the type $\mathcal{T}_h(K) = \hbar\mathbb{T}^2 + iK$ is called a *tube domain* over the quadrant K.

We will work with holomorphic functions $f(x + i\tau)$ in such tube domains $\mathcal{T}_h(K)$. Let us note that a lot of results of similar theory of holomorphic functions in radial tube domains over cones are presented in the book [11].

Definition 3. *Periodic wave factorization* of the symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form $A_d(\xi) = A_{d,\neq}(\xi) A_{d,=}(\xi)$, where the factors $A_{d,\neq}(\xi)$, $A_{d,=}(\xi)$ admit holomorphic continuation into tube domains $\mathcal{T}_h(K), \mathcal{T}_h(-K)$ respectively, satisfying the estimates

$$c_1(1 + |\hat{\zeta}^2|)^{\frac{æ}{2}} \leq |A_{d,\neq}(\xi + i\tau)| \leq c_1'(1 + |\hat{\zeta}^2|)^{\frac{æ}{2}},$$

$$c_2(1 + |\hat{\zeta}^2|)^{\frac{\alpha-æ}{2}} \leq |A_{d,=}(\xi - i\tau)| \leq c_2'(1 + |\hat{\zeta}^2|)^{\frac{\alpha-æ}{2}},$$

with positive constants c_1, c_1', c_2, c_2' non-depending on h;

$$\hat{\zeta}^2 \equiv \hbar^2 \left((e^{-ih(\xi_1+i\tau_1)} - 1)^2 + (e^{-ih(\xi_2+i\tau_2)} - 1)^2\right), \quad \xi = (\xi_1, \xi_2) \in \hbar\mathbb{T}^2, \quad \tau = (\tau_1, \tau_2) \in K.$$

The number $æ \in \mathbb{R}$ is called an *index* of periodic wave factorization.

Everywhere below we assume that we have this periodic wave factorization of the symbol $A_d(\xi)$ with the index æ. Using methods developed in [15], we can prove the following result.

Theorem 1. *Let $æ - s = n + \delta$, $n \in \mathbb{N}$, $|\delta| < 1/2$. Then, a general solution of the equation (4) has the following form*

$$\tilde{u}_d(\xi) = A^{-1}_{d,\neq}(\xi)\left(\sum_{k=0}^{n-1} \tilde{c}_k(\xi_1)\zeta_2^k + \tilde{d}_k(\xi_2)\zeta_1^k\right),$$

where $\tilde{c}_k(\xi_1), \tilde{d}_k(\xi_2), k = 0, 1, \cdots, n-1$, are arbitrary functions from $\widetilde{H}^{s_k}(h\mathbb{T}), s_k = s - æ + k - 1/2$. The a priori estimate

$$||u_d||_s \leq const \sum_{k=0}^{n-1}([c_k]_{s_k} + [d_k]_{s_k}),$$

holds, where $[\cdot]_{s_k}$ denotes a norm in the space $H^{s_k}(h\mathbb{T})$, and const doesn't depend on h.

3. DISCRETE NEUMANN PROBLEM

As we see a general solution of the equation (4) includes some arbitrary functions. In this section we will assume that $æ - s = 1 + \delta, |\delta| < 1/2$, and will consider discrete boundary Neumann conditions. For this case a general solution of the equation (4) has the form

$$\tilde{u}_d(\xi) = A^{-1}_{d,\neq}(\xi)(\tilde{c}_0(\xi_1) + \tilde{d}_0(\xi_2)), \tag{5}$$

where $c_0, d_0 \in H^{s-æ-1/2}(\hbar\mathbb{Z})$ are arbitrary functions. To define these functions uniquely we add the following conditions on angle sides

$$(\Delta_1^{(1)}u_d)_{|\tilde{x}_1=0} = f_d(\tilde{x}_2), \quad (\Delta_2^{(1)}u_d)_{|\tilde{x}_2=0} = g_d(\tilde{x}_1); \tag{6}$$

these conditions are discrete Neumann conditions. Let us remind [3, 15] that

$$(\Delta_1^{(1)}u_d)(\tilde{x}) = h^{-1}(u_d(x_1+h, x_2) - u_d(x_1, x_2)), \quad (\Delta_2^{(1)}u_d)(\tilde{x}) = h^{-1}(u_d(x_1, x_2+h) - u_d(x_1, x_2)),$$

and their discrete Fourier transforms are

$$(\widetilde{\Delta_k^{(1)}u_d})(\xi) = \zeta_k\tilde{u}_d(\xi), \quad \zeta_k = h^{-1}(e^{ih\cdot\xi_k} - 1), \quad k = 1, 2.$$

Let us introduce

$$\int_{-\hbar\pi}^{\hbar\pi} \zeta_1 A^{-1}_{d,\neq}(\xi)d\xi_1 \equiv \tilde{a}_0(\xi_2), \quad \int_{-\hbar\pi}^{\hbar\pi} \zeta_2 A^{-1}_{d,\neq}(\xi)d\xi_2 \equiv \tilde{b}_0(\xi_1).$$

If we assume that $\tilde{a}_0(\xi_2), \tilde{b}_0(\xi_1) \neq 0, \forall\xi_1 \neq 0, \xi_2 \neq 0$, then we can correctly define

$$\tilde{F}_d(\xi_2) = \tilde{f}_d(\xi_2)\tilde{a}_0^{-1}(\xi_2), \quad \tilde{G}_d(\xi_1) = \tilde{g}_d(\xi_1)\tilde{b}_0^{-1}(\xi_1),$$

$$k_1(\xi) = \zeta_1 A^{-1}_{d,\neq}(\xi)\tilde{a}_0^{-1}(\xi_2), \quad k_2(\xi) = \zeta_2 A^{-1}_{d,\neq}(\xi)\tilde{b}_0^{-1}(\xi_1).$$

Using new notations, we lead to the the following system of linear integral equations $\tilde{c}_0(\xi_1), \tilde{d}_0(\xi_2)$

$$\begin{cases} \int_{-\hbar\pi}^{\hbar\pi} k_1(\xi)\tilde{c}_0(\xi_1)d\xi_1 + \tilde{d}_0(\xi_2) = \tilde{F}_d(\xi_2) \\ \tilde{c}_0(\xi_1) + \int_{-\hbar\pi}^{\hbar\pi} k_2(\xi)\tilde{d}_0(\xi_2)d\xi_2 = \tilde{G}_d(\xi_1), \end{cases} \tag{7}$$

with respect to unknowns functions $\tilde{c}_0(\xi_1)$, $\tilde{d}_0(\xi_2)$. Thus, we can suggest the following property for the discrete Neumann problem (4), (6).

Theorem 2. *If $f_d, g_d \in H^{s-3/2}(\mathbb{R}_+)$, $s > 3/2$, $\inf|\tilde{a}_0(\xi_2)| \neq 0$, $\inf|\tilde{b}_0(\xi_1)| \neq 0$, then the discrete Neumann boundary value problem (4), (6) is equivalent to the system of linear integral equation (7).*

Proof. Let us apply the discrete Fourier transform to the discrete conditions (6). We obtain the following Fourier images

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \zeta_1 \tilde{u}_d(\xi_1,\xi_2)d\xi_1 = \tilde{f}_d(\xi_2), \quad \int\limits_{-\hbar\pi}^{\hbar\pi} \zeta_2 \tilde{u}_d(\xi_1,\xi_2)d\xi_2 = \tilde{g}_d(\xi_1). \tag{8}$$

Taking into account (8) in the equality (5), we obtain the following relations

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \zeta_1 \tilde{u}_d(\xi)d\xi_1 = \int\limits_{-\hbar\pi}^{\hbar\pi} \zeta_1 A^{-1}_{d,\neq}(\xi)\tilde{c}_0(\xi_1)d\xi_1 + \tilde{d}_0(\xi_2)\int\limits_{-\hbar\pi}^{\hbar\pi} \zeta_1 A^{-1}_{d,\neq}(\xi)d\xi_1,$$

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \zeta_2 \tilde{u}_d(\xi)d\xi_2 = \tilde{c}_0(\xi_1)\int\limits_{-\hbar\pi}^{\hbar\pi} \zeta_2 A^{-1}_{d,\neq}(\xi)d\xi_2 + \int\limits_{-\hbar\pi}^{\hbar\pi} \zeta_2 A^{-1}_{d,\neq}(\xi)\tilde{d}_0(\xi_2)d\xi_2.$$

If we will remind our notations, then we obtain the required system (7). □

3.1. What We Know for Continuous Case

Here we discuss a similar approach for the continuous analogue of considered discrete Neumann problem (4), (6). It also reduces to a system of linear integral equations by the fourier transform (all details can be found in [10]).

We study the pseudo-differential equation

$$(Au)(x) = 0, \quad x \in K, \tag{9}$$

with the symbol $A(\xi)$ satisfying the condition

$$c_1(1+|\xi|)^\alpha \leq |A(\xi)| \leq c_2(1+|\xi|)^\alpha \tag{*}$$

and admitting the wave factorization with respect to K

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi)$$

with the index æ, $æ - s = 1 + \delta$, $|\delta| < 1/2$. Then, we have a general solution

$$\tilde{u}(\xi) = A^{-1}_{\neq}(\xi)(\tilde{C}_0(\xi_1) + \tilde{D}_0(\xi_2))$$

with arbitrary functions $\tilde{C}_0(\xi_1)$, $\tilde{D}_0(\xi_2) \in \widetilde{H}^{s-æ-1/2}(\mathbb{R})$. These functions can be determined from the following system of integral equations

$$\begin{cases} \int\limits_{-\infty}^{\infty} K_1(\xi)\tilde{C}_0(\xi_1)d\xi_1 + \tilde{D}_0(\xi_2) = \tilde{F}(\xi_2), \\ \tilde{C}_0(\xi_1) + \int\limits_{-\infty}^{\infty} K_2(\xi)\tilde{D}_0(\xi_2)d\xi_2 = \tilde{G}(\xi_1), \end{cases} \tag{10}$$

if we use the Neumann boundary conditions

$$\left(\frac{\partial u}{\partial x_1}\right)|_{x_1=0} = f(x_2), \left(\frac{\partial u}{\partial x_2}\right)|_{x_2=0} = g(x_1) \tag{11}$$

and assume that the system (10) has unique solution; additionally we need to require the following conditions, namely $\inf|\tilde{A}_0(\xi_2)| \neq 0$, $\inf|\tilde{B}_0(\xi_1)| \neq 0$.

The following notations are used here

$$\int_{-\infty}^{\infty} \xi_1 A_{\neq}^{-1}(\xi) d\xi_1 \equiv \tilde{A}_0(\xi_2), \quad \int_{-\infty}^{\infty} \xi_2 A_{\neq}^{-1}(\xi) d\xi_2 \equiv \tilde{B}_0(\xi_1),$$

$$\tilde{F}(\xi_2) = \tilde{f}(\xi_2)\tilde{A}_0^{-1}(\xi_2), \quad \tilde{G}(\xi_1) = \tilde{g}(\xi_1)\tilde{B}_0^{-1}(\xi_1),$$

$$K_1(\xi) = \xi_1 A_{\neq}^{-1}(\xi)\tilde{A}_0^{-1}(\xi_2), \quad K_2(\xi) = \xi_2 A_{\neq}^{-1}(\xi)\tilde{B}_0^{-1}(\xi_1).$$

The following result is proved in the book [10].

Theorem 3. *Let $s > 3/2$ and the symbol $A(\xi)$ satisfies the condition $(*)$ and admits the wave factorization with respect to K with the index æ such that $æ - s = 1 + \delta, |\delta| < 1/2$. If the following conditions*

$$\inf |\tilde{A}_0(\xi_2)| \neq 0, \quad \inf |\tilde{B}_0(\xi_1)| \neq 0, \tag{**}$$

hold then the Neumann problem (9), (11) with boundary functions $f, g \in H^{s-3/2}(\mathbb{R}_+)$ is equivalent to the system of integral equations (10) with unknowns $\tilde{C}_0, \tilde{D}_0 \in \widetilde{H}^{s_0}(\mathbb{R})$.

4. DISCRETE AND CONTINUOUS

This section is devoted to a comparison of solutions (7) and (10) although these solutions are defined in different spaces. We will consider truncations of integral operators from (10) on $\hbar\mathbb{T}$ and then will compare it with operator from (7). Here we will use very important result from [14] related to a general concept of projectional methods.

Everywhere below we take into account that the condition (**) holds.

4.1. Properties of Integral Operators

Let us introduce the space $\mathbf{H}^s(\mathbb{R})$ of vector-functions $f = (f_1, f_2)$, $f_j \in H^s(\mathbb{R})$, $j = 1, 2$, $||f||_s \equiv ||f_1||_s + ||f_2||_s$, and the following operators

$$K = \begin{pmatrix} K_1 & I \\ I & K_2 \end{pmatrix}, \quad k = \begin{pmatrix} k_1 & I_h \\ I_h & k_2 \end{pmatrix},$$

which act in spaces $\mathbf{H}^{s-æ-1/2}(\mathbb{R})$ and $\mathbf{H}^{s-æ-1/2}(\hbar\mathbb{T})$.

We consider here the case $æ - s = 1 + \delta, |\delta| < 1/2$ and, according to Theorem 1, we have $s_0 = s - æ - 1/2$.

Lemma 1. *If $s > 2$, $æ > 2$, then the operator K is bounded in the space $\mathbf{H}^{s_0}(\mathbb{R})$, $K : \mathbf{H}^{s_0}(\mathbb{R}) \to \mathbf{H}^{s_0}(\mathbb{R})$.*

Proof. We consider the $K_1 f$ since $K_2 f$ is almost the same.

$$||K_1 f||_{s_0}^2 = \int_{-\infty}^{+\infty} (1 + |\xi_2|)^{2s_0} |(K_1 f)(\xi_2)|^2 d\xi_2 = \int_{-\infty}^{+\infty} (1 + |\xi_2|)^{2s_0} \left| \int_{-\infty}^{+\infty} K_1(\xi_1, \xi_2) f(\xi_1) d\xi_1 \right|^2 d\xi_2$$

$$\leq \int_{-\infty}^{+\infty} (1 + |\xi_2|)^{2s_0} \left(\int_{-\infty}^{+\infty} |K_1(\xi_1, \xi_2)||f(\xi_1)| d\xi_1 \right)^2 d\xi_2$$

$$\leq \ const \int_{-\infty}^{+\infty} (1 + |\xi_2|)^{2s_0} \left(\int_{-\infty}^{+\infty} (1 + |\xi_1| + |\xi_2|)^{-æ+1} ||f(\xi_1)| d\xi_1 \right)^2 d\xi_2.$$

Further, we apply Cauchy–Bunyakovskii inequality for the inner integral and add the factors $(1+|\xi_1|)^{-s_0}$ and $(1+|\xi_1|)^{s_0}$ for the first term and the second one and use the inequality $(1+|\xi_1|)^{-s_0} \leq (1+|\xi_1|+|\xi_2|)^{-s_0}$. Then, we have

$$||K_1 f||^2_{s_0} \leq \ const||f||^2_{s_0} \int\limits_{-\infty}^{+\infty} (1+|\xi_2|)^{2s_0} \left(\int\limits_{-\infty}^{+\infty} (1+|\xi_1|+|\xi_2|)^{-2(s_0+æ-1)} d\xi_1 \right) d\xi_2$$

$$\leq \ const||f||^2_{s_0} \int\limits_{0}^{+\infty} (1+|\xi_2|)^{2s_0} \left(\int\limits_{0}^{+\infty} (1+|\xi_1|+|\xi_2|)^{-2s+3} d\xi_1 \right) d\xi_2$$

$$\leq \ const||f||^2_{s_0} \int\limits_{0}^{+\infty} (1+|\xi_2|)^{2s_0-2s+4} d\xi_2 \leq \ const||f||^2_{s_0} \int\limits_{0}^{+\infty} (1+|\xi_2|)^{-2æ+3} d\xi_2 \leq \ const||f||^2,$$

since $s > 2$, $æ > 2$. □

We introduce new notations, the operator $\chi_h : H^s(\mathbb{R}) \to H^s(\hbar\mathbb{T})$ is a restriction on the segment $\hbar\mathbb{T}$, and the restriction operator on the segment $\hbar\mathbb{T}$ in the vector-space $\mathbf{H}^s(\mathbb{R})$ will be denoted by Ξ_h so that for $f = (f_1, f_2) \in \mathbf{H}^s(\mathbb{R})$ we can write $\Xi_h f = (\chi_h f_1, \chi_h f_2)$. Of course, everywhere below we consider the parameter h enough small, $0 < h < 1$.

Lemma 2. *For $s > 2$, $æ > 2$ the operator K has the following property*

$$||\Xi_h K - K\Xi_h||_{\widetilde{\mathbf{H}}^{s_0}(\mathbb{R}) \to \widetilde{\mathbf{H}}^{s_0}(\mathbb{R})} \leq \ const\ h^{s-2}.$$

Proof. We start from the following representation

$$\Xi_h K - K\Xi_h = \begin{pmatrix} \chi_h K_1 - K_1\chi_h & 0 \\ 0 & \chi_h K_2 - K_2 \chi_h \end{pmatrix}.$$

The first difference is

$$((\chi_h K_1 - K_1 \chi_h)f)(\xi_2) = \begin{cases} \left(\int\limits_{-\infty}^{-\hbar\pi} + \int\limits_{\hbar\pi}^{+\infty} \right) K_1(\xi_1,\xi_2) f(\xi_1) d\xi_1, & \xi_2 \in \hbar\mathbb{T}, \\ -\int\limits_{-\hbar\pi}^{+\hbar\pi} K_1(\xi_1,\xi_2) f(\xi_1) d\xi_1, & \xi_2 \notin \hbar\mathbb{T}. \end{cases}$$

We estimate this difference, the second integral looks the same.

$$\left| \int\limits_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1 \right| \leq \int\limits_{\hbar\pi}^{+\infty} |K_1(\xi)||f(\xi_1) d\xi_1 \leq const \int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-æ+1} |f(\xi_1)| d\xi_1$$

(we apply the Cauchy–Bunyakovskii inequality once again)

$$\leq \ const \left(\int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-2æ+2} (1+|\xi_1|)^{-2s_0} d\xi_1 \right)^{1/2} \left(\int\limits_{\hbar\pi}^{+\infty} |f(\xi_1)|^2 (1+|\xi_1|)^{2s_0} d\xi_1 \right)^{1/2}$$

$$\leq \ const \left(\int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-2(æ+s_0-1)} d\xi_1 \right)^{1/2} ||f||_{s_0},$$

We have

$$\int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-2(æ+s_0-1)} d\xi_1 \sim (1+|\xi_2|+\hbar\pi)^{-2(æ+s_0)+3},$$

taking into account that $-2(æ + s_0 - 1) + 1 = -2(s - 3/2) + 1 = -2s + 4 < 0$. Thus, we can write the inequality

$$\left|\int_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1\right| \leq \ const||f||_{s_0}(1 + |\xi_2| + \hbar\pi)^{-(æ+s_0)+3/2}.$$

Squaring the latter inequality and multiplying by $(1 + |\xi_2|)^{2s_0}$ after integrating over $\hbar\mathbb{T}$, we find

$$\int_{\hbar\mathbb{T}} (1 + |\xi_2|)^{2s_0} \left|\int_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1\right|^2 d\xi_2$$

$$\leq \ const||f||_{s_0}^2 \int_{\hbar\mathbb{T}} (1 + |\xi_2| + \hbar\pi)^{-2(æ+s_0)+3}(1 + |\xi_2|)^{2s_0} d\xi_2$$

$$\leq \ const||f||_{s_0}^2 \hbar^{-2(s-2)} \int_{\hbar\mathbb{T}} (1 + |\xi_2|)^{2s_0} d\xi_2 \leq \ const||f||_{s_0}^2 h^{2(s-2)},$$

because $1 + |\xi_2| + \hbar\pi \geq 1 + \hbar\pi, -2(æ + s_0) + 3 = -2s + 4 < 0; s_0 < -1$. Thus, we obtain

$$\int_{\hbar\mathbb{T}} (1 + |\xi_2|)^{2s_0} \left|\int_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1\right|^2 d\xi_2 \leq \ const||f||_{s_0}^2 h^{2(s-2)}.$$

The second case ($|\xi_2| > \hbar\pi$):

$$\left|\int_{-\hbar\pi}^{+\hbar\pi} K_1(\xi_1, \xi_2) f(\xi_1) d\xi_1\right| \leq const \int_{-\hbar\pi}^{+\hbar\pi} (1 + |\xi|)^{-æ+1} |f(\xi_1)| d\xi_1$$

$$\leq \ const \left(\int_{-\hbar\pi}^{\hbar\pi} (1 + |\xi|)^{-2æ+2}(1 + |\xi_1|)^{-2s_0} d\xi_1\right)^{1/2} \left(\int_{-\hbar\pi}^{\hbar\pi} |f(\xi_1)|^2 (1 + |\xi_1|)^{2s_0} d\xi_1\right)^{1/2}.$$

We have applied the Cauchy–Bunyakovskii inequality once again and with the inequality $1 + |\xi| \geq 1 + |\xi_1|$ we have the estimate

$$\int_{-\hbar\pi}^{\hbar\pi} (1 + |\xi|)^{-2æ+2}(1 + |\xi_1|)^{-2s_0} d\xi_1 \leq \ 2\int_0^{\hbar\pi} (1 + \xi_1 + |\xi_2|)^{-2(s_0+æ)+2} d\xi_1 \leq \ const(1 + |\xi_2|)^{-2s+4}$$

$$\leq \ const(1 + \hbar\pi)^{-2(s-2)},$$

because $-2(s_0 + æ) + 2 = -2(s - 1/2) + 2 = -2s + 3$. Therefore, we have obtained the inequality

$$\left|\int_{-\hbar\pi}^{+\hbar\pi} K_1(\xi_1, \xi_2) f(\xi_1) d\xi_1\right| \leq \ const||f||_{s_0} h^{s-2}.$$

Multiplying the latter inequality by $(1 + |\xi_2|)^{s_0}$, squaring and integrating over $\mathbb{R} \setminus \hbar\mathbb{T}$, we find

$$\int_{\mathbb{R}\setminus\mathbb{T}} (1 + |\xi_2|)^{2s_0} \left|\int_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1\right|^2 d\xi_2 \leq \ const||f||_{s_0}^2 h^{2(s-2)} \int_{\hbar\pi}^{+\infty} (1 + \xi_2)^{2s_0} d\xi_2.$$

The latter integral converges since $s_0 < -1$. The same estimates are valid for the operator K_2. □

Corollary 1. *If $s > 2$, $æ > 2$ and the operator K is invertible, then the operator K^{-1} admits the same estimate*

$$||\Xi_h K^{-1} - K^{-1}\Xi_h||_{\widetilde{\mathbf{H}}^{s_0}(\mathbb{R})\to\widetilde{\mathbf{H}}^{s_0}(\mathbb{R})} \leq \ const\ h^{s-2}.$$

Proof. Indeed, this property follows from inequalities

$$\Xi_h K^{-1} - K^{-1}\Xi_h = K^{-1}K\Xi_h K^{-1} - K^{-1}\Xi_h K K^{-1} = K^{-1}(\Xi_h K - K\Xi_h)K^{-1},$$

so that

$$||\Xi_h K^{-1} - K^{-1}\Xi_h||_{\widetilde{\mathbf{H}}^{s_0}(\mathbb{R})\to\widetilde{\mathbf{H}}^{s_0}(\mathbb{R})} \leq ||K^{-1}|| \cdot ||\Xi_h K - K\Xi_h||_{\widetilde{\mathbf{H}}^{s_0}(\mathbb{R})\to\widetilde{\mathbf{H}}^{s_0}(\mathbb{R})} \cdot ||K^{-1}||,$$

that is required. □

4.2. Special Discrete Symbols and Discrete Boundary Conditions

Since Theorems 2 and 3 give an equivalence for boundary value problems (4), (6) and (9), (11) to systems of integral equations (7) and (10) respectively, then we will assume that continuous boundary value problem (9), (11) is uniquely solvable for arbitrary boundary functions $f, g \in H^{s-3/2}(\mathbb{R}_+)$. In other words it means that there is the bounded inverse operator K^{-1} or the system of integral equations (10) has unique solution for arbitrary right hand sides $(\tilde{F}, \tilde{G})^T$.

To obtain a comparison between discrete and continuous solutions we need special choice of a discrete operator and discrete boundary conditions.

The symbol $A_d(\xi)$ of the discrete operator A_d will be constructed in the following way. If we have the wave factorization for $A(\xi)$

$$A(\xi) = A_{\neq}(\xi)\, A_{=}(\xi),$$

then we take restrictions of factors $A_{\neq}(\xi), A_{=}(\xi)$ on $\hbar\mathbb{T}^2$ and periodically continue them on $\mathbb{R}^2$. We denote these elements by $A_{d,\neq}(\xi), A_{d,=}(\xi)$ and construct the periodic symbol $A_d(\xi)$ which admits periodic wave factorization with respect to K

$$A_d(\xi) = A_{d,\neq}(\xi)\, A_{d,=}(\xi)$$

with the same index æ. We construct the boundary functions f_d X g_d in the same way. Thus, we have the corresponding discrete boundary value problem (4), (6). The solution of the problem (9),(11) will be compared with the discrete solution of such discrete boundary value problem.

Now we will introduce special discrete operators $d_1^{(1)}, d_2^{(1)}$ instead of divided differences of first order $\Delta_1^{(1)}, \Delta_2^{(1)}$.

If we have a discrete function $u_d(\tilde{x}), \tilde{x} \in h\mathbb{Z}^2$, then we take its discrete Fourier transform $\tilde{u}_d(\xi), \xi \in \hbar\mathbb{T}^2$. Further, we define the following periodic function $\tilde{r}(t)$ of one variable in the following way. If $t \in \mathbb{R}$ we put $r(t) = t$ and then we take its restriction on $(-\hbar\pi, \hbar\pi)$ and periodically continue it into a whole $\mathbb{R}$. This periodic function will be denoted by $\tilde{r}(t)$. Thus, we have the periodic function $\tilde{r}(\xi_k)\tilde{u}_d(\xi), k = 1, 2$ with basic quadrat of periods $\hbar\mathbb{T}^2$. By definition, $(d_k^{(1)}u_d)(\tilde{x}), \tilde{x} \in h\mathbb{Z}^2$ is inverse discrete Fourier transform of the function $\tilde{r}(\xi_k)\tilde{u}_d(\xi), k = 1, 2$. Taking into account this construction we introduce the following boundary conditions instead of (6)

$$(d_1^{(1)}u_d)_{|\tilde{x}_1=0} = f_d(\tilde{x}_2), \quad (d_2^{(1)}u_d)_{|\tilde{x}_2=0} = g_d(\tilde{x}_1). \tag{12}$$

For this case we will obtain the following corrections for k_1, k_2, $\tilde{a}_0$, $\tilde{b}_0$:

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \xi_1 A_{d,\neq}^{-1}(\xi)d\xi_1 \equiv \tilde{a}_0(\xi_2), \quad \int\limits_{-\hbar\pi}^{\hbar\pi} \xi_2 A_{d,\neq}^{-1}(\xi)d\xi_2 \equiv \tilde{b}_0(\xi_1).$$

$$k_1(\xi) = \xi_1 A_{d,\neq}^{-1}(\xi)\tilde{a}_0^{-1}(\xi_2), \quad k_2(\xi) = \xi_2 A_{d,\neq}^{-1}(\xi)\tilde{b}_0^{-1}(\xi_1),$$

Everywhere below we take into account that this choice is done.

Lemma 3. *For* $æ > 2$ *we have*

$$|K_1(\xi) - k_1(\xi)| \leq \; const \; (1 + |\xi|)^{-æ+1} h^{æ-2} \;\; , \xi \in \hbar\mathbb{T}^2.$$

Proof. According to our choice of the symbol $A^{-1}_{d,\neq}(\xi)$,

$$|K_1(\xi) - k_1(\xi)| = |\xi_1 A^{-1}_{\neq}(\xi)\tilde{A}^{-1}_0(\xi_2) - \xi_1 A^{-1}_{d,\neq}(\xi)\tilde{a}^{-1}_0(\xi_2)| \leq \; const(1 + |\xi|)^{-æ+1}|\tilde{A}_0(\xi_2) - \tilde{a}_0(\xi_2)|.$$

We estimate the $|\tilde{A}_0(\xi_2) - \tilde{a}_0(\xi_2)|$ as follows

$$|\tilde{A}_0(\xi_2) - \tilde{a}_0(\xi_2)| = \left| \int\limits_{-\infty}^{\infty} \xi_1 A^{-1}_{\neq}(\xi) d\xi_1 - \int\limits_{-\hbar\pi}^{\hbar\pi} \xi_1 A^{-1}_{d,\neq}(\xi) d\xi_1 \right|$$

$$\leq \; const \int\limits_{\hbar\pi}^{+\infty} (1 + |\xi|)^{-æ+1} d\xi_2 \leq \; const \; (1 + |\xi_1| + \hbar)^{-æ+2} \leq const \; h^{æ-2}$$

for enough small h. By the way it gives the following $\inf |\tilde{A}_0(\xi_2)| \neq 0 \Longrightarrow \inf |\tilde{a}_0(\xi_2)| \neq 0$ for enough small h. All estimates permit us to complete the proof. □

We introduce the operator $\Xi_h K \Xi_h$. Lemma 2 implies that for enough small h an invertibility of the operator $\Xi_h K \Xi_h$ in the space $\widetilde{\mathbf{H}}^{s-æ-1/2}(\hbar\mathbb{T})$ follows from an invertibility of the operator K in the space $\widetilde{\mathbf{H}}^{s-æ-1/2}(\mathbb{R})$ [14]. Moreover,

$$||(\Xi_h K \Xi_h)^{-1}||_{\widetilde{\mathbf{H}}^{s_0}(\hbar\mathbb{T}) \to \widetilde{\mathbf{H}}^{s_0}(\hbar\mathbb{T})} \leq \; const$$

for enough small h.

Lemma 4. *If* $s > 2$*,* $æ > 2$ *then we have the following estimate for operators* $\Xi_h K \Xi_h$ *and* k

$$||\Xi_h K \Xi_h - k||_{\widetilde{\mathbf{H}}^{s_0}(\hbar\mathbb{T}) \to \widetilde{\mathbf{H}}^{s_0}(\hbar\mathbb{T})} \leq \; const \; h^{æ-2}.$$

Proof. We have the following identity

$$\Xi_h K \Xi_h - k = \begin{pmatrix} \chi_h K_1 \chi_h - k_1 & 0 \\ 0 & \chi_h K_2 \chi_h - k_2 \end{pmatrix}.$$

We need to estimate the norm of the operators $\chi_h K_j \chi_h - k_j, j = 1, 2$. We will estimate only one term, for example K_1 using Lemma 3. Then we have

$$||\chi_h K_1 \chi_h f - k_1 f||^2_{s_0} = \int\limits_{\hbar\mathbb{T}} (1 + |\xi_2|)^{2s_0} \left| \int\limits_{\hbar\mathbb{T}} [K_1(\xi) - k_1(\xi)] f(\xi_1) d\xi_1 \right|^2 d\xi_2$$

$$\leq \int\limits_{\hbar\mathbb{T}} (1 + |\xi_2|)^{2s_0} \left(\int\limits_{\hbar\mathbb{T}} |K_1(\xi) - k_1(\xi)||f(\xi_1)| d\xi_1 \right)^2 d\xi_2$$

$$\leq \; const \; h^{2æ-4} \int\limits_{\hbar\mathbb{T}} (1 + |\xi_2|)^{2s_0} \left(\int\limits_{\hbar\mathbb{T}} (1 + |\xi|)^{-æ+1} |f(\xi_1)| d\xi_1 \right)^2 d\xi_2.$$

In the inner integral, we introduce the factor $(1 + |\xi_1|)^{s_0}$ and apply the Cauchy—Bunyakovskii inequality

$$\int\limits_{\hbar\mathbb{T}} (1 + |\xi|)^{-æ+1} |f(\xi_1)| d\xi_1 \leq ||f||_{s_0} \left(\int\limits_{\hbar\mathbb{T}} (1 + |\xi|)^{-2æ+2} (1 + |\xi_1|)^{-2s_0} d\xi_1 \right)^{1/2}$$

$$\leq ||f||_{s_0}\left(\int\limits_0^{+\infty}(1+|\xi|)^{-2(æ+s_0-1)}d\xi_1\right)^{1/2} \leq ||f||_{s_0}(1+|\xi_2|)^{-(æ+s_0-1)+1/2} = ||f||_{s_0}(1+|\xi_2|)^{-s+2},$$

because $s_0 = s - æ - 1/2$, it gives

$$||\chi_h K_1 \chi_h f - k_1 f||^2_{s_0} \leq \ const\ h^{2æ-4}||f||^2_{s_0}\int\limits_{-\hbar\pi}^{\hbar\pi}(1+|\xi_2|)^{2s_0-2s+4}d\xi_2$$

$$\leq \ const\ h^{2æ-4}||f||^2_{s_0}\int\limits_0^{+\infty}(1+|\xi_2|)^{-2æ+3}d\xi_2 \leq \ const\ h^{2æ-4}||f||^2_{s_0},$$

$s_0+1-s = -æ+1/2$, taking a square root we obtain the assertion of Lemma 4. □

4.3. A Comparison

This section is devoted to a comparison between discrete and continuous solutions.

Theorem 4. *If the conditions of Theorem 3 hold and $s > 2$, $æ > 2$ then a comparison for solutions of problems (4), (12) and (9), (11) for enough small h is given by the estimate*

$$||u - u_d||_{\widetilde{H}^s(\hbar\mathbb{T}^2)} \leq \ const\ h^{s-2}(||f||_{s-3/2} + ||g||_{s-3/2}),$$

where const does not depend on h.

Proof. We will compare solutions of systems (7) and (10). There are two solutions

$$\tilde{u}(\xi) = A^{-1}_{\neq}(\xi)(\tilde{C}_0(\xi_1) + \tilde{D}_0(\xi_2))$$

and

$$\tilde{u}_d(\xi) = A^{-1}_{d,\neq}(\xi)(\tilde{c}_0(\xi_1) + \tilde{d}_0(\xi_2)),$$

Keeping in mind $\xi \in \hbar\pi$ we denote by $\tilde{\Phi}_d$ and $\tilde{\Phi}$ vectors with coordinates $(\tilde{F}_d, \tilde{G}_d)^T$ and $(\tilde{F}, \tilde{G})^T$, $\tilde{C}$ and $\tilde{c}$ are vectors with coordinates $(\tilde{C}_0, \tilde{D}_0)^T$ and $(\tilde{c}_0, \tilde{d}_0)^T$ respectively. Then, we use vector notation

$$\tilde{C} = K^{-1}\tilde{\Phi}, \quad \tilde{c} = k^{-1}\tilde{\Phi}_d,$$

and for simplicity we denote by C_1, C_2 and c_1, c_2, jth coordinates of vectors $\tilde{C}, \tilde{c}$, $j = 1, 2$. Thus,

$$(\chi_h\tilde{u})(\xi) - \tilde{u}_d(\xi) = \chi_h A^{-1}_{\neq}(\xi)\left((\tilde{C}_0(\xi_1) - \tilde{c}_0(\xi_1)) + (\tilde{D}_0(\xi_2) - \tilde{d}_0(\xi_2)\right)$$

$$= \chi_h A^{-1}_{\neq}(\xi)\left((K^{-1}\tilde{\Phi})_1(\xi_1) - (k^{-1}\tilde{\Phi}_d)_1(\xi_1) + (K^{-1}\tilde{\Phi})_2(\xi_2) - (k^{-1}\tilde{\Phi}_d)_2(\xi_2)\right);$$

we conclude that it is enough to estimate the norm $||\Xi_h K^{-1}\Phi - k^{-1}\Phi_d||_{\mathbf{H}^{s_0}(\hbar\mathbb{T})}$. Let us write

$$\Xi_h K^{-1}\Phi - k^{-1}\Phi_d = (\Xi_h K^{-1}\Phi - K^{-1}\Xi_h\Phi) + (K^{-1}\Xi_h\Phi - k^{-1}\Phi_d).$$

We use Corollary 1 to estimate the first summand,

$$||\Xi_h K^{-1}\Phi - K^{-1}\Xi_h\Phi||_{s_0} \leq \ const\ h^{s-2}||\Phi||_{s_0}.$$

The second summand is represented as follows

$$K^{-1}\Xi_h\Phi - k^{-1}\Phi_d = (K^{-1}\Xi_h\Phi - k^{-1}\Xi_h\Phi) + (k^{-1}\Xi_h\Phi - k^{-1}\Phi_d),$$

and we estimate each summand separately.

Let us consider $k^{-1}\Xi_h\Phi - k^{-1}\Phi_d$. In view of boundedness for the norm $||k^{-1}||$ by a constant non-depending on h we have

$$||k^{-1}\Xi_h\Phi - k^{-1}\Phi_d||_{s_0} \leq \ const||\Xi_h\Phi - \Phi_d||_{s_0} \leq \ const(||\chi_h F - F_d||_{s_0} + ||\chi_h G - G_d||_{s_0})$$

and it is left to estimate $||\chi_h F - F_d||_{s_0}$ and $||\chi_h G - G_d||_{s_0}$. For F we obtain

$$||\chi_h F - F_d||_{s_0}^2 = \int_{-\hbar\pi}^{\hbar\pi} |\tilde{f}(\xi_2)A_0^{-1}(\xi_2) - \tilde{f}_d(\xi_2)a_0^{-1}(\xi_2)|^2(1+|\xi_2|)^{2s_0}d\xi_2$$

$$\leq \ const\, h^{2æ-4} \int_{-\hbar\pi}^{\hbar\pi} |\tilde{f}(\xi_2)|^2(1+|\xi_2|)^{2s_0}d\xi_2 \leq \ const\, h^{2æ-4}||f||_{s_0}^2$$

since f_d and f are the same on $\hbar\mathbb{T}$ and Lemma 3 can be used.

We apply the following operator identity to estimate the left summand

$$K^{-1} - k^{-1} = K^{-1}(k-K)k^{-1}$$

(let us remind that an invertibility of the operator k is stipulated by an invertibility of the operator K). Thus, for $\hbar\mathbb{T}$

$$K^{-1}\Xi_h\Phi - k^{-1}\Xi_h\Phi = \Xi_h(K^{-1} - k^{-1})\Xi_h\Phi = \Xi_h K^{-1}(k-K)k^{-1}\Xi_h\Phi,$$

and, according to Lemma 4, we have

$$||K^{-1}\Xi_h\Phi - k^{-1}\Xi_h\Phi||_{s_0} \leq \ const\, h^{æ-2}||\Phi||_{s_0} \leq \ const\, h^{æ-2}(||f||_{s_0} + ||g||_{s_0}).$$

All obtained estimates give the assertion of Theorem 4 if we take into account mapping properties of pseudo-differential operators which permit to obtain the result with H^s-norm. □

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