Discrete Operators and Equations: Analysis and Comparison

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Abstract We develop a discrete variant of a theory of pseudo-differential equations and boundary value problems in canonical domains which are model situations for manifolds with non-smooth boundaries. Using digitization process for ordinary functional spaces we construct certain discrete functional spaces or spaces of functions of a discrete variable and define discrete pseudo-differential operators acting in such spaces. A main problem in which we are interested is to establish a correspondence between continual and discrete solutions of considered continual and discrete equations and in future boundary value problems. We have illustrated our considerations by certain examples of Calderon–Zygmund operators for which we have some interesting conclusions.

Keywords Discrete operator · Solvability

1 Introduction

We deal with some special operators namely pseudo-differential operators. Our global main goal is to construct a theory of discrete pseudo-differential operators and corresponding boundary value problems on smooth manifolds with a boundary which may be non-smooth.

A basic equation in an operator form is the following

$$(Au)(x) = v(x), \quad x \in D, \tag{1}$$

where $D \subset \mathbf{R}^m$ is a some domain, A is a pseudo-differential operator which is defined by the formula

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$$(Au)(x) = (2\pi)^{-m} \int\limits_{D} \int\limits_{\mathbf{R}^m} e^{i(y-x)\cdot\xi} \tilde{A}(x,\xi)\tilde{u}(\xi)dyd\xi, \quad x \in D, \tag{2}$$

and a sign $\sim$ over the function u denotes its Fourier transform

$$\tilde{u}(\xi) = \int\limits_{\mathbf{R}^m} e^{-ix\cdot\xi} u(x)dx.$$

Definition 1 The function $\tilde{A}(x,\xi)$ is called a symbol of a pseudo-differential operator A. A symbol $\tilde{A}(x,\xi)$ is called an elliptic symbol if $ess \inf\limits_{(x,\xi)\in\mathbf{R}^m\times\mathbf{R}^m} |\tilde{A}(x,\xi)| > 0$.

As far as I know it is impossible to find an exact solution of the equation (1) for an arbitrary domain D. Therefore all researches are interested in describing Fredholm properties of the equation at least. But for simplest cases it can very easy by the Fourier transform.

Example 1 Let K(x) be a Calderon–Zygmund kernel and the operator A is defined by the formula [4]

$$(Ku)(x) = v.p. \int\limits_{\mathbf{R}^m} K(x-y)u(y)dy, \tag{3}$$

so that it can represented in the form (6)

$$(Ku)(x) = (2\pi)^{-m} \int\limits_{\mathbf{R}^m} \int\limits_{\mathbf{R}^m} e^{i(y-x)\cdot\xi} \sigma(\xi)\tilde{u}(\xi)dyd\xi,$$

and the function $\sigma(\xi)$ is called a symbol of the operator A. It is well known that for the operator A to be invertible in the space $L_2(\mathbf{R}^m)$ necessary and sufficient its symbol $\sigma(\xi)$ should be an elliptic [4].

Let $D_d = D \cap h\mathbf{Z}^m, h > 0$. We are interested in studying some discrete equations which we call discrete pseudo-differential equations and which are related to the Eq. (1). Let us define a discrete pseudo-differential operator by the formula

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in D_d} \int\limits_{\hbar\mathbf{T}^m} e^{i(\tilde{y}-\tilde{x})\cdot\xi} A_d(\tilde{x},\xi)\tilde{u}_d(\xi)d\xi, \quad \tilde{x} \in D_d,$$

where $u_d(\tilde{x})$ is a function of a discrete variable $\tilde{x} \in h\mathbf{Z}^m$, $\tilde{u}_d(\xi)$ denotes its discrete Fourier transform

$$\tilde{u}_d(\xi) \equiv (F_d u_d)(\xi) = \sum_{\tilde{y}\in h\mathbf{Z}^m} e^{i\tilde{y}\cdot\xi} \tilde{u}_d(y), \quad \xi \in \hbar\mathbf{T}^m, \tag{4}$$

$\mathbf{Z}^m$ is an integer lattice in $\mathbf{R}^m$, $\mathbf{T}^m$ is m-dimensional cube $[-\pi, \pi]^m$, $\hbar = \frac{h^{-1}}{2\pi}$, and given function $A_d(\tilde{x}, \xi), \tilde{x} \in h\mathbf{Z}^m, \xi \in \hbar\mathbf{T}^m$, is called a symbol of the discrete pseudo-differential operator A_d.

We would like to study the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{5}$$

in some discrete functional spaces. Since it is difficult to study such general operators (as it was said above) for discrete cases also we'll consider certain model situations.

2 The Concept of the Research

We'll present here main ideas for studying this large problem. In contrast of algebraic approaches [2, 3, 5] we use analytical methods based on properties of the Fourier transform and considered operators. A plan of the studying is the following:

- infinite discrete and finite discrete Fourier transform
- discrete functional spaces
- solvability of infinite discrete equation
- solvability of finite discrete equation
- comparison of continual and infinite discrete solution
- comparison of infinite and finite discrete solution.

2.1 Local Discrete Operators

We'll illustrated the above scheme with very simple model pseudo-differential operator namely operator A from example 1 because many our results are related to this operator. In addition we assume that kernel $K(x)$ of the operator A is differentiable on $\mathbf{R}^m \setminus \{0\}$.

2.2 Discrete and Continual

Discrete Fourier Transform To obtain a good approximation for the integral equation (1) we will use the following reduction. First instead of the integral in (1) we introduce the series

$$\sum_{\tilde{y} \in h\mathbf{Z}^m} K(\tilde{x} - \tilde{y}) u_d(\tilde{y}) h^m, \tag{6}$$

which generates a discrete operator

$$(K_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbf{Z}^m} K(\tilde{x} - \tilde{y}) u_d(\tilde{y}) h^m, \quad \tilde{x} \in h\mathbf{Z}^m, \tag{7}$$

defined on functions u_d of discrete variable $\tilde{x} \in h\mathbf{Z}^m$. Since the Calderon–Zygmund kernel has a strong singularity at the origin we mean $K(0) = 0$. Convergence for the series (6) means that the following limit

$$\lim_{N \to +\infty} \sum_{\tilde{y} \in h\mathbf{Z}^m \cap Q_N} K(\tilde{x} - \tilde{y}) u_d(\tilde{y}) h^m$$

exists, where $Q_N = \{x \in \mathbf{R}^m : \max_{1 \le k \le m} |x_k| < N\}$. It was shown earlier that a norm of the operator $K_d : L_2(h\mathbf{Z}^m) \to L_2(h\mathbf{Z}^m)$ does not depend on h [11]. But although the operator is a discrete object it is an infinite one.

Let us define the infinite discrete Fourier transform for functions u_d of a discrete variable $\tilde{x} \in h\mathbf{Z}^m$

$$(F_d u_d)(\xi) = \sum_{\tilde{x} \in h\mathbf{Z}^m} u_d(\tilde{x}) e^{i\tilde{x} \cdot \xi} h^m, \quad \xi \in \hbar\mathbf{T}^m.$$

Such discrete Fourier transform preserves all basic properties of the classical Fourier transform, particularly for a discrete convolution of two discrete functions u_d, v_d

$$(u_d * v_d)(\tilde{x}) \equiv \sum_{\tilde{y} \in h\mathbf{Z}^m} u_d(\tilde{x} - \tilde{y}) v_d(\tilde{y}) h^m$$

we have the well known multiplication property

$$(F_d(u_d * v_d))(\xi) = (F_d u_d)(\xi) \cdot (F_d v_d)(\xi).$$

If we apply this property to the operator K_d we obtain

$$(F_d(K_d u_d))(\xi) = (F_d K_d)(\xi) \cdot (F_d u_d)(\xi).$$

Let us denote $(F_d K_d)(\xi) \equiv \sigma_d(\xi)$ and give the following

Definition 2 The function $\sigma_d(\xi)$, $\xi \in \hbar\mathbf{T}^m$, is called a symbol of the discrete operator K_d.

We will assume below that the symbol $\sigma_d(\xi) \in C(\hbar\mathbf{T}^m)$ therefore we have immediately the following

Property 1 The operator K_d is invertible in the space $L_2(h\mathbf{Z}^m)$ iff $\sigma_d(\xi) \neq 0, \forall \xi \in \hbar\mathbf{T}^m$.

We say that a continuous symbol is called an **elliptic symbol** if $\sigma_d(\xi) \neq 0, \forall \xi \in \hbar\mathbf{T}^m$.

So we see that an arbitrary elliptic symbol $\sigma_d(\xi)$ corresponds to an invertible operator K_d in the space $L_2(h\mathbf{Z}^m)$.

A very interesting fact was proved in [8, 9].

Theorem 1 *Operators (3) and (7) are invertible or non-invertible in spaces $L_2(\mathbf{R}^m)$ and $L_2(h\mathbf{Z}^m)$ simultaneously $\forall h > 0$.*

If we consider the equation

$$(K_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in h\mathbf{Z}^m,$$

in the space $L_2(h\mathbf{Z}^m)$ then we solve the equation by the discrete Fourier transform F_d. Indeed after applying the Fourier transform we have the trivial equation

$$\sigma_d(\xi)\tilde{u}_d(\xi) = \tilde{v}_d(\xi), \quad \xi \in \hbar\mathbf{T}^m,$$

in the dual space $L_2(\hbar\mathbf{T}^m)$.

We have first difficulties when consider this equation in the space $L_2(h\mathbf{Z}^m_+)$, where $\mathbf{Z}^m_+ = \{\tilde{x} \in \mathbf{Z}^m : \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_m), \tilde{x}_m > 0\}$. We can not apply the Fourier transform directly as above because the functions under consideration are defined not on a whole space. Thus we need to describe images of such function after the discrete Fourier transform, ant it leads us to the next extensions.

A Half-Space Case If we consider Eqs. (3) and (7) in spaces $L_2(\mathbf{R}^m_+)$ and $L_2(h\mathbf{Z}^m_+)$ or in other words operators $K : L_2(\mathbf{R}^m_+) \to L_2(\mathbf{R}^m_+)$ and $K_d : L_2(h\mathbf{Z}^m_+) \to L_2(h\mathbf{Z}^m_+)$ then for studying invertibility of the operator K_d one has constructed a special periodic Riemann boundary value problem [10]. A solvability of mentioned Riemann problem depends on a certain topological invariant æ related to a symbol of an elliptic operator. This number æ is called an index of periodic Riemann boundary value problem. It was shown these topological numbers for elliptic operators K and K_d are the same and it implies the following [8, 9]

Theorem 2 *Operators (3) and (7) are invertible or non-invertible in spaces $L_2(\mathbf{R}^m_+)$ and $L_2(h\mathbf{Z}^m_+)$ simultaneously $\forall h > 0$.*

Studying more complicated situations related to cones [6] was started in [14], first steps were done.

Discrete Boundary Value Problems These arise first in the case $h\mathbf{Z}^m_+$ then we have a boundary, and it is possible the mentioned index æ is not a zero. To exclude a non-uniqueness of solution one needs some boundary conditions [1, 6]. Some similar situations were considered for difference equations in papers [12, 13, 15].

2.3 Infinite and Finite

Finite Discrete Fourier Transform Here we will introduce a special discrete periodic kernel $K_{d,N}(\tilde{x})$ which is defined in the following way. We take a restriction of the discrete kernel $K_d(\tilde{x})$ on the set $Q_N \cap h\mathbf{Z}^m \equiv Q_N^d$ and periodically continue it to a whole $h\mathbf{Z}^m$. Further we consider discrete periodic functions $u_{d,N}$ with discrete cube of periods Q_N^d. We can define a cyclic convolution for a pair of such functions $u_{d,N}$, $v_{d,N}$ by the formula

$$(u_{d,N} * v_{d,N})(\tilde{x}) = \sum_{\tilde{y}\in Q_N^d} u_{d,N}(\tilde{x}-\tilde{y})v_{d,N}(\tilde{y})h^m. \tag{8}$$

Further we introduce finite discrete Fourier transform by the formula

$$(F_{d,N}u_{d,N})(\tilde{\xi}) = \sum_{\tilde{x}\in Q_N^d} u_{d,N}(\tilde{x})e^{i\tilde{x}\cdot\tilde{\xi}}h^m, \quad \tilde{\xi} \in R_N^d,$$

where $R_N^d = \hbar\mathbf{T}^m \cap \hbar\mathbf{Z}^m$. Let us note that here $\tilde{\xi}$ is a discrete variable.

Finite Discrete Operator According to the formula (8) one can introduce the operator

$$K_{d,N}u_{d,N}(\tilde{x}) = \sum_{\tilde{y}\in Q_N^d} K_{d,N}(\tilde{x}-\tilde{y})u_{d,N}(\tilde{y})h^m$$

on periodic discrete functions $u_{d,N}$ and a finite discrete Fourier transform for its kernel

$$\sigma_{d,N}(\tilde{\xi}) = \sum_{\tilde{x}\in Q_N^d} K_{d,N}(\tilde{x})e^{i\tilde{x}\cdot\tilde{\xi}}h^m, \quad \tilde{\xi} \in R_N^d.$$

Definition 3 A function $\sigma_{d,N}(\tilde{\xi})$, $\tilde{\xi} \in R_N^d$, is called s symbol of the operator $K_{d,N}$. This symbol is called an elliptic symbol if $\sigma_{d,N}(\tilde{\xi}) \neq 0$, $\forall\tilde{\xi} \in R_N^d$.

Theorem 3 *Let $\sigma_d(\xi)$ be an elliptic symbol. Then for enough large N the symbol $\sigma_{d,N}(\tilde{\xi})$ is elliptic symbol also.*

A proof of the theorem follows immediately.

As before an elliptic symbol $\sigma_{d,N}(\tilde{\xi})$ corresponds to the invertible operator $K_{d,N}$ in the space $L_2(Q_N^d)$.

3 Discrete Functional Spaces

Since we'll use projectors on points of lattice we need subspaces of continuous functions instead of Lebesgue spaces. We introduce the space C_h which is the space of functions u_d of discrete variable $\tilde{x} \in h\mathbf{Z}^m$ with the norm

$$||u_d||_{C_h} = \max_{\tilde{x} \in h\mathbf{Z}^m} |u_d(\tilde{x})|.$$

In other words, the space C_h is the space of functions $u \in C(\mathbf{R}^m)$ restricted on lattice points $\mathbf{Z}_h^m$. Here we remind, that the operator K isn't bounded in the space $C(\mathbf{R}^m)$, but it is bounded in the space $L_2(\mathbf{R}^m)$, and it is well-known, that if the right hand side of the equation

$$(Ku)(x) = v(x)$$

has some smoothness properties (for example, it satisfies the Hölder condition), then the solution of this (if it exists in the space $L_2(\mathbf{R}^m)$) has the same smoothness property [4].

Further we define the discrete space $C_h(\alpha, \beta)$ as a functional space of discrete variable $\tilde{x} \in h\mathbf{Z}^m$ with finite norm

$$||u_d||_{C_h(\alpha,\beta)} = ||u_d||_{C_h} + \sup_{\tilde{x},\tilde{y} \in h\mathbf{Z}^m} |u_d(\tilde{x}) - u_d(\tilde{y})|,$$

and additional assumptions

$$|u_d(\tilde{x}) - u_d(\tilde{y})| \leq c \frac{|\tilde{x} - \tilde{y}|^\alpha}{(\max\{1 + |\tilde{x}|,\ 1 + |\tilde{y}|\})^\beta},$$

$$|u_d(\tilde{x})| \leq \frac{c}{(1 + |\tilde{x}|)^{\beta-\alpha}}, \qquad \forall \tilde{x}, \tilde{y} \in h\mathbf{Z}^m,\ \alpha, \beta > 0,\ 0 < \alpha < 1.$$

4 Approximate Solutions

4.1 *Infinite Discrete Solutions*

Let's denote P_h the restriction operator on the lattice $h\mathbf{Z}^m$, i.e. the operator, which an arbitrary function, defined on $\mathbf{R}^m$, maps to the set of its discrete values in lattice points $h\mathbf{Z}^m$.

Definition 4 The approximation rate for the operators K and K_d in vector normed space X of functions defined on $\mathbf{R}^m$, is called the operator norm

$$||P_h K - K_d P_h||_{X \to X_d},$$

where X_d is the normed space of functions defined on the lattice $h\mathbf{Z}^m$ with norm, which is induced by the norm of the space X.

For the space $C_h(\alpha, \beta)$ we have

Theorem 4 *If $m < \beta < \alpha + m$, then the estimate*

$$||K_d u_d||_{C_h(\alpha,\beta)} \le c||u_d||_{C_h(\alpha,\beta)},$$

is valid, and c doesn't depend on h.

The continual analogue of such spaces is the space $H_\beta^\alpha(\mathbf{R}^m)$ of functions, which are continuous in $\mathbf{R}^m$ and satisfy the Hölder condition of order $0 < \alpha < 1$ and with weight $(1+|x|)^\beta$. It is well known from results of S.K. Abdullaev (Sov. Math., Dokl. 40, No.2, 417-421,1990) that the operator K is a linear bounded operator $K : H_\beta^\alpha(\mathbf{R}^m) \to H_\beta^\alpha(\mathbf{R}^m)$ under the condition $m < \beta < \alpha + m$.

We will give the approximation rate for the operators K and K_d in the space $C_h(\alpha, \beta)$. It will permit to obtain the error estimate for approximate solution, if we will change the continual operator K by its discrete analogue K_d.

Theorem 5 *The approximate rate for the operators K and K_d is the following*

$$||P_h K - K_d P_h||_{C_h(\alpha,\beta)} \le c h^{\tilde{\alpha}},$$

where c doesn't depend on h, $\tilde{\alpha} < \alpha$, $\tilde{\beta} > \beta$.

Some of these results were obtained in [7].

4.2 Finite Discrete Solutions

Let us denote P_N the projector $L_2(h\mathbf{Z}^m) \to L_2(Q_N^d)$.

Theorem 6 *For operators K_d and $K_{d,N}$ we have the following estimate*

$$||(P_N K_d - K_{d,N} P_N)u_d||_{L_2(Q_N^d)} \le C N^{m+2(\alpha-\beta)}$$

for arbitrary $u_d \in C_h(\alpha, \beta)$, $\beta > \alpha + m/2$.

Now we consider the equation

$$K_{d,N} u_{d,N} = P_N v_d \tag{9}$$

instead of the equation

$$K_d u_d = v_d \tag{10}$$

and give a comparison for these two solutions assuming that operator K_d is invertible in $L_2(h\mathbf{Z}^m)$.

Theorem 7 *If $v_d \in C_h(\alpha, \beta)$, $\beta > \alpha + m/2$, u_d is a solution of the Eq. (10), $u_{d,N}$ is a solution of (9) then the estimate*

$$||u_d - u_{d,N}||_{L_2(h\mathbf{Z}^m)} \leq CN^{m+2(\alpha-\beta)}$$

is valid, and C is a constant non-depending on N.

Conclusion

These considerations are first steps to realize the declared programm. We hope that obtained results will help us to study more general discrete operators and equations and to describe a correspondence between discrete and continual objects, and also between finite and infinite discrete objects.

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