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Solution of the Goursat Problem for a Fourth-Order Hyperbolic Equation with Singular Coefficients by the Method of Transmutation Operators

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Abstract: In this paper, the method of transmutation operators is used to construct an exact solution of the Goursat problem for a fourth-order hyperbolic equation with a singular Bessel operator. We emphasise that in many other papers and monographs the fractional Erdélyi-Kober operators are used as integral operators, but our approach used them as transmutation operators with additional new properties and important applications. Specifically, it extends its properties and applications to singular differential equations, especially with Bessel-type operators. Using this operator, the problem under consideration is reduced to a similar problem without the Bessel operator. The resulting auxiliary problem is solved by the Riemann method. On this basis, an exact solution of the original problem is constructed and analyzed.

Keywords: Goursat problem; Bessel operator; transmutation operator; Erdélyi-Kober operator; Riemann method; fourth-order equation

MSC: 26D07; 26D10; 26D15; 26A33

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1. Introduction: Formulation of the Problem

The study of more complex higher-order equations with singular coefficients is a natural next step on the path of theoretical generalizations. The value of the theoretical results obtained in this way increases substantially in connection with the fact that such equations or their special cases occur in applications.

We especially note the class of partial differential equations with singularities in the coefficients, typical representatives of which are equations with Bessel operators of the form

$$B_{\eta}^{x} = x^{-2\eta-1}\frac{d}{dx}x^{2\eta+1}\frac{d}{dx} = \frac{d^2}{dx^2} + \frac{2\eta+1}{x}\frac{d}{dx} \tag{1}$$

For equations of elliptic, hyperbolic, and parabolic types with the Bessel operator in single or several variables, I.A. Kipriyanov [1] introduced the terminology B-elliptic, B-hyperbolic, and B-parabolic equations, respectively. The importance of equations from these classes is also determined by their use in applications to problems of generalized axially symmetric potential theory (GASPT) [2,3], Euler-Poisson-Darboux (EPD) equations [4,5], Radon transform and tomography [6–8], gas dynamics and acoustics [9], the theory of jets in hydrodynamics [10], the linearized Maxwell-Einstein equations [11,12], mechanics, the theory of elasticity and plasticity [13], and many others.

In a certain approximation, we can say that these three classes of differential equations according to the terminology of I.A. Kipriyanov were once considered in three well-known monographs: B-elliptic equations in the monograph by I.A. Kipriyanov [1], B-hyperbolic equations in the monograph by R. Carroll and R. Showalter [14], and B-parabolic equations in the monograph by M.I. Matiichuk [15]. Of course, many other problems are covered in these books.

The entire range of questions for equations with Bessel operators was most fully studied by the Voronezh mathematician I.A. Kipriyanov and his students. Note also that the 2023rd year is a centennial jubilee of Professor Ivan Alexandrovich Kipriyanov. More detailed information about this direction can be found in the monographs of V.V. Katrakhov and S.M. Sitnik [16], S.M. Sitnik, and E.L. Shishkina [17,18]. These types of equations is also deeply connected with fractional type operators, via Riemann–Liouville, Gerasimov, Erdélyi-Kober and other classes of operators, different classical integral transforms, and integral transforms with special function kernels, cf. [17–22].

The theory of equations with singular coefficients is closely related to the theory of equations degenerating on the boundary of a domain. By means of a change of variables, one can reduce a rather wide class of degenerate equations to equations with singular coefficients.

The difficulty to solve problems in the theory of partial differential equations with singular coefficients, as well as the resulting equations that degenerate at the boundary of the domain under consideration, has been extremely stimulating and continues to stimulate intensive research in this area. This is confirmed by numerous scientific publications over the past fifty years, noted the monographs of M.S. Salakhitdinov and A.K. Urinov [20], T.D. Dzhuraev and A. Sapuev [23], V.I. Zhegalov, A.N. Mironov, E.A. Utkina [24], A.M. Nakhushev [25], M.S. Salakhitdinov and M. Mirsaburov [26], V. V. Katrakhov and S. M. Sitnik [16], S. M. Sitnik and E. L. Shishkina [17,18], M.S. Salakhitdinov and B. Islamov [27], O.A. Marichev, A.A. Kilbas and O.A. Repin [28], A.K. Urinov [29,30], A. K. Urinov and Sh.T. Karimov [21], and others.

Note that in this paper, we consider only mathematical problems of differential equations and transmutations, and do not consider computational aspects of related problems or applications to physics, mechanics, etc.

In this paper in the domain $\Omega = \{(x, y) : 0 < x < l,\ 0 < y < h\}$, we consider the equation

$$L^{c}_{a,b}(u) \equiv \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{a}{x}\frac{\partial^3 u}{\partial x \partial y^2} + \frac{b}{y}\frac{\partial^3 u}{\partial y \partial x^2} + \frac{ab}{xy}\frac{\partial^2 u}{\partial x \partial y} + cu = 0, \tag{2}$$

where $l, h, a, b, c \in R$, and $l, h > 0, 0 < a, b < 1$.

Equation (2) for $a = b = 0$ was studied in [23] and, according to the classification of this work, it belongs to the hyperbolic type. The straight lines $x = const,\ \ y = const$ are real double characteristics of Equation (2).

In [23], the problems were considered in the characteristic quadrangle, and the coefficients of the equation were smooth enough to ensure the existence of the Riemann function, in terms of which, ultimately, the solutions of the problems were written. However, the problems for Equation (2), whose coefficients have singularities, are almost not studied. The coefficients of Equation (2) have a singularity on the lines $x = 0,\ \ y = 0$; such equations are called equations with singular coefficients. In addition, the singularity lines are simultaneously the characteristics of this equation.

A systematic study of two-dimensional equations of the fourth order was considered in the works of M.S. Salakhitdinov [31], T.D. Dzhuraev and A. Sopuev [23], V.I. Zhegalov, A.N. Mironov, E.A. Utkina [24], M.M. Smirnov [32], M.M. Meredov [33], A.K. Urinov [29] and their students. In the works of T.D. Dzhuraev and A. Sopuev [23], the questions of complete classification and reduction to the canonical form of a general linear fourth-order equation with two independent variables were studied. It is known that degenerate and singular equations of the second order have the peculiarity that the well-posedness of classical problems does not always hold for them. The formulation of the problem is significantly affected by lower coefficients. Such questions for high-order equations with singular coefficients have hardly been studied. In this paper, in domain Ω, we study an analogue of the Goursat problem for Equation (2).

Problem G. It is required to find a function $u(x,y) \in C(\bar{\Omega})$ satisfying Equation (1) and boundary conditions

$$u(0,y) = \varphi_1(y), \quad \lim_{x\to 0} x^a u_x(x,y) = \varphi_2(y), \quad 0 \leqslant y \leqslant h \tag{3}$$

$$u(x,0) = \psi_1(x), \quad \lim_{y\to 0} y^b u_y(x,y) = \psi_2(x), \quad 0 \leqslant x \leqslant l. \tag{4}$$

where $\varphi_k(y)$, $\psi_k(x)$, $(k = 1,2)$ are given smooth functions, such that $\varphi_1(0) = \psi_1(0)$, $\varphi_2(0) = \psi_2(0) = 0$. In this paper, in contrast to the cited sources, we use a different approach to solve the problem. Namely, taking into account the specifics of equations with singular coefficients, we use the Erdélyi-Kober transmutation operator.

Definition 1 ([16,17,34–36])**.** *Let a pair of operators (A, B) be given. A non-zero operator T is called a transmutation operator if the relation holds*

$$TA = BT. \tag{5}$$

In order for (5) to be a rigorous definition, it is necessary to specify spaces or sets of functions on which the operators A, B, and, consequently, T, act; various transmutation operators' issues are also considered in [17,36].

The Erdélyi-Kober operators for a certain choice of parameters are a generalization of the classical Sonin and Poisson transmutation operators [16,17,21,34–36]. We emphasise that in many of the above mentioned papers and monographs, the fractional Erdélyi-Kober operators are used as integral operators, but our approach uses them as transmutation operators with additional new properties and important applications. To be specific, it extends its properties and applications to singular differential equations, especially with Bessel-type operators.

Therefore, we first consider some properties of this operator.

2. Erdélyi—Kober Transmutation Operator

To construct a solution to the problem posed, we apply the multidimensional Erdélyi-Kober operator. This is for Erdélyi-Kober operators cf. [16–19,21,22] and namely for multidimensional ones and its generalizations cf. [37–39].

Let us emphasise the role of *multidimensional* Erdélyi-Kober operators with Bessel functions as kernels. They provide an instrument to transform more complex *multidimensional* singular differential equations with a spectral parameter into more simple ones. As a consequence, in this way we receive connection formulas for solutions to more complex *multidimensional* singular differential equations via solutions to more simple equations. As an example, this property of generalized Erdélyi-Kober operators was successfully applied to particular types of *multidimensional* singular differential equations in [37–39].

Therefore, we first consider some properties of this operator. Exactly, let us introduce the multidimensional generalized Erdélyi-Kober operator

$$J_\lambda \begin{pmatrix} \alpha \\ p \end{pmatrix} f(x) = J_{\lambda_1,\ldots,\lambda_n} \begin{pmatrix} \alpha_1, & \ldots, & \alpha_n \\ p_1, & \ldots, & p_n \end{pmatrix} f(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n} \left[\frac{2x_k^{-2(\alpha_k+p_k)}}{\Gamma(\alpha_k)} \right]$$

$$\int_0^{x_1}\int_0^{x_2} \ldots \int_0^{x_n} \prod_{k=1}^{n} \left[\frac{\bar{J}_{\alpha_k-1}\left(\lambda_k\sqrt{x_k^2-t_k^2}\right)}{\left(x_k^2-t_k^2\right)^{1-\alpha_k}} t_k^{2p_k+1} \right] f(t_1, \ldots, t_n)dt_1 \ldots dt_n \tag{6}$$

its properties and their application to multidimensional equations of hyperbolic and parabolic types with singular coefficients, where $\alpha_k > 0$, $p_k \geqslant -1/2$, $k = \overline{1,n}$, $\bar{J}_\nu(z)$ is the Bessel-Clifford function [22], which is expressed in terms of the Bessel functions

$J_\nu(z)$ according to the formula $\bar{J}_\nu(z) = \Gamma(\nu+1)(z/2)^{-\nu} J_\nu(z)$, $\Gamma(\nu)$ is the gamma function, parameters λ_k, $k \geq 1$ are all real or purely imaginary.

Note that in this paper, we use the term "Bessel-Clifford function", which is introduced by A.A.Kilbas in [22]. Often, another term is used for this function—"normalized" or "small" Bessel function, cf. [1,16–18].

Integral (6) is a multidimensional analogue of the one-dimensional generalized Erdélyi-Kober operator with the Bessel function in the kernel [22], ch. 7, §37.2, pp. 737–741. The integral (6) satisfies the following theorem [21], ch.2–3; [19], ch. 3; [37,38].

Theorem 1. *Let $\alpha_k > 0$, $p_k \geqslant -1/2$, $k = \overline{1,n}$, $f(x) \in C^{2n}(\Omega_n)$, $x_k^{2p_k+1} B_{p_k}^{x_k} f(x)$ are integrable in a neighborhood of $x_k = 0$ and $\lim\limits_{x_k \to 0} x_k^{2p_k+1} f_{x_k}(x) = 0$, $k = \overline{1, n}$.*

Then, the next equality is valid

$$\prod_{k=1}^{n} (B_{p_k+\alpha_k}^{x_k} + \lambda_k^2) J_\lambda \begin{pmatrix} \alpha \\ p \end{pmatrix} f(x) = J_\lambda \begin{pmatrix} \alpha \\ p \end{pmatrix} \prod_{k=1}^{n} B_{p_k}^{x_k} f(x),$$

where $\Omega_n = \prod\limits_{k=1}^{n} (0, a_k)$ is the Cartesian product, $a_k > 0$, $k = \overline{1,n}$ is the Bessel operator with respect to x_k, parameters, and $\lambda_k, k \geq 1$ are all real or purely imaginary.

Note that the theorem is also true for some or all of

$$\lambda_k = 0, \; k = \overline{1,m}, \; m \leqslant n.$$

3. Application of the Erdélyi-Kober Operator to the Solution of the Problem

Theorem 1 allows us to apply operator (6) as a transmutation operator that allows one to transform equations of high even order with singular coefficients into equations without singular coefficients. This fact is applicable to the study of problem G for Equation (2).

Due to the linearity of Equation (2), we first consider the following auxiliary problem.

Problem G_0. It is required to find a function $u_1(x,y) \in C(\bar{\Omega})$ satisfying Equation (2) and boundary conditions

$$u_1(0,y) = \varphi_1(y), \;\; u_{1x}(0,y) = 0, \;\; 0 \leqslant y \leqslant h, \tag{7}$$

$$u_1(x,0) = \psi_1(x), \;\; u_{1y}(x,0) = 0, \;\; 0 \leqslant x \leqslant l, \tag{8}$$

where $\varphi_1(y)$, $\psi_1(x)$ are given smooth functions, and $\varphi_1(0) = \psi_1(0)$. Assume that a solution to problem (2), (7) and (8) exists. This solution is sought in the form

$$u_1(x,y) = J_{0,0} \begin{pmatrix} \alpha, & \beta \\ -\frac{1}{2}, & -\frac{1}{2} \end{pmatrix} U(x,y), \tag{9}$$

where $U(x,y)$ is an unknown differentiable function, $\alpha = a/2$, $\beta = b/2$.

Substituting (9) into Equation (2) and boundary conditions (7) and (8), and then, using Theorem 1 for $n = 2, \lambda_1 = 0, \lambda_2 = 0, \; p_1 = p_2 = -1/2$, we obtain the problem of finding a solution $U(x,y)$ of the equation

$$U_{xxyy} + cU = 0, \;\; (x,y) \in \Omega \tag{10}$$

satisfying the boundary conditions

$$U(0,y) = \Phi_1(y), \;\; U_x(0,y) = 0, \;\; 0 \leqslant y \leqslant h, \tag{11}$$

$$U(x,0) = \Psi_1(x), \;\; U_y(x,0) = 0, \;\; 0 \leqslant x \leqslant l, \tag{12}$$

where

$$\Phi_1(y) = A_0 \frac{d}{dy} \int_0^y \left(y^2 - s^2\right)^{-\alpha} s^{2\alpha} \varphi_1(s) ds, \quad A_0 = \Gamma(\alpha + 1/2)/[\sqrt{\pi}\Gamma(1-\beta)], \tag{13}$$

$$\Psi_1(x) = B_0 \frac{d}{dx} \int_0^x \left(x^2 - s^2\right)^{-\alpha} s^{2\alpha} \psi_1(s) ds, \quad B_0 = \Gamma(\beta + 1/2)/[\sqrt{\pi}\Gamma(1-\alpha)]. \tag{14}$$

To construct a solution to problem (10)–(12), we apply the Riemann method. The Riemann function $R(x, y; \xi, \eta)$ is the solution—the adjoint Equation [23]

$$L^*(R) = R_{\eta\eta\xi\xi} + cR = 0, \tag{15}$$

satisfying the conditions

$$R(x, y; \xi, \eta)|_{\xi=x} = 0, \quad R_\xi(x, y; \xi, \eta)\big|_{\xi=x} = \eta - y,$$

$$R(x, y; \xi, \eta)|_{\eta=y} = 0, \quad R_\eta(x, y; \xi, \eta)\big|_{\eta=y} = \xi - x.$$

If the Riemann function is known, then the solution of the problem G_0 can be represented as [23]

$$u(x, y) = R_{\eta\xi}(x, y; 0, y)\varphi_1(y) - \int_0^y R_{\xi\eta\eta}(x, y; 0, \eta)\varphi_1(\eta) d\eta - \int_0^x R_\eta(x, y; \xi, 0)\psi''_1(\xi) d\xi. \tag{16}$$

We are looking for the Riemann function in the form

$$R(x, y; \xi, \eta) = pw(\sigma), \tag{17}$$

where $p = (\xi - x)(\eta - y), \sigma = \lambda(\xi - x)^2(\eta - y)^2, \quad \lambda = -c/16, w(\sigma)$ is an unknown function.

Calculating the derivatives of the expression (17) and substituting into the conjugate Equation (15), we find the equation

$$\sigma^3 w''''(\sigma) + 7\sigma^2 w'''(\sigma) + \frac{41}{4}\sigma w''(\sigma) + \frac{9}{4}w'(\sigma) - w(\sigma) = 0. \tag{18}$$

Generalized hypergeometric function [40]

$${}_0F_3(a, b, c; z) = \sum_{n=0}^{\infty} \frac{z^n}{(a)_n (b)_n (c)_n n!},$$

satisfies the equation

$$z^3 w''''(z) + (3 + a + b + c)z^2 w'''(z) + (1 + a + b + c + ab + ac + bc)zw''(z)$$
$$+ abcw'(z) - w(z) = 0. \tag{19}$$

Comparing (18) and (19) with respect to the parameters, we obtain the system of equations

$$\begin{cases} 3 + a + b + c = 7, \\ 1 + a + b + c + ab + ac + bc = 41/4, \\ abc = 9/4, \end{cases}$$

Solving this system by Vieta formulas for a cubic equation, we find the solution of Equation (18) in the form

$$w(\sigma) = {}_0F_3(3/2,\ 3/2,\ 1;\ \sigma)$$

and substituting this solution into representation (17), we determine the Riemann function for problem G:

$$R(x,y;\,\xi,\eta) = p_0F_3(3/2,\ 3/2,\ 1;\ \sigma). \tag{20}$$

By virtue of the equalities $(3/2)_n = 2^{-2n}\frac{(2n+1)!}{n!}$ and $(1)_n = n!$, function (20) coincides with the Riemann function from [23], constructed as a series

$$R(x,y;\,\xi,\eta) = \sum_{m=0}^{\infty} \frac{(-1)^m c^m (\xi - x)^{2m+1}(\eta - y)^{2m+1}}{[(2m+1)!]^2}.$$

Calculating the corresponding derivatives of function (20) and substituting them into equality (16), we obtain the solution of problem (10)–(12) in the form

$$\begin{aligned} U(x,y) &= \Phi_1(y) + \Psi_1(x) - \Psi_1(0)_0F_3(1/2,\,1/2,\,1;\lambda x^2y^2) \\ &+ \frac{cx^2}{2}\int_0^y [(t-y)_0F_3(3/2,\,3/2,\,2;\sigma_0)]\Phi_1(t)dt + \\ &+ \frac{cy^2}{2}\int_0^x [(s-x)_0F_3(3/2,\,3/2,\,2;\omega_0)]\Psi_1(s)ds, \end{aligned} \tag{21}$$

where $\sigma_0 = \lambda x^2(t-y)^2$, $\omega_0 = \lambda y^2(s-x)^2$, $\lambda = -c/16$.

Substituting (21) into (9) and taking into account (13) and (14), we change the order of integration, and then, after calculating the internal integrals, we obtain

$$\begin{aligned} u_1(x,y) &= \psi_1(x) + \varphi_1(y) - \varphi_1(0)_0F_3(\alpha+1/2,\,\beta+1/2,2;\,\lambda x^2y^2\,) - \\ &- \gamma_1 y^2\int_0^x K_1(x,y,s;\alpha,\beta)\psi_1(s)ds - \gamma_2 x^2\int_0^y K_2(x,y,s;\alpha,\beta)\varphi_1(s)ds, \end{aligned} \tag{22}$$

where $\gamma_1 = c2^{2\alpha-2}/(1+2\beta)$, $\gamma_2 = c2^{2\beta-2}/(1+2\alpha)$,

$$K_1(x,y,s;\alpha,\beta) = \frac{s^{2\alpha}(x-s)}{(x+s)^{2\alpha}} F^{1;0;1}_{1;3;0}\left[\begin{matrix} 1/2+\alpha;\ -;\ \alpha; \\ 3/2;\ 3/2+\beta,\ 1/2+\alpha,2;\ -; \end{matrix}\ \sigma_1,\ \omega_1\right],$$

$$K_2(x,y,s;\alpha,\beta) = \frac{s^{2\beta}(y-s)}{(y+s)^{2\beta}} F^{1;0;1}_{1;3;0}\left[\begin{matrix} 1/2+\beta;\ -;\ \beta; \\ 3/2;\ 3/2+\alpha,\ 1/2+\beta,2;\ -; \end{matrix}\ \sigma_2,\ \omega_2\right],$$

$\sigma_1 = \lambda y^2(x-s)^2$, $\omega_1 = (x-s)^2/(x+s)^2$, $\sigma_2 = \lambda x^2(y-s)^2$, $\omega_2 = (y-s)^2/(y+s)^2$, and $0 \le \omega_k \le 1$, $k = 1,2$; $F^{p;q;k}_{l;m;n}$ is the hypergeometric function of Kampé de Fériet, which has the form

$$F^{p;q;k}_{l;m;n}\left[\begin{matrix} (a_p); & (b_q); & (c_k); \\ (\alpha_l); & (\beta_m); & (\gamma_n); \end{matrix}\ x,y\right] =$$

$$= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!},$$

here, $(a_p) = a_1, a_2, \ldots, a_p$.

This series converges at

1. $p+q < l+m+1$, $p+k < l+n+1$, $|x| < \infty$, $|y| < \infty$ or
2. $p+q = l+m+1$, $p+k = l+n+1$ and

$\begin{cases} |x|^{1/(p-l)} + |y|^{1/(p-l)} < 1, \ \text{at} p > l, \\ \max\{|x|, |y|\} < 1, \ \text{at} p \leq l, \end{cases}$ and besides that $\alpha_j \neq 0, -1, -2, \ldots, \ j = \overline{1, l}$; $\beta_j \neq 0, -1, -2, \ldots, \ j = \overline{1, m}; \ \gamma_j \neq 0, -1, -2, \ldots, j = \overline{1, n}$.

To construct a solution to Equation (20) that satisfies the conditions

$$u(0, y) = 0, \ \lim_{x \to 0} x^{2\alpha} u_x(x, y) = \varphi_2(y), \ 0 < y < h,$$

$$u(x, 0) = 0, \ \lim_{y \to 0} y^{2\beta} u_y(x, y) = \psi_2(x), \ 0 < x < l,$$

we use the following, easily proven property of the solution to Equation (20): if $u_1(x, y; 1 - \alpha, 1 - \beta)$ is a solution to the equation $L^c_{1-\alpha,1-\beta}(u_1) = 0$, satisfying conditions (7) and (8), then the function $u_2(x, y; \alpha, \beta) = x^{1-2\alpha} y^{1-2\beta} u_1(x, y; 1 - \alpha, 1 - \beta)$ for $0 < \alpha, \beta < 1/2$ will be a solution of the equation $L^c_{\alpha,\beta}(u_2) = 0$ satisfying the conditions

$$u_2(0, y) = 0, \ \lim_{x \to 0} x^{2\alpha} u_{2x}(x, y) = (1 - 2\alpha)\varphi_1(y), \ 0 < y < h;$$

$$u_2(x, 0) = 0, \ \lim_{y \to 0} y^{2\beta} u_{2y}(x, y) = (1 - 2\beta)\psi_1(x), \ 0 < x < l.$$

Taking into account this property and replacing $(1 - 2\alpha)\varphi_1(y)$ and $(1 - 2\beta)\psi_1(x)$, respectively, by $\varphi_2(y)$ and $\psi_2(x)$, from equality (22), we obtain

$$\begin{aligned} u_2(x, y) = {} & \frac{y^{1-2\beta}}{1 - 2\beta}\psi_2(x) + \frac{x^{1-2\alpha}}{1 - 2\alpha}\varphi_2(y) - \\ & -\widetilde{\gamma}_1 y^{3-2\beta} \int_0^x K_1(x, y, s; \alpha, 1 - \beta)\psi_2(s) ds - \\ & -\widetilde{\gamma}_2 x^{3-2\alpha} \int_0^y K_2(x, y, s; 1 - \alpha, \beta)\varphi_2(s) ds, \end{aligned} \tag{23}$$

where $\widetilde{\gamma}_1 = c2^{2\alpha-2}/(3 - 2\beta)(1 - 2\beta)$, $\widetilde{\gamma}_2 = c2^{2\beta-2}/(3 - 2\alpha)(1 - 2\alpha)$.

By virtue of the principle of the linear superposition, the solution of Problem G can be represented as $u(x, y) = u_1(x, y) + u_2(x, y)$.

Note that problem G is equivalent to a Volterra integral equation of the second kind of the form

$$u(x, y) + c \int_0^x \int_0^y K(x, y; s, t) u(s, t) ds dt = F(x, y) \tag{24}$$

where

$$K(x, y; s, t) = \frac{s^a t^b}{(1 - a)(1 - b)} [x^{1-a} - s^{1-a}][y^{1-b} - t^{1-b}],$$

$$F(x, y) = \frac{x^{1-a}}{1 - a}[\varphi_2(y) - \varphi_2(0)] + \frac{y^{1-b}}{1 - b}[\psi_2(y) - \psi_2(0)] + \psi_1(x) + \varphi_1(y) - \varphi_1(0).$$

By virtue of the general theory of Volterra integral equations of the second kind [41], the integral Equation (24) has a unique solution. At the same time, it is fair

Theorem 2. *If $0 < a, b < 1$ and $\varphi_k(y) \in C[0, h] \bigcap C^2(0, h)$, $\psi_k(x) \in C[0, l] \bigcap C^2(0, l)$, $k = 1, 2$, where $\varphi_2'(y)$ and $\psi_2'(x)$, respectively, can have order singularities less than b at $y \to 0$ and less than a at $x \to 0$, then there exists a unique classical solution to Problem G.*

For other values of the parameters a and b, the problem is solved by the method of analytic continuation of the operator (6) with respect to these parameters for $a = 2\alpha$ and $b = 2\beta$.

This method can also be applied to solving the boundary value problems for a multidimensional equation, a high-order equation of the (2) type, and a nonclassical Sobolev-type equation with many singular coefficients.

4. Conclusions

Using the Erdélyi-Kober transmutation operator, an exact solution of the problem is constructed. Despite the development of modern computer technology, the construction of exact solutions to boundary value problems for partial differential equations is still an important and urgent task. These solutions allow a deeper understanding of the qualitative features of the described processes and phenomena, the properties of mathematical models, and can also be used as test cases for asymptotic, approximate and numerical methods.

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