ISSN 0001-4346, Mathematical Notes, 2024, Vol. 115, No. 5, pp. 692–705. © Pleiades Publishing, Ltd., 2024.
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Two-Sided Estimates of Solutions with a Blow-Up Mode for a Nonlinear Heat Equation with a Quadratic Source

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Received June 30, 2023; revised August 11, 2023; accepted August 17, 2023

Abstract—We study compactly supported solutions $u(x,t) \geq 0$, $x \in \mathbb{R}$, $t \geq 0$, of a one-dimensional quasilinear heat transfer equation. The equation has a transport coefficient linear in u and a self-consistent source $\alpha u + \beta u^2$ of general form. For the blow-up time of compactly supported solutions, we establish two-sided estimates functionally depending on the initial conditions $u(x,0)$.

DOI: 10.1134/S0001434624050055

Keywords: *approximation of solutions, compact support, nonlinear heat transfer equation, blow-up mode, model solution*

1. INTRODUCTION

The quasilinear parabolic equation of the general form

$$\dot{u} = (k(u)u_x)_x + f(u_x, u) \tag{1}$$

for functions $u(x,t), x \in \mathbb{R}$, $t \geq 0$, where $u_x \equiv \partial u/\partial x$ and $\dot{u} \equiv \partial u/\partial t$, with a *transport coefficient* $k(u) \geq 0$ and a measurable function $f(u_x, u)$ of the current values of $u(x,t)$ and $u_x(x,t)$ is the basis for various models studied in mathematical physics (e.g., see [1]). In this case, as a rule, one is interested in nonnegative solutions $u(x,t) \geq 0$ of this equation with zero boundary conditions. A special case of such an equation is one in which $k(0) = 0$. In this case, solutions with a weak discontinuity may arise that have the solution $u(x,t) = 0$ on one side of the discontinuity [1]. This phenomenon was predicted in [2] and was subsequently studied in various publications (e.g., see [3]); in particular, the conditions for stopping the front were studied in [4]. Assume that the function $f(u_x, u)$ in the equation can be represented as $f(u_x, u) = (F(u))_x + g(u)$, where $F(u)$ is differentiable with respect to u and the function $g(u)$ is measurable. If $F(u)$ is other than linear and tends to infinity as $u \to \infty$, then, as is well known, such an equation no longer has global solutions owing to the formation of discontinuities in the solution $u(x,t)$ in finite time. Moreover, weak solutions of the Cauchy problem are no longer determined in a unique way. The term $(F(u))_x$ in this case describes the transfer phenomenon. The concept of an entropy solution is introduced to ensure uniqueness [5]. A substantial number of papers have already been devoted to the study of such solutions (e.g., see [6]–[8]). Depending on the properties of the function $g(u)$, the solutions of Eq. (1) may manifest different qualitative behavior [1]. Of particular interest is the study of solutions $u(x,t)$ of Eq. (1) for $f(u_x, u) = (F(u))_x + g(u)$ with $g(u) > 0$. In this case, the solutions of this equation can go into the so-called blow-up mode. Such solutions $u(x,t)$ exist only on a finite time interval $t \in [0, t_*)$ so that $u(x,t) \to \infty$ at some point $x \in \mathbb{R}$ as $t \to t_*$, where t_* is called the blow-up time. This mode is realized under a certain asymptotic behavior of the function $g(u)$, namely, $g(u) \sim u^\gamma$, $\gamma > 1$. The main direction of these publications is related to the study of the conditions under which a blow-up mode arises and to determining the global characteristics of the corresponding solutions. We will further be interested in the special case of Eq. (1) in which

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$F \equiv 0$. In this case, we refer to the function $g(u)$ as a self-consistent source. The study of blow-up modes for such an equation has a long history [9]–[15]. The research into these modes is summarized in the monograph [16]. Among the more recent advancements in the study of blow-up modes, we note [17], [18], where it was studied how to determine the critical index of nonlinearity that causes the blow-up and how to estimate the blow-up time and the size of the compact domain in which such a blow-up occurs.

In the present paper, we consider the degenerate equation (1) with $F \equiv 0$ taking into account terms that are at most quadratic in the function u, namely,

$$\dot{u} = (uu_x)_x + \alpha u + \beta u^2, \tag{2}$$

where $\alpha, \beta \in \mathbb{R}$. We study solutions compactly supported in $[c_-, c_+] \subset \mathbb{R}$. They have a blow-up for $\beta > 0$ and also "vanish" in finite time for $\beta < 0$. We propose an alternative (in our opinion) method for two-sided estimation of blow-up time that was first used by one of the authors in [19], [20] without sufficient mathematical justification. Here we fill the gap, and we hope that this method will be developed further.

2. MAXIMUM PRINCIPLE

In this section, we prove the maximum principle for quasilinear parabolic equations in the form presented in the monograph [16], where, however, its proof is not given, and then we present a generalization of this principle to compactly supported weak solutions of degenerate equations of this type.

Let $T > 0$ and $K > 0$ be arbitrary constants. Consider the Banach space $C_{2,1}([-K, K], [0, T])$ of functions $u(x, t)$ of two variables $\langle x, t\rangle$ twice continuously differentiable with respect to $x \in [-K, K]$ and continuously differentiable with respect to $t \in [0, T]$ with the norm

$$\|u\| = \max_{j\in\{0,1\},k\in\{0,1,2\}} \max_{x\in[-K,K],t\in[0,T]} \left|\frac{\partial^{j+k}u(x,t)}{\partial t^j \partial x^k}\right|.$$

Lemma 2.1. *Let $k'(u)$ be a function satisfying the Lipschitz condition in $u \in \mathbb{R}$, and let $f(u_x, u)$ be a function satisfying the same condition in each of the variables $\langle u_x, u\rangle \in \mathbb{R}^2$. Then the sets*

$$S_1 = \{u \in \mathrm{C}_{2,1}([-K, K], [0, T]) : \dot{u}(x, t) < \mathrm{L}[u](x, t)\}, \tag{3}$$

$$S_2 = \{u \in \mathrm{C}_{2,1}([-K, K], [0, T]) : \dot{u}(x, t) > \mathrm{L}[u](x, t)\} \tag{4}$$

of functions, where

$$\mathrm{L}[u](x, t) \equiv [(k(u)u_x)_x + f(u_x, u)](x, t), \tag{5}$$

are open in the space $\mathrm{C}_{2,1}([-K, K], [0, T])$, and their closures are, respectively, the sets

$$\{u \in \mathrm{C}_{2,1}([-K, K], [0, T]) : \dot{u}(x, t) \le \mathrm{L}[u](x, t)\}$$

and

$$\{u \in \mathrm{C}_{2,1}([-K, K], [0, T]) : \dot{u}(x, t) \ge \mathrm{L}[u](x, t)\ .$$

Proof. It suffices to prove the first part of the assertion. Assume that the function $u(x, t)$ satisfies the inequality $\dot{u}(x, t) < \mathrm{L}[u](x, t)$. Since the function $\dot{u}(x, t) - \mathrm{L}[u](x, t)$ is jointly continuous on the rectangle $[-K, K] \times [0, T]$, it follows that

$$-\varepsilon = \max_{x\in[-K,K],t\in[0,T]} \{(\dot{u} - \mathrm{L}[u])(x, t)\} < 0.$$

Adding an arbitrary function $\delta(x, t)$ with a sufficiently small norm such that

$$\dot{\delta}(x, t) + (\mathrm{L}[u] - \mathrm{L}[u + \delta])(x, t) < \varepsilon \tag{6}$$

to the function $u(x, t)$, we conclude that all functions $(u + \delta)(x, t)$ in $\mathrm{C}_{2,1}([-K, K], [0, T])$ form a neighborhood of $u(x, t)$ contained in a set of the type (3). At the same time, the possibility of choosing such a function $\delta(x, t)$ follows from the estimates

$$|\dot{\delta}(x, t) + (\mathrm{L}[u] - \mathrm{L}[u + \delta])(x, t)| \le \|\delta\| + \|\mathrm{L}[u] - \mathrm{L}[u + \delta]\|_0$$

$$\le \|\delta\| + \|u_{xx}\| \cdot \|k(u) - k(u+\delta)\|_0 + \|\delta\| \cdot \|k(u+\delta)\|_0 + \|u_x\|^2 \cdot \left\|k'(u) - k'(u+\delta)\right\|_0 + \|\delta\|(2\|u\| + \|\delta\|)\left\|k'(u+\delta)\right\|_0 + \left\|f\left(u_x + \delta_x, u+\delta\right) - f\left(u_x, u\right)\right\|_0,$$

where by $\|\cdot\|_0$ we denote the norm on the space $\mathrm{C}([-K,K]\times[0,T])$ of continuous functions $u(x,t)$,

$$\|u\|_0 = \max_{x\in[-K,K],t\in[0,T]} |u(x,t)|.$$

The norm $\|\delta\|$, just as the norms

$$\|k(u) - k(u+\delta)\|_0 \le \left\|k'(u)\right\|_0 \|\delta\|_0,$$

$$\left\|k'(u) - k'(u+\delta)\right\|_0 \le K(u)\|\delta\|_0,$$

$$\left\|f\left(u_x+\delta_x, u+\delta\right) - f\left(u_x, u\right)\right\|_0 \le L\left(u_x, u\right)\|\delta\|_0$$

of the differences for $\|\delta\|_0 < \varepsilon$ can be made arbitrarily small, because the functions $k'(u)$ and $f(u_x, u)$ satisfy the Lipschitz condition with constants $K(u)$ and $L(u_x, u)$, respectively, depending on u_x and u.

In the same way, one proves that the set (4) is open. In this case, one takes

$$\min_{x\in[-K,K],t\in[0,T]} \{(\dot u - \mathrm{L}[u])(x,t)\} = \varepsilon > 0$$

and chooses the function $\delta(x,t)$ in the same way so that inequality (6) holds. □

Theorem 2.1. *Let $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ be functions twice continuously differentiable with respect to $x \in \mathbb{R}$ and satisfying the inequalities*

$$\dot u^{(1)}(x,t) \le \left[\left(k\left(u^{(1)}\right)u_x^{(1)}\right)_x + f\left(u_x^{(1)}, u^{(1)}\right)\right](x,t), \tag{7}$$

$$\dot u^{(2)}(x,t) \ge \left[\left(k\left(u^{(2)}\right)u_x^{(2)}\right)_x + f\left(u_x^{(2)}, u^{(2)}\right)\right](x,t), \tag{8}$$

respectively, for $t \ge t_0 \in \mathbb{R}_+$. If $u^{(1)}(x,t_0) \le u^{(2)}(x,t_0)$ for all $x \in \mathbb{R}$, then for each $t > t_0$ one also has the inequality $u^{(1)}(x,t) \le u^{(2)}(x,t)$ for all $x \in \mathbb{R}$.

Proof. First, consider the case in which the function $u^{(1)}(x,t)$ satisfies the strict inequality in (7). Assume the contrary: for some $t' > t_0$, there exists a point x' of contact between the graphs of $u^{(1)}(x,t')$ and $u^{(2)}(x,t')$; that is, the equalities $u^{(1)}(x',t') = u^{(2)}(x',t')$ and $u_x^{(1)}(x',t') = u_x^{(2)}(x',t')$ hold. In addition, the inequality $u_{xx}^{(1)}(x',t') \le u_{xx}^{(2)}(x',t')$ must be satisfied in this case. Without loss of generality, we assume that the point t' is the first of all possible points of this type. Consequently, based on (7) and (8) we have the inequality $\dot u^{(2)}(x',t') > \dot u^{(1)}(x',t')$, and therefore, for sufficiently small $\varepsilon > 0$ we have $u^{(2)}(x',t'+\varepsilon) > u^{(1)}(x',t'+\varepsilon)$. Thus, there is no intersection of the graphs of $u^{(1)}(x,t')$ and $u^{(2)}(x,t')$ at the point x' at $t'+\varepsilon$.

In the same way, it is proved that the graphs of $u^{(1)}(x,t')$ and $u^{(2)}(x,t')$ do not meet at any point $t' > t_0$ if the function $u^{(2)}(x,t)$ satisfies the strict inequality in (8).

Now let us extend the proof to the general case. To this end, in the space

$$\mathrm{C}_{2,1}([-K,K],[0,T]) \times \mathrm{C}_{2,1}([-K,K],[0,T])$$

consider pairs of functions $\left\langle u^{(1)}(x,t), u^{(2)}(x,t)\right\rangle$ that belong to the cone S in which the inequality $u^{(1)}(x,t) \le u^{(2)}(x,t)$ holds. In this space, the sets

$$S_1 \times \mathrm{C}_{2,1}([-K,K],[0,T]), \qquad \mathrm{C}_{2,1}([-K,K],[0,T]) \times S_2$$

are open by Lemma 2.1. It follows from the first part of the proof that the set

$$S \cap (S_1 \times \mathrm{C}_{2,1}([-K,K],[0,T])) \cap (\mathrm{C}_{2,1}([-K,K],[0,T]) \times S_2)$$

is nonempty and open. Then the theorem holds on the closure of this set for $\langle x,t\rangle \in [-K,K]\times[0,T]$. By passing to the limit, first as $K \to \infty$ and then as $T \to \infty$, we conclude that the theorem holds for all pairs $\left\langle u^{(1)}(x,t), u^{(2)}(x,t)\right\rangle$ of functions with the properties specified in the condition of the theorem. □

Remark. The presented proof of the maximum principle does not assume the uniqueness of the solution of the Cauchy problem for Eq. (1); this is important in the case of the considered degenerate equation with $u = 0$, where the solution of such a problem may not be unique.

Further, we will study solutions of degenerate hyperbolic equations (1) for which the functions $g(u) = f(u_x, u)$ and $k'(u)$ satisfy the Lipschitz condition and at the same time $k(0) = 0$ and $k'(0) > 0$. We denote the class of such equations by $\mathfrak{B}$. For these equations, consider compactly supported weak solutions $u(x,t)$ of a special type. They are constructed as follows. Let $w(x,t)$ be an "exact" solution of the equation (a classical solution); that is, $\dot{w}(x,t) = (k(w)w_x)_x + g(w)$. Assume that $w(x_-(t),t) = w(x_+(t),t) = 0$ and define

$$u(x,t) = \begin{cases} w(x,t), & x \in [x_-(t), x_+(t)], \\ 0, & \mathbb{R}\setminus(x_-(t), x_+(t)). \end{cases}$$

If $u(x,t)$ is not an exact solution, then at least at one of the points $x' = x_\pm(t)$ the derivative $(du/dx)_{\langle x_\pm(t),t\rangle}$ is not zero.

It is obvious that the functions $u(x,t)$ are weak solutions if $\lim_{x\to x_\pm(t)} |w_x(x,t)| < \infty$, because one has

$$\int_{-\infty}^{\infty} (\dot{w} - (k(w)w_x)_x - g(w))(x,t)\,dx = 0.$$

We denote the class of all such weak solutions by $\mathfrak{R}$. Then the following assertion holds.

Theorem 2.2. *Let $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ be weak solutions in the class $\mathfrak{R}$ of an equation in the class $\mathfrak{B}$, and let the following inequalities hold for $t \ge t_0 \in \mathbb{R}_+$:*

$$\dot{u}^{(1)}(x,t) \le \big[\big(k\big(u^{(1)}\big)u_x^{(1)}\big)_x + g\big(u^{(1)}\big)\big](x,t), \tag{9}$$

$$\dot{u}^{(2)}(x,t) \ge \big[\big(k\big(u^{(2)}\big)u_x^{(2)}\big)_x + g\big(u^{(2)}\big)\big](x,t). \tag{10}$$

If $u^{(1)}(x,t_0) \le u^{(2)}(x,t_0)$ for all $x \in \mathbb{R}$, then for each $t > t_0$ and for all $x \in \mathbb{R}$ one also has the inequality $u^{(1)}(x,t) \le u^{(2)}(x,t)$.

Proof. Let functions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ of the class $\mathfrak{R}$ with supports $\big[x_-^{(j)}(t), x_+^{(j)}(t)\big]$, $j \in \{1,2\}$, satisfy the inequality $u^{(1)}(x,t_0) \le u^{(2)}(x,t_0)$, so that $\big[x_-^{(1)}(t_0), x_+^{(1)}(t_0)\big] \subset \big[x_-^{(2)}(t_0), x_+^{(2)}(t_0)\big]$. Assume that for some $t' > t_0$ there exists a point x' at which $u^{(1)}(x',t') = u^{(2)}(x',t')$ and for sufficiently small $\varepsilon > 0$ one has $u^{(1)}(x',t'+\varepsilon) > u^{(2)}(x',t'+\varepsilon)$. Moreover, t' is the first of all possible points of this type, and therefore, at the point t' one has the inclusion $\big[x_-^{(1)}(t'), x_+^{(1)}(t')\big] \subset \big[x_-^{(2)}(t'), x_+^{(2)}(t')\big]$.

If $x' \in \big[x_-^{(1)}(t'), x_+^{(1)}(t')\big]$, then the inequality $u^{(1)}(x',t'+\varepsilon) > u^{(2)}(x',t'+\varepsilon)$ is impossible for any sufficiently small $\varepsilon > 0$ according to the proof of Theorem 2.1. Consider the case in which $x' \in \big\{x_-^{(1)}(t'), x_+^{(1)}(t')\big\}$. To be definite, assume that $x' = x_+^{(1)}(t)$. If in a neighborhood of the point $x_+^{(1)}(t)$ the values of both functions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ are represented as exact solutions of the equation, that is, this point is an extreme point of the support of exact solutions, then the reasoning given in the proof of the statement of Theorem 2.1 is again valid for it. Therefore, we assume that, at least for one of the functions, the point $x_+^{(1)}(t)$ is not an extreme point of the support of the exact solution. It follows that

$$\big(du^{(j)}/dx\big)_{\big\langle x_+^{(1)}(t'),t'\big\rangle} \ne 0, \qquad j \in \{1,2\}.$$

Then one has $u^{(1)}\big(x_+^{(1)}(t'),t'\big) = u^{(2)}\big(x_+^{(1)}(t'),t'\big) = 0$ and

$$0 \le \big(du^{(1)}/dx\big)^2\big(x_+^{(1)}(t'),t'\big) < \big(du^{(2)}/dx\big)^2\big(x_+^{(1)}(t'),t'\big).$$

Consequently, in view of the properties of the function $k(u)$, at this point one has the inequality

$$\left[\dot{u}^{(1)}\left(x,t'\right)\right]_{x=x_+^{(1)}} = k'(0)\left[\left(\frac{du^{(1)}}{dx}\right)^2_{\langle x_+(t'),t'\rangle} - \left(\frac{du^{(2)}}{dx}\right)^2_{\langle x_+(t'),t'\rangle}\right] < 0.$$

Therefore, the inequality

$$u^{(1)}\left(x_+^{(1)}\left(t'\right),t\right) > u^{(2)}\left(x_+^{(1)}\left(t'\right),t\right), \qquad 0 < t - t' < \varepsilon,$$

is also impossible at this point, because the change of each of the functions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ at the point $x = x_+^{(1)}\left(t'\right)$ with time t is determined based on the exact solution of Eq. (1). □

Theorems 2.1 and 2.2 admit obvious generalizations. The conditions for the Lipschitz property of the functions $k'(u)$, $f\left(u_x,u\right)$, and $g(u)$ can be, without changing the strategy for proving the theorems, relaxed by replacing them with the conditions for the Hölder property of these functions with respect to their arguments with an arbitrary, however small Hölder exponent, and moreover, further weakening of the conditions imposed on these functions is possible.

3. ASYMPTOTICS OF BLOW-UP SOLUTIONS

In this section, we determine the possible type of the asymptotic behavior as $t \to t_*$ of a solution $u(x,t)$ of Eq. (2) assuming that the solution is compactly supported in some interval $[c_-,c_+]$ and tends to infinity at some point $x \in [c_-,c_+]$. Here t_* may be either finite or infinite.

First, we prove a statement related to the theorem on the differentiation of function sequences.

Lemma 3.1. *Let $\langle y_n(x); n \in \mathbb{N}\rangle$ be a sequence of continuously differentiable functions on an interval $[c_-,c_+] \subset \mathbb{R}$ such that the sequence $\langle y_n'(x); n \in \mathbb{N}\rangle$ of their derivatives is uniformly bounded, $\max_{x\in[c_-,c_+]}|y_n'(x)| < M$, and converges at each point $x \in [c_-,c_+]$ to a bounded measurable function $v(x)$, $x \in [c_-,c_+]$. In addition, assume that there exists a point $c \in [c_-,c_+]$ at which there exists a limit*

$$\lim_{n\to\infty} y_n(c) \equiv y(c). \tag{11}$$

Then the sequence $\langle y_n(x); n \in \mathbb{N}\rangle$ uniformly converges at each point $x \in [c_-,c_+]$ to a differentiable function

$$\lim_{n\to\infty} y_n(x) \equiv y(x) \tag{12}$$

such that $y'(x) = v(x)$.

Proof. Let us write the expression for the functions $y_n(x)$, $n \in \mathbb{N}$, in the form

$$y_n(x) = y_n(c) + \int_c^x v(\xi)\,d\xi + \int_c^x \left[y_n'(\xi) - v(\xi)\right]\,d\xi.$$

By Egorov's theorem (e.g., see [22]), for any $\varepsilon > 0$ there exists a set E_ε such that $\operatorname{mes}\left([c_-,c_+]\setminus E_\varepsilon\right) < \varepsilon$ and the function sequence $\langle y_n'(x); n \in \mathbb{N}\rangle$ uniformly converges as $x \in E_\varepsilon$ to the function $v(x)$. We can assume that the set E_ε is closed. From the estimate of the integral

$$\left|\int_c^x \left[y_n'(\xi) - v(\xi)\right]\,d\xi\right| \le \left(\max_{x\in[c_-,c_+]}|v(x)| + M\right)\operatorname{mes}\left([c_-,c_+]\setminus E_\varepsilon\right) + \operatorname{mes}\left(E_\varepsilon\right)\max_{x\in E_\varepsilon}\left|y_n'(x) - v(x)\right|,$$

in view of (12), after passing to the limit as $n \to \infty$, we find that at each point $x \in [c_-,c_+]$ one has the inequality

$$\limsup_{n\to\infty}\left|y_n(x) - y(c) - \int_x^x v(\xi)\,d\xi\right| \le \varepsilon\Big(\max_{x\in[c_-,c_+]}|v(x)| + M\Big).$$

Since the number $\varepsilon > 0$ is arbitrary, it follows that the sequence $\langle y_n(x); n \in \mathbb{N}\rangle$ converges uniformly at each point $x \in [a, b]$ to the function

$$y(x) = y(c) + \int_0^x v(\xi)\, d\xi. \tag{13}$$

Since in this case the sequence $\langle y_n'(x); n \in \mathbb{N}\rangle$ converges uniformly on E_ε, we can apply the theorem on the differentiation of function sequences and find that $y'(x) = v(x)$ for $x \in E_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we find that this equality holds everywhere on $[c_-, c_+]$. □

Let $u(x,t)$ be a solution of Eq. (2) supported inside $[c_-, c_+]$ and such that at some points of this interval it tends to infinity as $t \to t_*$, so that the solution $u(x,t)$ no longer makes sense for $t > t_*$. Let us also assume that the asymptotic behavior as $t \to t_*$ of the solution $u(x,t)$ is uniform in $x \in [c_-, c_+]$. This means that as $t \to t_*$ there exists a function $\varphi(t) \to \infty$ for which there exists a finite limit $\lim_{t\to t_*} u(x,t)/\varphi(t)$ which is a bounded function measurable with respect to $x \in [c_-, c_+]$,

$$u(x) = \limsup_{t\to t_*} \frac{u(x,t)}{\varphi(t)} \ge 0,$$

which is not identically zero. It follows from the defining formulas (12) and (13) that, in the case of uniform asymptotics, one has $u(x,t) = u(x)\varphi(t)(1 + o(1))$, where the function $o(t)$ tends to zero as $t \to t_*$ uniformly with respect to $x \in [c_-, c_+]$. The following assertion holds.

Theorem 3.1. *If $u(x,t)$ is a solution of Eq. (2) compactly supported in the interval $[c_-, c_+]$ and has the asymptotic behavior $u(x,t) = u(x)\varphi(t)(1 + o(1)) \to \infty$ as $t \to t_*$ uniformly in $x \in [c_-, c_+]$, then the nonnegative function $u(x)$ is twice differentiable and satisfies the differential equation*

$$(uu_x)_x + \beta u^2 = A^2 u, \quad A = \text{const}, \tag{14}$$

so that $\varphi(t) = \varphi(0)\left(1 - A^2\varphi(0)t\right)^{-1}$ and $t_* = \left[A^2\varphi(0)\right]^{-1}$.

Proof. Since the asymptotic formula can be differentiated with respect to t, it follows from Eq. (2) that

$$\left[\left(\frac{u(x,t)}{\varphi(t)}\left[\frac{u(x,t)}{\varphi(t)}\right]_x\right)_x - \varphi^{-2}(t)\dot\varphi(t)u(x) + \alpha\frac{u(x)}{\varphi(t)}\right](1 + o(1)) = -\beta u^2(x).$$

Based on this, passing to the limit as $t \to t_*$ and taking into account the fact that $u(x) \not\equiv 0$, the right-hand side of the equality is independent of t, and $\varphi(t) \to \infty$, we find that there exist limit values

$$\lim_{t\to t_*}\left(\frac{u(x,t)}{\varphi(t)}\left[\frac{u(x,t)}{\varphi(t)}\right]_x\right)_x \equiv v(x), \qquad \lim_{t\to t_*}\varphi^{-2}(t)\dot\varphi(t) \equiv C = \text{const},$$

which must satisfy the relation $v(x) = Cu(x) - \beta u^2(x)$, while the function $\varphi(t)$ must satisfy the equation $\dot\varphi(t) = C\varphi^2(t)$. It follows from the last equation that $\varphi(t) = \varphi(0)(1 - C\varphi(0)t)^{-1}$. Since $\varphi(t) > 0$ by construction, we have $\varphi(0) > 0$, and for $\varphi(t)$ to tend to infinity it is necessary that $C = A^2 > 0$. In this case, $t_* = \left[A^2\varphi(0)\right]^{-1}$.

Since the function $u(x)$ is bounded and measurable on $[a, b]$ by assumption, we see that the function $v(x)$ must have the same property. By the same pattern, applying Lemma 2.1 to an arbitrary sequence $\langle t_n; n \in \mathbb{N}\rangle$ monotonically tending to t_* as $n \to \infty$ and to the corresponding function sequence

$$\langle y_n(x) = [u(x, t_n)/\varphi(t_n)]\,[u_x(x, t_n)/\varphi(t_n)]\,; n \in \mathbb{N}\rangle,$$

we conclude that the latter sequence uniformly converges to the limit function

$$y(x) = u(x)\lim_{n\to\infty}[u_x(x, t_n)/\varphi(t_n)]$$

and $y'(x) = v(x)$. Moreover, for the point c in the statement of Lemma 2.1 for the functions $y_n(x)$, $n \in \mathbb{N}$, we take $c = c_-$, because $u(c_-, t_n) = 0$ by assumption.

Since the function $y(x)$ is bounded and measurable, it follows that so is the function $y(x)/u(x)$ at the points where $u(x) \neq 0$. Consider the function sequence $\langle [u_x(x,t_n)/\varphi(t_n)]_x; n \in \mathbb{N}\rangle$, which tends to $y(x)/u(x)$ as $n \to \infty$ and, at the same time, satisfies $u(c_-, t_n) = 0$. Since

$$\lim_{n\to\infty} \frac{u_x(x,t_n)}{\varphi(t_n)} = u'(x),$$

we can again apply Lemma 2.1 to this sequence and find that $u'(x) = y(x)/u(x)$.

It follows from the argument in the last two paragraphs that $[u(x)u'(x)]' = v(x)$. Using the equality $v(x) = Cu(x) - \beta u^2(x)$, we obtain formula (14). □

Applying the standard method for reducing the second-order autonomous equation (14) to quadratures, we obtain the following assertion.

Corollary. *The class of all admissible functions uniformly and asymptotically accurately approximating nonnegative solutions $u(x,t)$ of Eq. (2) with compact support $\operatorname{supp} u(x,t) \subset [c_-, c_+]$ and with $u(x,t) \to \infty$ as $t \to t_*$ is nonempty only if $\beta > 0$ and $c_+ - c_- \geq \pi\sqrt{2\beta}$, and it is described by the formula*

$$u(x,t) = \frac{2A^2}{3\beta}\left(1 - A^2\varphi(0)t\right)^{-1}\left(1 + \cos\left(L + (\beta/2)^{1/2}x\right)\right), \tag{15}$$

in which $t_* = \left[A^2\varphi(0)\right]^{-1}$ and the extreme points $x_\pm$ of the solution support must satisfy the conditions $x_- > c_-$ and $x_+ < c_+$, where

$$c_-(\beta/2)^{1/2} + L < \pi\,(2n_- + 1)\,, \quad c_+(\beta/2)^{1/2} + L > \pi\,(2n_+ + 1)\,, \tag{16}$$

and $n_+ > n_-$. Further, it is necessary that $c_+ - c_- \geq \pi(2\beta)^{1/2}$.

Proof. Formula (15) can be obtained by a straightforward computation of the general solution of Eq. (14). The extreme points $x_\pm$ must satisfy the conditions $L + x_\pm(\beta/2)^{1/2} = (2n_\pm + 1)\,\pi$, whence the constraints (16) followiw. For $n_- = 0$ and $n_+ = 1$, we obtain $c_+ - c_- > \pi\sqrt{2\beta}$. □

4. WEAK MODEL SOLUTIONS

The aim of this section is to construct model weak solutions $u_\pm(x,t)$ of Eq. (2) on the basis of which we will find two-sided estimates of the blow-up time t_* of compactly supported solutions satisfying the condition $c_+ - c_- > \pi\sqrt{2\beta}$ with some $\beta > 0$. Here the functions $u_+(x,t)$ and $u_-(x,t)$ give upper and lower bounds, respectively, for the localized solutions $u(x,t)$.

We seek solutions $w(x,t)$ of Eq. (2) in the form

$$w(x,t) = a(t) + b(t)\cos\pi L_*^{-1}\,(x + x_0)\,, \tag{17}$$

where the functions $a(t)$ and $b(t)$ and the parameter L_* are unknown. Substituting this expression into Eq. (2), we arrive at the identity

$$\begin{aligned}\dot a(t) + \dot b(t)\cos\pi L_*^{-1}\,(x+x_0) = \alpha a(t) + \beta a^2(t) + \pi^2 L_*^{-2} b^2(t) \\ + \left(\alpha b(t) + 2\beta a(t)b(t) - \kappa\pi^2 L_*^{-2} a(t)b(t)\right)\cos\pi L_*^{-1}\,(x+x_0) \\ + \left(\beta - 2\pi^2 L_*^{-2}\right) b^2(t)\cos^2\pi L_*^{-1}\,(x+x_0)\,.\end{aligned}$$

The harmonic balance with respect to the variable x necessarily leads to the equality $L_* = \pi(2/\beta)^{1/2}$ and to a conservative system of ordinary differential equations for the functions $a(t)$ and $b(t)$,

$$\dot a = \alpha a + \beta\left(a^2 + \frac{b^2}{2}\right), \qquad \dot b = \alpha b + \frac{3}{2}\beta ab. \tag{18}$$

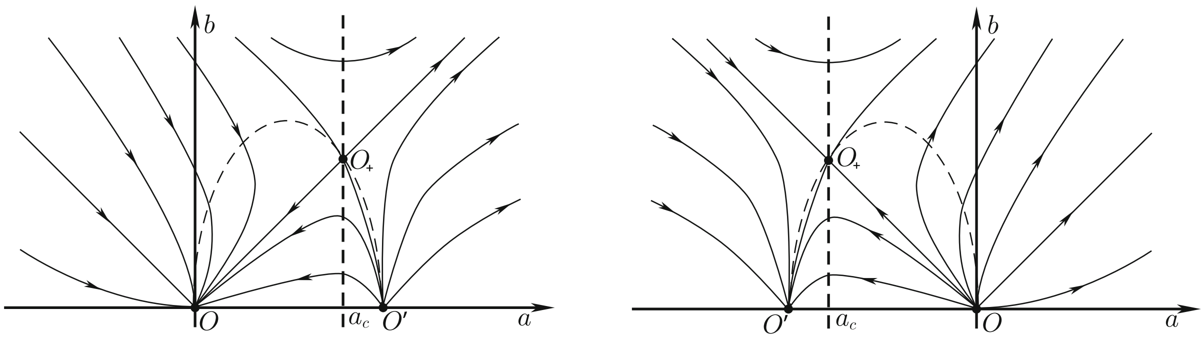

Fig. 1. Phase plane of the dynamical system $\langle a(t), b(t)\rangle$.

Thus, the family of model solutions $w(x,t)$ is completely described by the initial data $a(0)$ and $b(0)$ of the solutions of system (18) and the coordinate x_0. The general view of the phase plane of the system depending on the sign of α is presented in Fig. 1.

Model solutions for estimating exact compactly supported nonnegative solutions $u(x,t)$ of Eq. (2) are constructed based on Theorem 2.2. First, note that the derivative $w_x(x,t)$ is bounded; this allows using the function $w(x,t)$ to construct weak solutions of the class $\mathfrak{R}$. Let us introduce compactly supported model solutions

$$u^{(j)}(x,t) = a_j(t) + b_j(t)\cos \pi L_*^{-1}\bigl(x + x_0^{(j)}\bigr), \qquad j \in \{1,2\},$$

according to formula (9), which are continuous nonnegative functions with supports $\bigl[x_-^{(j)}(t), x_+^{(j)}(t)\bigr]$. The additional conditions $\bigl|a_j(t)\bigr| \le b_j(t)$ and $b_j(t) > 0$, $j \in \{1,2\}$, are imposed on the pairs of coefficients to ensure that there exists a range of the coordinate x where, by construction, the model solutions are positive and their continuity can be guaranteed. If these inequalities are satisfied, then $u^{(j)}(x,t) > 0$ at the point $-x_0^{(j)}$, $u^{(1,2)}\bigl(-x_0^{(j)}\bigr) = a_j(t) + b_j(t) > 0$, and the graphs of the functions $w(x,t)$ with the pairs of coefficients $\bigl\langle a_j(t), b_j(t)\bigr\rangle$ have points of intersection with the level $u = 0$; i.e., there exist solutions of the equation

$$a_j(t) + b_j(t)\cos \pi L_*^{-1}\bigl(x + x_0^{(j)}\bigr) = 0,$$

in particular, for $t = 0$. In this case, the boundaries of the supports are defined as the solutions of the equation

$$a_j(t) + b_j(t)\cos \pi L_*^{-1}\left(x_\pm^{(j)} + x_0^{(j)}\right) = 0, \qquad j \in \{1,2\},$$

on the part of the phase plane $\langle a, b\rangle$ defined by the inequality $b \ge |a|$.

We assume that the solution $u(x,t)$ has compact support whose initial size r is less than L_*. Then there exist a point $x_0^{(1)}$ and parameter values $a_1(0)$, $b_1 > 0$, and $|a_1| \le b_1$ for which the inequality $u^{(1)}(x,0) \le u(x,0)$ holds. One can choose admissible values of the parameters $x_0^{(1)}$, $a_1(0)$, and $b_1(0)$ so as to ensure that the function $u^{(1)}(x,0)$ approximates the function $u(x,0)$ from below in the most optimal way.

In exactly the same way, we construct a model solution $u^{(2)}(x,t)$ with parameters $a_2(0)$, $b_2(0)$, and $x_0^{(2)}$ in such a way that the inequality $u^{(2)}(x,0) \ge u(x,0)$ holds but the size of its support does not exceed L_*. By choosing the parameter values $a_j(0)$, $b_j(0)$, and $x_0^{(j)}$, we have thereby fixed the solutions $\langle a_j(t), b_j(t)\rangle$, $j \in \{1,2\}$, of the dynamical system $\langle a(t), b(t)\rangle$ with these values as the initial data. As a result, by Theorem 2.2, the exact solution of Eq. (2) to be approximated satisfies the inequalities $u^{(1)}(x,t) \le u(x,t) \le u^{(2)}(x,t)$.

Let this solution have a blow-up mode with blow-up time $t_* < \infty$. Note that for $\alpha \geq 0$ any model solution has a blow-up mode. If $\alpha < 0$ and in this case the parameters $\langle a_-(0), b_-(0)\rangle$ can be chosen so that the model solution $u^{(1)}(x,t)$ also has a blow-up mode with some blow-up time $t_*^{(1)}$, then, owing to the indicated inequality, we have $t_*^{(1)} \geq t_*$. In addition, the model solution $u_2(x,t)$ has a blow-up mode with blow-up time $t_*^{(2)}$ satisfying the inequality $t_*^{(2)} \leq t_*$.

In view of the inequality $u^{(1)}(x,t) \leq u(x,t) \leq u^{(2)}(x,t)$, the time-dependent support size $r(t)$ of the solution $u(x,t)$ is also subject to the inequalities $r_1(t) \leq r(t) \leq r_+(t) \leq L_*$, where $r_j(t)$ are the sizes of the supports of the model solutions, $j \in \{1,2\}$. Therefore, $r(t_*) \leq L_*$. Taking into account Theorem 3.1, we can state that the limit sizes of all supports coincide and are equal to L_*.

Thus, we now can estimate the blow-up time and the size of the localization domain of an arbitrary solution of Eq. (2) with compact support whose size does not exceed L_*.

5. ESTIMATES OF THE BLOW-UP TIME

Let us analyze the behavior of trajectories of the system on the phase plane $\langle a, b\rangle$. The system has singular points whose coordinates $\langle a, b\rangle$ are solutions of the system of equations $\alpha a + \beta\left(a^2 + b^2/2\right) = 0$, $(\alpha + 3\beta a/2)b = 0$.

The point of intersection $\langle a_c, 0\rangle, a_c = -2\alpha/3\beta$ of the straight lines $b = 0$ and $a = a_c$ lies inside the ellipse defined by the first equation, because the ellipse meets the a-axis at the points 0 and $-\alpha/\beta$, and therefore, the system has four singular points $O = \langle 0, 0\rangle$, $O' = \langle -\alpha/\beta, 0\rangle$, and $O_\pm = \langle a_c, \pm a_c\rangle$. The matrix of the system linearized at an arbitrary point $\langle a, b\rangle$ has the form

$$\begin{pmatrix} \alpha + 2\beta a & \beta b \\ 3\beta b/2 & \alpha + 3\beta a/2 \end{pmatrix}.$$

It is diagonal at the points O and O', where $b = 0$, and hence has the pairs $\langle \alpha + 2\beta a, \alpha + 3\beta a/2\rangle$ of eigenvalues. They are equal to $\langle \alpha, \alpha\rangle$ at the point O and$\langle -\alpha, -\alpha/2\rangle$ at the point O'. Thus, the points O and O' are nodes whose stability is determined by the sign of α. At the points $O_\pm$, the matrix is nondiagonal,

$$\begin{pmatrix} -\alpha/3 & 2\alpha/3 \\ \alpha & 0 \end{pmatrix}.$$

The eigenvalues of the matrix at these points are $\langle \alpha, -2\alpha/3\rangle$. Since they have opposite signs, it follows that the points $O_\pm$ are saddle ones, in which the orientation of the saddle (the set of directions on the separatrix arcs issuing from these points) is also determined by the sign of α as shown in Figs. 1, a and 1, b. The turning points of a trajectory of the system on the phase plane in the direction of the b-axis can only lie on the straight line $a = a_c$, while the straight line $b = 0$ can only be an asymptote of the trajectories. The turning points in the direction of the a-axis lie on the ellipse. The system has solutions with the trajectories $b = \pm a$, since after the substitutions $b = \pm a$ both equations in system (18) coincide. Hence the trajectories passing through any point $\langle a, b\rangle$ with $b > |a|$ are completely contained in the quadrant defined by this inequality. The rays $a = b$, $a \geq 0$, and $a = -b$, $a \leq 0$, are trajectories of the system. They contain the singular point O with $a = 0$, which is an unstable node for $\alpha > 0$. On the contrary, for $\alpha < 0$ this singular point is a stable node with $a = 0$. In addition, for $\alpha < 0$ the ray $a = b$ contains the saddle point O_+ with $a = 2|\alpha|/3\beta = -a_c$, and for $\alpha > 0$ the saddle point O_+ is contained by the ray $a = -b$. This analysis shows that the vector field of the system on the plane $\langle a, b\rangle$ has the form shown in Fig. 1, a for $\alpha < 0$ and Fig. 1, b for $\alpha > 0$.

In the general case, system (18) can be integrated by the substitution $a(t) = e^{\alpha t}A(t)$, $b(t) = e^{\alpha t}B(t)$, which reduces it to the system

$$\dot{A} = \beta e^{\alpha t}\left(A^2 + \frac{1}{2}B^2\right), \quad \dot{B} = \frac{3}{2}\beta e^{\alpha t}AB \tag{19}$$

with scale-invariant trajectories. The last property allows one to determine the trajectories of system (18) by reduction to the homogeneous equation

$$\frac{dA}{dB} = \frac{2A^2 + B^2}{3AB}. \tag{20}$$

To construct model solutions $u_j(x,t)$, $j \in \{1,2\}$, one needs to determine solutions $\langle a(t), b(t)\rangle$ of system (18) satisfying the condition $b(t) > |a(t)|$. Then the corresponding pairs of functions $\langle A(t), B(t)\rangle$ satisfy the condition $B(t) > |A(t)|$ as well. The trajectories defined by Eq. (20) are independent of the constants α and β. For $A > 0$, it follows from (20) that A increases with B; i.e., a trajectory starting in the right half-plane remains in it. If $A < 0$, then it follows from (20) that B is a decreasing function of A. If the trajectory is in the quadrant $\{\langle A, B\rangle : B > |A|\}$, then it cannot meet the line $B = 0$. It can only reach the point $\langle 0, 0\rangle$ or cross the line $A = 0$ at the right angle (since $dB/dA = 0$) and go to the right half-plane. Let us show that the second case holds.

Equation (20) can be integrated by the standard substitution $w(B) = A/B$, so that for the function $w(B)$ we obtain the equation $B(dw/dB) = \left(1 - w^2\right)/3w$. By separating the variables, we obtain the family of solutions

$$\left|1 - w^2\right| = (b_0/B)^{2/3} \tag{21}$$

with an arbitrary constant $b_0 > 0$.

The point $\langle 0, b_0\rangle$ at which $w(B) = 0$ is the point where the trajectory meets the straight line $A = 0$. The value $b_0 = 0$ corresponds to the degenerate case of the trajectory consisting of the half-lines $B = -A$ for $A < 0$ and $B = A$ for $A > 0$. For $w(B) < 1$, we have the following formula for the trajectories in the quadrant $\{\langle A, B\rangle : B > |A|\}$ of the plane $\langle A, B\rangle$:

$$A = \pm B\left(1 - (b_0/B)^{2/3}\right)^{1/2}, \tag{22}$$

where the signs $(+)$ and $(-)$ correspond to the parts of the trajectory in the right and left half-planes, respectively. It follows from (22) that asymptotically $B \sim \pm A$ as $B \to \infty$.

Along with the trajectories (22), there exist trajectories described by formula (21) for $w^2 > 1$,

$$A = \pm B\left(1 + (b_0/B)^{2/3}\right)^{1/2}.$$

However, $|A| > B$ and $B > 0$ for such trajectories, and therefore, they are located outside the domain $\{\langle A, B\rangle : B > |A|\}$; this is of interest when constructing the model functions $u^{(j)}(x,t)$.

The time dependence of the solutions of interest to us can be found by substituting formula (22) into the second equation in system (19),

$$\dot{B} = \pm\frac{3}{2}\beta e^{\alpha t}\cdot B^2\left(1 - (b_0/B)^{2/3}\right)^{1/2}.$$

As a result, we obtain an equation implicitly determining the function $B(t)$,

$$\begin{aligned} \pm b_0^{-1}\int_{b_0/B}^{b_0/b(0)} \frac{d\xi}{\sqrt{1-\xi^{2/3}}} = \frac{3\beta}{2\alpha}\left(e^{\alpha t} - 1\right)\left(\arcsin\left(b_0/b(0)\right)^{1/3} - \arcsin\left(b_0/B\right)^{1/3}\right) \\ + \left((b_0/b(0))^{1/3}\sqrt{1-(b_0/b(0))^{2/3}} - (b_0/B)^{1/3}\sqrt{1-(b_0/B)^{2/3}}\right) = \pm\left(\beta b_0/\alpha\right)\left(e^{\alpha t} - 1\right). \end{aligned} \tag{23}$$

Expressing the parameter b_0 in this equation via the initial data $a(0)$, $b(0)$, and

$$(b_0/b(0))^{1/3} = [1 - (a(0)/b(0))^2]^{1/2},$$

we find the following equation for the implicit function $B(t)$:

$$\arcsin\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right]^{1/2} - \arcsin\left(\frac{b(0)}{B}\right)^{1/3}\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right]^{1/2} + \frac{|a(0)|}{b(0)}\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right]^{1/2}$$
$$-\left(\frac{b(0)}{B}\right)^{1/3}\left[\left(1-\left(\frac{a(0)}{b(0)}\right)^2\right)\left(1-\left(\frac{b(0)}{B}\right)^{2/3}\left(1-\left(\frac{a(0)}{b(0)}\right)^2\right)\right)\right]^{1/2} \tag{24}$$
$$= \pm(\beta b(0)/\alpha)\left(e^{\alpha t}-1\right)\left(1-\left(\frac{a(0)}{b(0)}\right)^2\right)^{3/2}.$$

Let $a(0) > 0$. Then $A(0) > 0$, and it follows from the first equation in (19) that $\dot{A}(t) > 0$, i.e., $A(t) > 0$ at subsequent time; that is, the motion occurs along the part of the trajectory in the right half-plane of the plane $\langle A, B\rangle$. Then we need to select the sign $(+)$ in (24). In this case, letting $B \to \infty$ on the left-hand side in the formula and setting $t = t_*$ on the right-hand side, we find a formula for the blow-up time,

$$t_* = \alpha^{-1}\ln\left(1 + \frac{\alpha\arcsin\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right]^{1/2} + \frac{a(0)}{b(0)}\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right]^{1/2}}{\beta b(0)\left(1-\left(\frac{a(0)}{b(0)}\right)^2\right)^{3/2}}\right), \tag{25}$$

expressing it via the initial data. The expressions for $t_*^{(j)}$, $j \in \{1, 2\}$, in the above and subsequent formulas are obtained by substituting $a(0) \Rightarrow a_j(0)$ and $b(0) \Rightarrow b_j(0)$.

For $\alpha > 0$, the argument of the logarithm is obviously positive, and the blow-up time is well defined for all initial data. For $\alpha < 0$, for the blow-up time to exist, the argument of the logarithm must be positive. This condition is expressed by the inequality

$$\arcsin\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right]^{1/2} + \frac{a(0)}{b(0)}\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right]^{1/2} < \frac{\beta b(0)}{|\alpha|}\left(1-\left(\frac{a(0)}{b(0)}\right)^2\right)^{3/2}.$$

Equating the argument of the logarithm with zero yields the equation of the separatrix in Fig. 1, b, which, on the plane $\langle a, b\rangle$, separates the domains with initial data that lead to a blow-up mode and the domain in which the motion of system (18) is bounded. For $a(0) > 0$, this equation has the form

$$\arcsin\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right] + \frac{a(0)}{b(0)}\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right] = \frac{\beta b(0)}{|\alpha|}\left(1-\left(\frac{a(0)}{b(0)}\right)^2\right)^{3/2}.$$

We introduce the following function C_+ of the initial data:

$$C_+ = \left(1-\left(\frac{a(0)}{b(0)}\right)^2\right)^{-3/2}\left[\arcsin\sqrt{1-\left(\frac{a(0)}{b(0)}\right)^2} + \frac{a(0)}{b(0)}\sqrt{1-\left(\frac{a(0)}{b(0)}\right)^2}\right].$$

Then formula (25) takes the form

$$t_* = \alpha^{-1}\ln\left(1 + \frac{\alpha}{\beta b(0)}C_+\right)$$

In particular, $t_* = C_+/\beta b(0)$ as $\alpha \to 0$.

Now let $a(0) < 0$. In this case, each trajectory of system (18) consists of two parts lying in the left and right half-planes on the phase plane $\langle a, b\rangle$. Consider a given trajectory for t so large that we can assume that the trajectory has already crossed the straight line $a = 0$. If we set $B = b_0$ in formula (23) with the sign $(-)$, then for the left part of the trajectory we find

$$\arcsin\,(b_0/b(0))^{1/3} - \pi/2 + (b_0/b(0))^{1/3}\sqrt{1-(b_0/b(0))^{2/3}} = -\left(\beta b_0/\alpha\right)\left(e^{\alpha t_0}-1\right), \qquad (26)$$

where t_0 is the time of intersection of the straight line $a = 0$. On the contrary, having replaced the initial condition $b(0)$ with b_0 in (23) with the sign $(+)$, we obtain

$$\left(\pi/2 - \arcsin\,(b_0/B)^{1/3}\right) - (b_0/B)^{1/3}\sqrt{1-(b_0/B)^{2/3}} = \left(\beta b_0/\alpha\right)\left(e^{\alpha t}-e^{\alpha t_0}\right). \qquad (27)$$

After passing to the limit as $B \to \infty$ and $t \to t_*$ in this formula, we obtain an equation for the blow-up time in the form $\pi/2 = (\beta b_0/\alpha)\left(e^{\alpha t_*} - e^{\alpha t_0}\right)$. From formulas (26) and (27), expressing the parameter b_0 via the initial data and taking into account the fact that $a(0) < 0$, we find

$$\frac{\beta b(0)}{\alpha}\left(e^{\alpha t_*}-1\right)\left(1-\left(\frac{a(0)}{b(0)}\right)^2\right)^{3/2} = \pi - \arcsin\sqrt{1-\left(\frac{a(0)}{b(0)}\right)^2} + \frac{a(0)}{b(0)}\sqrt{1-\left(\frac{a(0)}{b(0)}\right)^2}. \qquad (28)$$

Introducing the function

$$C_- = \left(1-\left(\frac{a(0)}{b(0)}\right)^2\right)^{-3/2}\left[\pi - \arcsin\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right]^{1/2} + \frac{a(0)}{b(0)}\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right]^{1/2}\right],$$

we represent the formula for the blow-up time in the form

$$t_* = \alpha^{-1}\ln\left(1+\frac{\alpha}{\beta b(0)}C_\infty\right).$$

This function is positive, because the last term in square brackets does not exceed 1 in absolute value. Therefore, for $\alpha > 0$ there always exists a finite blow-up time. If $\alpha < 0$, then for the existence of a time t_* it is necessary that the argument of the logarithm be positive; i.e., $C_- < \beta b_0/|\alpha|$. When passing to the limit as $\alpha \to 0$, we obtain the similar formula $t_* = C_-/\beta b(0)$. The equation for the separatrix in the domain $a(0) < 0$ has the form

$$\frac{\beta b(0)}{|\alpha|}\left(1-\left(\frac{a(0)}{b(0)}\right)^2\right)^{3/2} = \pi - \arcsin\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right]^{1/2} + \frac{a(0)}{b(0)}\left[1-\left(\frac{a(0)}{b(0)}\right)^2\right]^{1/2}.$$

6. CONCLUSIONS

As was already mentioned in the introduction, the main focus in the study of differential equations whose solutions may experience a blow-up mode is on determining the conditions for the occurrence of a blow-up. Nevertheless, it is desirable to obtain more detailed information about the solutions of initial—boundary value problems for such equations. It should be noted that in applications in the fields where these equations are used to describe evolution instabilities with blow-up modes, namely, in plasma physics [4] as well as in the physics of semiconductor materials [19], [20], detailed information about the initial states is usually missing. Order-of-magnitude data are used for rough characterization of the initial conditions $u(x, 0)$. In such a situation, it is important to be able to solve initial—boundary value problems, at least in the form of asymptotic expansions, with random initial conditions that take values in a fairly large set of possible random implementations. In this case, of course, one cannot restrict oneself to solutions with compact supports of limited size. It is necessary to study solutions in the set of functions that sufficiently rapidly tend to some positive constants on the boundaries of the admissible domain. Finally, it is natural to generalize the results of the analysis of solutions of evolution equations similar to (2) in the multidimensional case, especially for dimensions 2 and 3; this is associated with the formulation of problems of mathematical physics related to specific applications.

In the present paper, we find two-sided estimates functionally dependent on the initial data for the blow-up time for solutions with compact support having the size of the so-called fundamental length [16]. The assumption that the solution support is compact has permitted us to exclude the influence of boundary conditions from consideration and focus on establishing the dependence of the blow-up time on the initial data. The results obtained can be considered as an important step in the solution of the stated general problems of the theory of equations with a blow-up mode.

In conclusion, note that the case of $\beta > 0$ was analyzed in the present paper. However, the opposite case may also be of interest. In this case, it is not a blow-up mode that appears but an evolution in which, in finite time, the solutions turn into the zero solution. The method we propose for estimating the solution vanishing time remains unchanged.

FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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