mathematics

MDPI

Article

On Hyperbolic Equations with a Translation Operator in Lowest Derivatives

Vladimir Vasilyev [1,*] and Natalya Zaitseva [2]

[1] Center of Applied Mathematics, Belgorod State National Research University, Pobedy St. 85, Belgorod 308015, Russia
[2] Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University, GSP-1, Leninskie Gory, Moscow 119991, Russia; zaitseva@cs.msu.ru
* Correspondence: vasilyev_v@bsu.edu.ru; Tel.: +7-472-230-1300

Abstract: In the half-plane, a solution to a two-dimensional hyperbolic equation with a translation operator in the lowest derivative with respect to a spatial variable varying along the entire real axis is constructed in an explicit form. It is proven that the solutions obtained are classical if the real part of the symbol of a differential-difference operator in the equation is positive.

Keywords: hyperbolic equation; differential-difference equation; translation operator; classical solution

MSC: 35L10

check for updates

Citation: Vasilyev, V.; Zaitseva, N. On Hyperbolic Equations with a Translation Operator in Lowest Derivatives. *Mathematics* **2024**, *12*, 1896. https://doi.org/10.3390/math12121896

Academic Editor: Alberto Ferrero

Received: 19 May 2024
Revised: 10 June 2024
Accepted: 17 June 2024
Published: 19 June 2024

1. Introduction

A differential equation containing differential operators and translation operators is called a differential-difference equation (see [1]). Ordinary differential-difference equations have been studied extensively and for a long time; see books [1–3] and the bibliographies in them. The first study of an ordinary differential equation with a translation operator (or with a deviating argument) appears to be found in J. Bernoulli (1728). He considered the problem of a weightless stretched string of finite length along which equal and equidistant masses are distributed, which led to the equation $y'(t) = y(t-1)$.

Partial differential equations with shift operators are generalizations of classical equations of mathematical physics. For these, many methods known for classical equations turn out to be inapplicable: for example, the maximum principle for elliptic and parabolic problems. Such equations are widely used in various applications in areas such as the mechanics of solid body deformations, relativistic electrodynamics, vortex formation processes, plasma-related problems, modeling of crystal lattice vibrations, nonlinear optics, neural networks, and many others. It turns out that when solving problems of differential-difference equations, qualitatively new effects arise in the solutions that do not occur in the classical case.

There are a sufficient number of works devoted to the study of boundary value problems for elliptic differential-difference equations; see for works [4–7]. Problems in both bounded and unbounded domains for parabolic differential-difference equations have been studied to a lesser extent. Let us note article [8].

And hyperbolic differential-difference equations have been studied to a much lesser extent. Thus, in [9,10], shift operators act on a temporary variable. And in articles [11–15], hyperbolic equations are considered in which shift operators act on spatial variables. Even in the study of ordinary differential-difference equations, it is noted that the use of integral transformations is one of the most effective methods for constructing their solutions. In the previously mentioned papers, the authors used the Fourier transform, since spatial variables take all real values. Note that in Fourier images the translation operator is a multiplier; thus, an exponential multiplier appears with the value of the translation in the exponent, which does not interfere with finding a general solution to the ordinary differential equation that arises after the Fourier transform. We also note that in the indicated articles [11–15], the translation operator with respect to a spatial variable in a two-dimensional equation or with respect to spatial variables in a multidimensional equation is present either in the highest derivative or in the free term. In this work, the lowest derivative appears in the equation for the first time, which already causes some difficulties in its study, in addition to having the translation operator in it. The authors plan to study initial problems for such an equation and foresee significant difficulties in justifying their solvability. Such tasks are new and previously unexplored.

The purpose of this article is to construct solutions in the half-plane of a hyperbolic equation with a translation operator with respect to a spatial variable in the lowest derivative, and to determine the conditions under which the constructed solution will turn out to be classical.

2. Statement of the Problem: Constructing Solutions to the Equation

We study the hyperbolic equation

$$u_{tt}(x,t) - a^2 u_{xx}(x,t) + b\, u_x(x-h,t) + c\, u(x,t) = 0, \quad x \in (-\infty,+\infty), \quad t > 0, \tag{1}$$

where $a > 0$ (for physical reasons), $b \neq 0$, $c \neq 0$, and $h \neq 0$ are given real numbers.

To find solutions to Equation (1), we use an operational scheme according to which we apply the Fourier transform to Equation (1) according to the formula

$$\widehat{u}(\xi,t) := F_x[u](\xi,t) = \int_{-\infty}^{+\infty} u(x,t) e^{i\xi x}\, dx.$$

Taking into account the formulas

$$F_x\left[\frac{\partial^{\alpha+\beta} u(x,t)}{\partial x^\alpha \partial t^\beta}\right] = (-i\xi)^\alpha \frac{\partial^\beta (F_x[u(x,t)])}{\partial t^\beta} = (-i\xi)^\alpha \frac{\partial^\beta \widehat{u}(\xi,t)}{\partial t^\beta}, \quad \alpha,\beta = 0,1,2,\ldots,$$

and

$$F_x[u(x-h,t)] = e^{ih\xi} F_x[u(x,t)] = e^{ih\xi} \widehat{u}(\xi,t),$$

we obtain the ordinary differential equation

$$\frac{d^2 \widehat{u}(\xi,t)}{dt^2} + \left(a^2\xi^2 - ib\xi e^{ih\xi} + c\right)\widehat{u}(\xi,t) = 0, \quad \xi \in (-\infty,+\infty), \tag{2}$$

for finding function $\widehat{u}(\xi,t)$.

Let us introduce the functions

$$\rho(\xi) := \left((a^2\xi^2 + b\xi \sin(h\xi) + c)^2 + (b\xi \cos(h\xi))^2\right)^{1/4}, \tag{3}$$

and

$$\varphi(\xi) := \frac{1}{2} \arctan \frac{b\xi \cos(h\xi)}{a^2\xi^2 + b\xi \sin(h\xi) + c}. \tag{4}$$

Remark 1. *Function ρ given by (3) is defined correctly for any real values a, b, c, and ξ.*

Then, the roots of the characteristic equation corresponding to Equation (2), taking into account notations (3) and (4), will have the form

$$k_{1,2} = \pm\sqrt{-(a^2\xi^2 - ib\xi e^{ih\xi} + c)} = \pm i\sqrt{a^2\xi^2 - ib\xi e^{ih\xi} + c}$$

$$= \pm i\sqrt{a^2\xi^2 + b\xi\sin(h\xi) + c - b\xi\cos(h\xi)} = \pm i\,\rho(\xi)e^{-i\varphi(\xi)},$$

and the general solution to Equation (2) will be determined by the formula

$$\widehat{u}(\xi,t) = C_1(\xi)\cos\left(\rho(\xi)e^{-i\varphi(\xi)}t\right) + C_2(\xi)\sin\left(\rho(\xi)e^{-i\varphi(\xi)}t\right),$$

where $C_1(\xi)$ and $C_2(\xi)$ are arbitrary constants depending on parameter $\xi \in (-\infty, +\infty)$. Let us choose the values of the constants $C_1(\xi) = 0$ and $C_2(\xi) = e^{i\varphi(\xi)}/\rho(\xi)$ and write down the final form of the solution to Equation (2):

$$\widehat{u}(\xi,t) = \frac{\sin\left(\rho(\xi)e^{-i\varphi(\xi)}t\right)}{\rho(\xi)e^{-i\varphi(\xi)}}. \tag{5}$$

Let us now apply the inverse Fourier transform to function (5) according to the formula

$$u(x,t) = F_\xi^{-1}[\widehat{u}](x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{u}(\xi,t)e^{-ix\xi}\,d\xi.$$

And as a result we will have the expression

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\sin\left(\rho(\xi)e^{-i\varphi(\xi)}t\right)}{\rho(\xi)} e^{i(\varphi(\xi)-x\xi)}\,d\xi.$$

Taking into account the evenness of function ρ given by (3): $\rho(-\xi) = \rho(\xi)$, and the oddness of function φ given by (4): $\varphi(-\xi) = -\varphi(\xi)$, we transform the last expression to the form

$$u(x,t) = \frac{1}{2\pi}\left[\int_{-\infty}^{0} \frac{\sin\left(\rho(\xi)e^{-i\varphi(\xi)}t\right)}{\rho(\xi)} e^{i(\varphi(\xi)-x\xi)}\,d\xi + \int_{0}^{+\infty} \frac{\sin\left(\rho(\xi)e^{-i\varphi(\xi)}t\right)}{\rho(\xi)} e^{i(\varphi(\xi)-x\xi)}\,d\xi\right]$$

$$= \frac{1}{2\pi}\left[\int_{0}^{+\infty} \frac{\sin\left(\rho(\xi)e^{i\varphi(\xi)}t\right)}{\rho(\xi)} e^{-i(\varphi(\xi)-x\xi)}\,d\xi + \int_{0}^{+\infty} \frac{\sin\left(\rho(\xi)e^{-i\varphi(\xi)}t\right)}{\rho(\xi)} e^{i(\varphi(\xi)-x\xi)}\,d\xi\right]$$

$$= \frac{1}{2\pi}\int_{0}^{+\infty} \frac{1}{\rho(\xi)}\Big[\sin(\rho(\xi)t\cos\varphi(\xi) + \varphi(\xi) - x\xi)e^{\rho(\xi)t\sin\varphi(\xi)}$$

$$+ \sin(\rho(\xi)t\cos\varphi(\xi) - \varphi(\xi) + x\xi)e^{-\rho(\xi)t\sin\varphi(\xi)}\Big]\,d\xi. \tag{6}$$

Remark 2. *For values $b = 0$ and $c = 0$ in Equation* (1)*, from formula* (6) *we obtain the fundamental solution to the wave operator.*

Indeed, if we put the values of constants $b = 0$ and $c = 0$ in Equation (1), then from formula (6) we obtain

$$u(x,t) = \frac{1}{2\pi a}\int_{0}^{+\infty} \left[\frac{\sin((at-x)\xi)}{\xi} + \frac{\sin((at+x)\xi)}{\xi}\right]d\xi.$$

Taking into account the well-known formula

$$\int_{0}^{+\infty} \frac{\sin(\alpha\xi)}{\xi}\,d\xi = \begin{cases} \pi/2, & \alpha > 0, \\ 0, & \alpha = 0, \\ -\pi/2, & \alpha < 0, \end{cases}$$

we have a function

$$u(x,t) = \frac{1}{2a}\theta(at - |x|),$$

which is a fundamental solution of the wave operator $\partial^2/\partial t^2 - a^2\partial^2/\partial x^2$. Here, θ is the Heaviside function.

3. Main Results

Based on the obtained function (6), we prove the following statement.

Theorem 1. *If the condition*

$$a^2\xi^2 + b\xi\sin(h\xi) + c > 0 \tag{7}$$

is satisfied for all $\xi \in (-\infty, +\infty)$*, the function*

$$\begin{aligned} G(x,t;\xi) := {} & \sin(\rho(\xi)t\cos\varphi(\xi) + \varphi(\xi) - x\xi)e^{\rho(\xi)t\sin\varphi(\xi)} \\ & + \sin(\rho(\xi)t\cos\varphi(\xi) - \varphi(\xi) + x\xi)e^{-\rho(\xi)t\sin\varphi(\xi)} \end{aligned} \tag{8}$$

satisfies Equation (1) for all values $\xi \in (-\infty, +\infty)$ *in the classical sense.*

Proof. Let us calculate the derivatives of function (8):

$$\begin{aligned} G_{tt}(x,t;\xi) = {} & -\rho^2(\xi)\sin(\rho(\xi)t\cos\varphi(\xi) - \varphi(\xi) - x\xi)e^{\rho(\xi)t\sin\varphi(\xi)} \\ & -\rho^2(\xi)\sin(\rho(\xi)t\cos\varphi(\xi) + \varphi(\xi) + x\xi)e^{-\rho(\xi)t\sin\varphi(\xi)}; \end{aligned} \tag{9}$$

$$\begin{aligned} G_x(x,t;\xi) = {} & -\xi\cos(\rho(\xi)t\cos\varphi(\xi) + \varphi(\xi) - x\xi)e^{\rho(\xi)t\sin\varphi(\xi)} \\ & +\xi\cos(\rho(\xi)t\cos\varphi(\xi) - \varphi(\xi) + x\xi)e^{-\rho(\xi)t\sin\varphi(\xi)}, \end{aligned}$$

then

$$\begin{aligned} G_x(x-h,t;\xi) = {} & -\xi\cos(\rho(\xi)t\cos\varphi(\xi) + \varphi(\xi) - x\xi + h\xi)e^{\rho(\xi)t\sin\varphi(\xi)} \\ & +\xi\cos(\rho(\xi)t\cos\varphi(\xi) - \varphi(\xi) + x\xi - h\xi)e^{-\rho(\xi)t\sin\varphi(\xi)}; \end{aligned} \tag{10}$$

$$\begin{aligned} G_{xx}(x,t;\xi) = {} & -\xi^2\sin(\rho(\xi)t\cos\varphi(\xi) + \varphi(\xi) - x\xi)e^{\rho(\xi)t\sin\varphi(\xi)} \\ & -\xi^2\sin(\rho(\xi)t\cos\varphi(\xi) - \varphi(\xi) + x\xi)e^{-\rho(\xi)t\sin\varphi(\xi)}. \end{aligned} \tag{11}$$

Substitute functions (8)–(11) into Equation (1) and obtain

$$G_1(x,t;\xi)e^{\rho(\xi)t\sin\varphi(\xi)} + G_2(x,t;\xi)e^{-\rho(\xi)t\sin\varphi(\xi)} = 0, \tag{12}$$

where

$$\begin{aligned} G_1(x,t;\xi) := {} & \rho^2(\xi)\sin(\rho(\xi)t\cos\varphi(\xi) - \varphi(\xi) - x\xi) \\ & -(a^2\xi^2 + c)\sin(\rho(\xi)t\cos\varphi(\xi) + \varphi(\xi) - x\xi) \\ & +b\xi\cos(\rho(\xi)t\cos\varphi(\xi) + \varphi(\xi) - x\xi + h\xi), \end{aligned} \tag{13}$$

and

$$\begin{aligned} G_2(x,t;\xi) := {} & \rho^2(\xi)\sin(\rho(\xi)t\cos\varphi(\xi) + \varphi(\xi) + x\xi) \\ & -(a^2\xi^2 + c)\sin(\rho(\xi)t\cos\varphi(\xi) - \varphi(\xi) + x\xi) \end{aligned}$$

$$-b\xi\cos\left(\rho(\xi)t\cos\varphi(\xi)-\varphi(\xi)+x\xi-h\xi\right). \tag{14}$$

Let us check the validity of the equality

$$\rho^2(\xi)\sin\left(\rho(\xi)t\cos\varphi(\xi)-\varphi(\xi)-x\xi\right)$$
$$=(a^2\xi^2+c)\sin\left(\rho(\xi)t\cos\varphi(\xi)+\varphi(\xi)-x\xi\right)$$
$$-b\xi\cos\left(\rho(\xi)t\cos\varphi(\xi)+\varphi(\xi)-x\xi+h\xi\right). \tag{15}$$

For convenience in calculations, we introduce the notation

$$\rho(\xi)t\cos\varphi(\xi)-\varphi(\xi)-x\xi=:\alpha=\alpha(x,t;\xi).$$

Now, taking into account the introduced notation α, we transform the right-hand side of equality (15) and obtain

$$(a^2\xi^2+c)\sin\left(\alpha+2\varphi(\xi)\right)-b\xi\cos\left(\alpha+(2\varphi(\xi)+h\xi)\right)$$
$$=(a^2\xi^2+c)[\sin\alpha\cos\left(2\varphi(\xi)\right)+\cos\alpha\sin\left(2\varphi(\xi)\right)]-b\xi[\cos\alpha\cos\left(2\varphi(\xi)\right)\cos\left(h\xi\right)$$
$$-\cos\alpha\sin\left(2\varphi(\xi)\right)\sin\left(h\xi\right)-\sin\alpha\sin\left(2\varphi(\xi)\right)\cos\left(h\xi\right)-\sin\alpha\cos\left(2\varphi(\xi)\right)\sin\left(h\xi\right)]$$
$$=(a^2\xi^2+b\xi\sin\left(h\xi\right)+c)\cos\left(2\varphi(\xi)\right)\sin\alpha$$
$$+(a^2\xi^2+b\xi\sin\left(h\xi\right)+c)\sin\left(2\varphi(\xi)\right)\cos\alpha$$
$$-b\xi\cos\left(h\xi\right)\cos\left(2\varphi(\xi)\right)\cos\alpha+b\xi\cos\left(h\xi\right)\sin\left(2\varphi(\xi)\right)\sin\alpha. \tag{16}$$

Let us find the values of $\cos\left(2\varphi(\xi)\right)$ and $\sin\left(2\varphi(\xi)\right)$, taking into account formula (4), and trigonometric formulas

$$\arctan x=\arccos\frac{1}{\sqrt{1+x^2}}=\arcsin\frac{x}{\sqrt{1+x^2}}.$$

As a result of calculations and taking into account condition (7) of the theorem, we obtain

$$\cos\left(2\varphi(\xi)\right)=\cos\left(\arctan\frac{b\xi\cos\left(h\xi\right)}{a^2\xi^2+b\xi\sin\left(h\xi\right)+c}\right)=\left(1+\left(\frac{b\xi\cos\left(h\xi\right)}{a^2\xi^2+b\xi\sin\left(h\xi\right)+c}\right)^2\right)^{-1/2}$$
$$=\left(\frac{(a^2\xi^2+b\xi\sin\left(h\xi\right)+c)^2+(b\xi\cos\left(h\xi\right))^2}{(a^2\xi^2+b\xi\sin\left(h\xi\right)+c)^2}\right)^{-1/2}$$
$$=\frac{|a^2\xi^2+b\xi\sin\left(h\xi\right)+c|}{\rho^2(\xi)}=\frac{a^2\xi^2+b\xi\sin\left(h\xi\right)+c}{\rho^2(\xi)}, \tag{17}$$

and

$$\sin\left(2\varphi(\xi)\right)=\sin\left(\arctan\frac{b\xi\cos\left(h\xi\right)}{a^2\xi^2+b\xi\sin\left(h\xi\right)+c}\right)$$
$$=\frac{b\xi\cos\left(h\xi\right)|a^2\xi^2+b\xi\sin\left(h\xi\right)+c|}{(a^2\xi^2+b\xi\sin\left(h\xi\right)+c)\rho^2(\xi)}=\frac{b\xi\cos\left(h\xi\right)}{\rho^2(\xi)}. \tag{18}$$

Let us substitute the obtained values (17) and (18) into equality (16), and as a result, we have

$$\frac{1}{\rho^2(\xi)}\Big[(a^2\xi^2+b\xi\sin\left(h\xi\right)+c)^2\sin\alpha+(a^2\xi^2+b\xi\sin\left(h\xi\right)+c)b\xi\cos\left(h\xi\right)\cos\alpha$$
$$-b\xi\cos\left(h\xi\right)(a^2\xi^2+b\xi\sin\left(h\xi\right)+c)\cos\alpha+(b\xi\cos\left(h\xi\right))^2\sin\alpha\Big]$$

$$= \frac{1}{\rho^2(\xi)}\Big[(a^2\xi^2 + b\xi\sin(h\xi) + c)^2 + (b\xi\cos(h\xi))^2\Big]\sin\alpha = \rho^2(\xi)\sin\alpha.$$

Thus, we have shown that equality (15) is true, which means function (13) is $G_1(x,t;\xi) = 0$.

Similarly, we can show that equality

$$\rho^2(\xi)\sin(\rho(\xi)t\cos\varphi(\xi) + \varphi(\xi) + x\xi)$$
$$= (a^2\xi^2 + c)\sin(\rho(\xi)t\cos\varphi(\xi) - \varphi(\xi) + x\xi)$$
$$+ b\xi\cos(\rho(\xi)t\cos\varphi(\xi) - \varphi(\xi) + x\xi - h\xi) \quad (19)$$

holds. Let us introduce the notation

$$\rho(\xi)t\cos\varphi(\xi) + \varphi(\xi) + x\xi =: \beta = \beta(x,t;\xi),$$

and, taking into account condition (7) of the theorem and formulas (17) and (18), transform the right-hand side of equality (19):

$$(a^2\xi^2 + c)\sin(\beta - 2\varphi(\xi)) + b\xi\cos(\beta - (2\varphi(\xi) + h\xi))$$
$$= (a^2\xi^2 + c)[\sin\beta\cos(2\varphi(\xi)) - \cos\beta\sin(2\varphi(\xi))] + b\xi[\cos\beta\cos(2\varphi(\xi))\cos(h\xi)$$
$$- \cos\beta\sin(2\varphi(\xi))\sin(h\xi) + \sin\beta\sin(2\varphi(\xi))\cos(h\xi) + \sin\beta\cos(2\varphi(\xi))\sin(h\xi)]$$
$$= (a^2\xi^2 + b\xi\sin(h\xi) + c)\cos(2\varphi(\xi))\sin\beta$$
$$-(a^2\xi^2 + b\xi\sin(h\xi) + c)\sin(2\varphi(\xi))\cos\beta$$
$$+ b\xi\cos(h\xi)\cos(2\varphi(\xi))\cos\beta + b\xi\cos(h\xi)\sin(2\varphi(\xi))\sin\beta$$
$$= \frac{1}{\rho^2(\xi)}\Big[(a^2\xi^2 + b\xi\sin(h\xi) + c)^2\sin\beta - (a^2\xi^2 + b\xi\sin(h\xi) + c)b\xi\cos(h\xi)\cos\beta$$
$$+ b\xi\cos(h\xi)(a^2\xi^2 + b\xi\sin(h\xi) + c)\cos\beta + (b\xi\cos(h\xi))^2\sin\beta\Big] = \rho^2(\xi)\sin\beta.$$

And, thus, it is proven that function (14) is $G_2(x,t;\xi) = 0$.

Substituting the obtained values $G_1(x,t;\xi) = 0$ and $G_2(x,t;\xi) = 0$ into equality (12), we verify its validity. Thus, by directly substituting function (8) into Equation (1), we show that it satisfies the equation.

In addition, since condition (7) of the theorem is satisfied, the function (4) is defined correctly for any of its parameters, and, therefore, the function (8) is defined correctly for any of its values. So, function (8) is indeed a classical solution to Equation (1), an infinitely smooth solution to Equation (1). The theorem is proven. □

Remark 3. *Condition (7) in the formulation of the theorem is the condition for the strong ellipticity of the differential-difference operator of Equation (1).*

4. Fulfillment of the Theorem Condition

Let us determine under what conditions of parameters a, b, c, and h of Equation (1) condition (7) of the theorem is satisfied for any $\xi \in (-\infty, +\infty)$.

Obviously, the condition $c > 0$ must be satisfied, which follows from inequality (7) with the value $\xi = 0$.

Next, consider a quadratic function

$$f(\xi) := a^2\xi^2 + b\xi\sin(h\xi) + c,$$

with respect to the variable ξ, the graph of which is a parabola with branches directed upward. The values of an even quadratic function $f(\xi)$ will be strictly positive if condition

$$b^2 \sin^2(h\xi) - 4a^2c < 0$$

is satisfied.

From the last inequality, we obtain

$$\sin^2(h\xi) < \frac{4a^2c}{b^2}.$$

Obviously, this relation will be satisfied for any values of $\xi \in (-\infty, +\infty)$ if condition $4ac > b^2$ is satisfied.

Thus, we have two conditions

$$4a^2c > b^2, \quad c > 0. \tag{20}$$

Remark 4. *Conditions* (20) *are sufficient for the existence of classical solutions to Equation* (1), *determined by formula* (8).

5. Conclusions

In this work, using an operational scheme in a half-plane, an explicit solution to a two-dimensional hyperbolic equation with a translation operator with respect to a spatial variable in the lowest derivative is constructed. A sufficient condition is obtained that guarantees the existence of classical solutions to this equation. Classes of equations for which the conditions of the theorem are satisfied are given. It is planned to use the results obtained for further research into initial problems in the half-plane for differential-difference hyperbolic equations with a translation operator in the lowest derivative. Note that such an equation has not been considered previously.

Author Contributions: Writing—original draft, N.Z.; Writing—review & editing, V.V. All authors have read and agreed to the published version of the manuscript.

Funding: The second author was financially supported by the Ministry of Science and Higher Education of the Russian Federation within the framework of the program for the Moscow Center for Fundamental and Applied Mathematics under agreement no. 075-15-2022-284.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Pinney, E. *Ordinary Difference-Differential Equations*; University California Press: Berkeley, CA, USA, 1958.
2. Bellman, R.; Cooke, K.L. *Differential-Difference Equations*; Academic Press: New York, NY, USA, 1967.
3. Hale, J. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
4. Skubachevskii, A.L. *Elliptic Functional-Differential Equations and Applications*; Birkhäuser: Basel, Switzerland, 1997.
5. Skubachevskii, A.L. Boundary-value problems for elliptic functional-differential equations and their applications. *Rus. Math. Surv.* **2016**, *71*, 801–906. [CrossRef]
6. Muravnik, A.B. Elliptic equations with translations of general form in a half-space. *Math. Notes* **2022**, *111*, 587–594. [CrossRef]
7. Muravnik, A.B. Elliptic differential-difference equations with nonlocal potentials in a half-space. *Comput. Math. Math. Phys.* **2022**, *62*, 955–961. [CrossRef]
8. Muravnik, A.B. Functional differential parabolic equations: Integral transformations and qualitative properties of solutions of the Cauchy problem. *J. Math. Sci.* **2016**, *216*, 345–496. [CrossRef]
9. Vlasov, V.V.; Medvedev, D.A. Functional-differential equations in Sobolev spaces and related problems of spectral theory. *J. Math. Sci.* **2010**, *164*, 659–841. [CrossRef]
10. Fallahi, A.A.; Yaakbarieh, A.; Sakbaev, V.Z. Conditions for hyperbolic differential-difference equations with shifts in the time argument. *Diff. Equ.* **2016**, *52*, 346–360. [CrossRef]
11. Vasilyev, V.; Zaitseva, N. Initial problem for two-dimensional hyperbolic equation with a nonlocal term. *Mathematics* **2023**, *11*, 130. [CrossRef]

12. Zaitseva, N.V. Classical solutions of hyperbolic equations with nonlocal potentials. In *Differential Equations, Mathematical Modeling and Computational Algorithms, Spreenger Proceedings in Mathematics and Statistics*; Vasilyev, V., Ed.; Springer: Cham, Switzerland, 2023; pp. 289–298.
13. Zaitseva, N.V.; Muravnik, A.B. Smooth solutions of hyperbolic equations with translation by an arbitrary vector in the free term. *Diff. Equ.* **2023**, *59*, 371–376. [CrossRef]
14. Vasilyev, V.; Zaitseva, N. Classical solutions of hyperbolic equation with translation operators in free terms. *Mathematics* **2023**, *11*, 3137. [CrossRef]
15. Zaitseva, N.V.; Muravnik, A.B. On one Cauchy problem for a hyperbolic differential-difference equation. *Diff. Equ.* **2023**, *59*, 1787–1792. [CrossRef]