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ORIGINAL ARTICLE

Recovery of the solution of the singular heat equation from measurement data

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Abstract

The problem of recovery of the solution of the singular heat equation over the positive part of the real line at a given time is solved from inaccurate measurements of this solution at other times. Explicit expressions for the optimal recovery method and its errors are obtained.

Keywords Singular heat equation · Bessel operator · Optimal recovery method

Mathematics Subject Classification 26A33 · 35Q92 · 35B40 · 43A32 · 35J15

1 Introduction: problem statement and interim results

It is well known that the temperature distribution in $\mathbb{R}^N$ is described by the equation

$$\frac{\partial u}{\partial t} = \Delta u + f(x, t),$$

where $\Delta = \partial^2/\partial x_1^2 + \ldots \partial^2/\partial x_n^2$ is the Laplace operator in $\mathbb{R}^N$.

The authors of [11] stated the following problem. Let there be temperature distributions $u(\cdot, t_1), \ldots, u(\cdot, t_p)$ at the instants of time $0 \leq t_1 < \cdots < t_p$ given approximately. More precisely, we know functions $y_j(\cdot) \in L_2(\mathbb{R}^N)$ such that

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$\|u(\cdot, t_j) - y_j(\cdot)\|_{L_2(\mathbb{R}^N)} \leq \delta_j$, where $\delta_j > 0$, $j = 1, \ldots, p$. For every set of such functions we want to find a function in $L_2(\mathbb{R}^N)$ which approximate a real temperature distribution in $\mathbb{R}^N$ at a fixed instant of time τ in a best way in some sense. We investigate a similar problem for the singular heat kind equation with the Bessel operator [4–10, 12, 13, 16, 17]. Singularities of the above type arise in models of mathematical physics such that the characteristic of the media (e.g., diffusion characteristics or heat-conductivity characteristics) have degenerate power-like heterogeneities.

Let's consider the initial-value Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = Bu,\ x \in \mathbb{R}_+,\ t > 0,$$

where B is the Bessel operator at $\mathbb{R}$, defined by the formula

$$B\,u = \frac{\partial^2 u}{\partial x^2} + \frac{\gamma}{x}\frac{\partial u}{\partial x},$$

with the initial condition

$$u(x, 0) = u_0(x),\quad x \in \mathbb{R}_+.$$

We assume that $u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+)$. The unique solution to this problem was gotten at [17], [13] by the next formula, generalizing the well-known Poisson formula

$$u(x, t) = \frac{1}{2tx^\nu} \int\limits_{\mathbb{R}+} \eta^{\nu+1}\, u_0(\eta)\, I_\nu\left(\frac{\eta x}{2t}\right) \exp\left(-\frac{\eta^2 + x^2}{4t}\right) \mathrm{d}\eta\ . \tag{1}$$

where

$$I_\nu\,(z) = \sum_{m=1}^{\infty} \frac{z^{2m+\nu}}{2^{2m+\nu} m!\Gamma(m+\nu+1)}$$

is the modified Bessel function of the first kind of order ν, $\Gamma(\cdot)$ is the Euler gamma function.

The following problem is set.

Let functions $y_j(\cdot) \in L_2^\gamma(\mathbb{R})$ be known at moments $0 \leq t_1 < \cdots < t_p$ and

$$\|u(\cdot, t_j) - y_j(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \leq \delta_j,\ j = 1, \ldots, p,$$

with $\delta_j > 0$, $j = 1, \ldots, p$. It is required for each such set of functions to match a function from $L_2^\gamma(\mathbb{R}_+)$, which in some sense would best approximate the true temperature distribution in $\mathbb{R}$ at a fixed point in time τ. In this regard, following [11], we call any mapping $m : L_2^\gamma(\mathbb{R}_+) \times \cdots \times L_2^\gamma(\mathbb{R}_+) \longrightarrow L_2^\gamma(\mathbb{R}_+)$ the recovery method (temperatures in $\mathbb{R}$ at time τ according to this information). The value

$$e(\tau, \overline{\delta}, m) = \sup_U \|u(\cdot, \tau) - m(y_j(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)},$$

where $\overline{y}(\cdot) = (y_1(\cdot), \dots, y_p(\cdot))$, $\overline{\delta} = (\delta_1(\cdot), \dots, \delta_p(\cdot))$,

$$U = \{(u_0(\cdot), \overline{y}(\cdot)) \in L_2^\gamma(\mathbb{R}_+) : \ \|u(\cdot, t_j) - y_j(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \le \delta_j, \ j = 1, \dots, p\},$$

is called the error of this method. The value

$$E(\tau, \overline{\delta}) = \inf_{m:(L_2^\gamma(\mathbb{R}))^p \longrightarrow L_2^\gamma(\mathbb{R}_+)} e(\tau, \overline{\delta}, m)$$

is called the error of optimal recovery. The method $\widehat{m}$, for which

$$E(\tau, \overline{\delta}) = e(\tau, \overline{\delta}, \widehat{m}),$$

is called the optimal method of the recovery.

2 The lower bound of the optimal method

Let's introduce the operator $P_t : L_2^\gamma(\mathbb{R}) \longrightarrow L_2^\gamma(\mathbb{R}_+)$, defined by formula (1):

$$P_t u_0(\cdot)(x, t) = \frac{1}{2tx^\nu} \int\limits_{\mathbb{R}_+} \eta^{\nu+1}\, u_0(\eta)\, I_\nu\left(\frac{\eta x}{2t}\right) \exp\left(-\frac{\eta^2 + x^2}{4t}\right) d\eta\ ,$$

$t > 0$ is a fixed value, P_0 is an identical operator.

Let $\tau \ge 0$. Let's consider the next problem

$$\|P_\tau u_0(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \longrightarrow \max, \tag{2}$$

$$\|P_{t_j} u_0(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \le \delta_j, \ \ j = 1, \dots, p, \ \ u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+). \tag{3}$$

A function that satisfies the condition (3) is called a valid function of the problem (2)–(3).

Let S mean the upper bound of $\|P_\tau u_0(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)}$ with conditions (3) and is called the value of the problem (2)–(3).

Lemma 1

$$E(\tau, \overline{\delta}) \ge S.$$

Proof Let $\overline{u}_0(\cdot)$ be a valid function of problem (2)–(3). Then $-\overline{u}_0(\cdot)$ is a valid function of problem (2)–(3) too. For any method $m : (L_2^\gamma(\mathbb{R}_+))^p \longrightarrow L_2^\gamma(\mathbb{R}_+)$, we have

$$\begin{aligned} 2\|P_\tau \overline{u}_0(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} &= \|P_\tau \overline{u}_0(\cdot) - m(0)(\cdot) + m(0)(\cdot) - P_\tau(-\overline{u}_0(\cdot))\|_{L_2^\gamma(\mathbb{R}_+)} \\ &\le \|P_\tau \overline{u}_0(\cdot) - m(0)(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} + \|m(0)(\cdot) - P_\tau(-\overline{u}_0(\cdot))\|_{L_2^\gamma(\mathbb{R}_+)} \\ &\le 2 \sup_{\substack{u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+) \\ \|P_{t_j} u_0(\cdot)\|_{L_2^\gamma(\mathbb{R})} \le \delta_j,\, j = 1, \dots, p}} \|P_\tau u_0(\cdot) - m(0)(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \end{aligned}$$

$$\leq 2 \sup_{U} \| P_\tau u_0(\cdot) - m(\overline{y}(\cdot))(\cdot) \|_{L_2^\gamma(\mathbb{R}_+)} .$$

In the left part of the resulting inequality, we pass to the supremum over admissible functions, and in the right one to the infimum over all methods. This step completes the proof of the lemma. □

Using formula 6.633 (4) from the book [2] it is easy to obtain that

$$F_\gamma [P_t u_0(\cdot)](\xi) = \exp(-|\xi|^2 t) F_\gamma u_0(\xi).$$

Therefore, by the Parseval–Plancherel theorem for the Fourier–Bessel transform the squared value of the problem (2)–(3) is equal to the value of the next problem

$$\frac{1}{2^{2\nu}\Gamma^2(\nu+1)} \int\limits_{\mathbb{R}_+} \xi^{2\nu+1} e^{-2|\xi|^2 \tau} |F_\gamma u_0(\xi)|^2 \, \mathrm{d}\xi \longrightarrow \max, \quad u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+), \tag{4}$$

$$\frac{1}{2^{2\nu}\Gamma^2(\nu+1)} \int\limits_{\mathbb{R}_+} \xi^{2\nu+1} e^{-2|\xi|^2 t_j} |F_\gamma u_0(\xi)|^2 \, \mathrm{d}\xi \leq \delta_j^2, \ \ j = 1, \ldots, p. \tag{5}$$

Let's move from the problem (4)–(5) to the extended problem (according the terminology [11]). To do this, let's replace $\frac{1}{2^{2\nu}\Gamma^2(\nu+1)} |F_\gamma u_0(\xi)|^2 \xi^{2\nu+1} \, \mathrm{d}\xi$ for the positive measure $\mathrm{d}\mu(\xi)$.

$$\int\limits_{\mathbb{R}_+} e^{-2|\xi|^2 \tau} \, \mathrm{d}\mu(\xi) \longrightarrow \max, \tag{6}$$

$$\int\limits_{\mathbb{R}_+} e^{-2|\xi|^2 t_j} \mathrm{d}\mu(\xi) \leq \delta_j^2, \ \ j = 1, \ldots, p. \tag{7}$$

The Lagrange function for this problem has the form

$$\mathcal{L}(\mathrm{d}\mu(\cdot), \lambda) = \lambda_0 \int\limits_{\mathbb{R}_+} e^{-2|\xi|^2 \tau} \, \mathrm{d}\mu(\xi) + \sum_{j=1}^{p} \lambda_j \left(\int\limits_{\mathbb{R}_+} e^{-2|\xi|^2 t_j} \mathrm{d}\mu(\xi) - \delta_j^2 \right),$$

where $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_p)$ is a set of the Lagrange multipliers. Extended problem (6)–(7) was solved in [11]. For the complete of the narrative, we will need to rewrite this solution, slightly changing the specific meanings for our needs. On a two-dimensional plane (t, y), we construct a set

$$M = \mathrm{co} \left\{ \left(t_j, \ln \left(\frac{1}{\delta_j} \right) \right) j = 1, \ldots, p \right\} + \{(t, 0) : t \geq 0\},$$

where $\mathrm{c\,o}\,A$ means the convex hall of set A. Let's introduce the function $\theta(t)$ on the ray $[0, +\infty)$:

$$\theta(t) = \max\{y : (t, y) \in M\},$$

assuming that $\theta(t) = -\infty$ if $(t, y) \notin M$, for all y. On the ray $[t_1, +\infty)$, the graph of the function $\theta(t)$ is an upward convex (concave) polygonal line. Let $t_1 = t_{s_1} < t_{s_2} < \cdots < t_{s_\varrho}$ be the essence of its breaking point. Obviously, $\{t_{s_1} < t_{s_2} < \cdots < t_{s_\varrho}\} \subseteq \{t_1 < t_2 < \cdots < t_p\}$.

We need to consider three cases.

(a) Let $\tau \geq t_1$, while to the right of τ there is a break point of the function $\theta(t)$. Suppose that $\tau \in [t_{s_j}, t_{s_{j+1}})$. Let $\mathrm{d}\widehat{\mu}(\xi) = x^\gamma T_\xi^{\xi_0}\delta_\gamma$, where the parameters A_0 and ξ_0 are determined from the conditions

$$\int\limits_{\mathbb{R}_+} e^{-2|\xi|^2\tau}\,\mathrm{d}\widehat{\mu}(\xi) = Ae^{-2|\xi_0|^2 t_k} = \delta_k^2, \quad k = s_j, s_{j+1}\,. \tag{8}$$

From conditions (8), we get

$$A = \delta_{s_j}^{2t_{s_{j+1}}/(t_{s_{j+1}}-t_{s_j})}\delta_{s_{j+1}}^{-2t_{s_j}/(t_{s_{j+1}}-t_{s_j})},$$
$$|\xi_0|^2 = \frac{\ln \delta_{s_j}/\delta_{s_{j+1}}}{t_{s_{j+1}} - t_{s_j}} = \frac{\ln(1/\delta_{s_{j+1}}) - \ln(1/\delta_{s_j})}{t_{s_{j+1}} - t_{s_j}}\,.$$

Let $\widehat{\lambda}_0 = -1$, $\widehat{\lambda}_k = 0$, $k \neq s_j,\ s_{j+1}$. In order to find numbers $\lambda_{s_j},\ \lambda_{s_{j+1}}$, let's make some preparations. Let

$$f(v) = \lambda_0 + \sum_{j=1}^{p} \lambda_j e^{-2v(t_j-\tau)}.$$

We require that $f(|\xi_0|^2) = f'(|\xi_0|^2) = 0$. From here, we obtain a system of linear equations with respect to $\lambda_{s_j},\ \lambda_{s_{j+1}}$

$$\lambda_{s_j} e^{-2|\xi_0|^2(t_{s_j}-\tau)} + \lambda_{s_{j+1}} e^{-2|\xi_0|^2(t_{s_{j+1}}-\tau)} = 1,$$
$$\lambda_{s_j}(t_{s_j} - \tau)e^{-2|\xi_0|^2(t_{s_j}-\tau)} + \lambda_{s_{j+1}}(t_{s_{j+1}} - \tau)e^{-2|\xi_0|^2(t_{s_{j+1}}-\tau)} = 0.$$

After solving this system, we get

$$\lambda_{s_j} = \frac{t_{s_{j+1}} - \tau}{t_{s_{j+1}} - t_{s_j}}\left(\frac{\delta_{s_{j+1}}}{\delta_{s_j}}\right)^{2(\tau-t_{s_j})/(t_{s_{j+1}}-t_{s_j})},$$
$$\lambda_{s_j} = \frac{\tau - t_{s_j}}{t_{s_{j+1}} - t_{s_j}}\left(\frac{\delta_{s_j}}{\delta_{s_{j+1}}}\right)^{2(t_{s_{j+1}}-\tau)/(t_{s_{j+1}}-t_{s_j})}.$$

For the measure $d\widehat{\mu}(\xi)$, we have:

$$\min_{d\mu(\cdot)\geq 0} \mathcal{L}(d\mu(\cdot), \widehat{\lambda}) = \mathcal{L}(d\mu(\cdot), \widehat{\lambda}), \tag{9}$$

$$\widehat{\lambda}_j \left(\int_{\mathbb{R}_+} e^{-2|\xi|^2\tau} \, d\widehat{\mu}(\xi) - \delta_j^2 \right) = 0, \;\; j = 1, \ldots, p. \tag{10}$$

Let

$$\rho(t) = \frac{\ln(1/\delta_{s_{j+1}}) - \ln(1/\delta_{s_j})}{t_{s_{j+1}} - t_{s_j}}(t - t_{s_j}) + \ln(1/\delta_{s_j}).$$

The straight line $y = \rho(t)$ passes through the points $(t_{s_j}, \ln(1/\delta_{s_j}))$ and $(t_{s_{j+1}}, \ln(1/\delta_{s_{j+1}}))$ and lies at least below the graph of the function $y = \theta(t)$. For the found values of A and $|\xi_0|^2$, we have:

$$\int_{\mathbb{R}_+} e^{-2|\xi|^2 t_i} \, d\widehat{\mu}(\xi) = Ae^{-2|\xi_0|^2 t_i} = \delta_{s_j}^{2(t_{s_{j+1}}-t_i)/(t_{s_{j+1}}-t_{s_j})} \delta_{s_{j+1}}^{2(t_i-t_{s_j})/(t_{s_{j+1}}-t_{s_j})}$$
$$= e^{-2\rho(t_i)} \leq e^{-2\ln(1/\delta_i)} = \delta_i^2, \;\; i = 1, \ldots, p.$$

These mean that $d\widehat{\mu}(\xi)$ is a valid measure in the extended problem (6)–(7) and is its solution. If we substitute $d\widehat{\mu}(\xi)$ into the functional defined in (6), we get the value of the problem (6)–(7), which is also the solution to the problem (4)–(5):

$$\int_{\mathbb{R}_+} e^{-2|\xi|^2\tau} \, d\widehat{\mu}(\xi) = Ae^{-2|\xi_0|^2\tau} = \delta_{s_j}^{2(t_{s_{j+1}}-\tau)/(t_{s_{j+1}}-t_{s_j})} \delta_{s_{j+1}}^{2(\tau-t_{s_j})/(t_{s_{j+1}}-t_{s_j})}$$
$$= e^{-2\rho(\tau)} = e^{-2\theta(\tau)}.$$

It means that the value of problem (2)–(3) is equal to $S = e^{-\theta(\tau)}$.

(b) Let $\tau \geq t_{s_\varrho}$. If the graph of the function $y = \theta(t)$ is a straight line, then $t_{s_\varrho} = t_1$. This time let's put $\widehat{\lambda}_0 = -1$, $\widehat{\lambda}_{s_\varrho} = 1$, $\widehat{\lambda}_{s_j} = 0$, when $j \neq \varrho$, $d\widehat{\mu}(\xi) = x^\gamma \delta_{s_\varrho} \delta_\gamma(\xi)$. The fulfillment of the condition (10) is quite obvious. In addition, for all $\xi \in \mathbb{R}_+$ the inequality

$$f(|\xi|^2) = -1 + e^{-2|\xi|^2(t_{s_\varrho}-\tau)} \geq 0$$

and the equality $f(0) = 0$ take place. Therefore, condition (9) is also met. On the ray $[t_{s_\varrho}, +\infty)$, the equality $\theta(t) \equiv \ln(1/\delta_{s_\varrho})$ is fulfilled identically. Therefore $\ln(1/\delta_j) \leq \ln(1/\delta_{s_\varrho})$, $j = 1, \ldots, p$. From here

$$\int_{\mathbb{R}_+} e^{-2|\xi|^2 t_j} \, d\widehat{\mu}(\xi) = \delta_{s_\varrho}^2 = e^{-2\ln(1/\delta_{s_\varrho})}.$$

Thus, the measure $d\widehat{\mu}(\xi)$ is valid in the problem (6)–(7) and is its solution. The value of this task is calculated as follows:

$$\int\limits_{\mathbb{R}_+} e^{-2|\xi|^2\tau}\, d\widehat{\mu}(\xi) = \delta_{s_\varrho}^2 = e^{-2\ln(1/\delta_{s_\varrho})} = e^{-2\theta(t)}.$$

This means again that the solution of the problem (2)–(3) is equal to $S = e^{-\theta(\tau)}$.

(c) Let $\tau < t_1$. For an arbitrary $y_0 > 0$, there is a straight line given by the equation $y = at + b$, $a > 0$, separating the point $(\tau, -y_0)$ and the set M. At the same time

$$-a\tau - y_0 \geq b \geq -at_j + \ln(1/\delta_{s_j}),\ \ j = 1, \dots, p.$$

Let $A = e^{-2b}$. Let's select $\xi_0 \in \mathbb{R}_+$ to provide $|\xi_0|^2 = a$. Then

$$Ae^{-2|\xi_0|^2 t_j} \leq \delta_j^2,\ \ j = 1, \dots, p.$$

It means that the measure $d\widehat{\mu}(\xi) = x^\gamma T_\xi^{\xi_0} \delta_\gamma(\xi)$ is valid for problem (6)–(7) and $Ae^{-2|\xi_0|^2\tau} \geq e^{2y_0}$. By virtue of arbitrariness of $y_0 > 0$ the value of the problem (6)–(7), and with it the solution of the problem (2)–(3) is $+\infty$.

In all three cases, for all $\tau \geq 0$, the error of optimal recovery is estimated from below $E(\tau, \overline{\delta}) \geq e^{-\theta(\tau)}$.

3 The upper estimation of the optimal recovery error

Let $\tau \geq t_1$, and $\widehat{\lambda}_1, \dots, \widehat{\lambda}_p$ be the Lagrange multipliers from cases (a), (b) for such values of τ.

Lemma 2 *Let for a set of functions* $\overline{y}(\cdot) = (y_1(\cdot), \dots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}_+))^p$ *the problem*

$$\sum_{j=1}^{p} \widehat{\lambda}_j \|P_{t_j} u_0(\cdot) - y_j(\cdot)\|^2_{L_2^\gamma(\mathbb{R}_+)} \longrightarrow \min,\ \ u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+), \tag{11}$$

have a solution $\widehat{u}_0(\cdot) = \widehat{u}_0(\cdot, \overline{y}(\cdot))$ *Then for any* $\sigma_1, \dots, \sigma_p$ *the value of problem*

$$\|P_\tau u_0(\cdot) - P_\tau \widehat{u}_0(\cdot)\|^2_{L_2^\gamma(\mathbb{R}_+)} \longrightarrow \max,\ \ u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+), \tag{12}$$

$$\|P_{t_j} u_0(\cdot) - y_j(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \leq \sigma_j\ \ j = 1, \dots, p, \tag{13}$$

is not more then the value of problem

$$\|P_\tau u_0(\cdot)\|^2_{L_2^\gamma(\mathbb{R}_+)} \longrightarrow \max,\ \ u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+), \tag{14}$$

$$\sum_{j=1}^{p} \widehat{\lambda}_j \| P_{t_j} u_0(\cdot) \|^2_{L_2^\gamma(\mathbb{R}_+)} \leq \sum_{j=1}^{p} \widehat{\lambda}_j \sigma_j^2. \tag{15}$$

Proof Equality to zero of the Frechet differential of the convex smooth target functional from (11) at the point $\widehat{u}_0(\cdot)$, that is, equality

$$2 \sum_{j=1}^{p} \widehat{\lambda}_j \int_{\mathbb{R}_+} x^\gamma (P_{t_j} \widehat{u}_0(x) - y_j(x)) P_{t_j} u_0(x) \mathrm{d}x = 0, \tag{16}$$

is a necessary and sufficient condition for the delivery the minimum to this functional by a function $\widehat{u}_0(\cdot)$. Taking this equality into account, it is easy to get that

$$\begin{aligned} &\sum_{j=1}^{p} \widehat{\lambda}_j \| P_{t_j} u_0(\cdot) - y_j(\cdot) \|^2_{L_2^\gamma(\mathbb{R}_+)} \\ &\quad = \sum_{j=1}^{p} \widehat{\lambda}_j \| P_{t_j} u_0(\cdot) - P_{t_j} \widehat{u}_0(\cdot) \|^2_{L_2^\gamma(\mathbb{R}_+)} + \sum_{j=1}^{p} \widehat{\lambda}_j \| P_{t_j} \widehat{u}_0(\cdot) - y_j(\cdot) \|^2_{L_2^\gamma(\mathbb{R}_+)} . \end{aligned}$$

Let the function $u_0(\cdot)$ be valid for the problem (12)–(13). Then

$$\begin{aligned} &\sum_{j=1}^{p} \widehat{\lambda}_j \| P_{t_j} \widehat{u}_0(\cdot) - y_j(\cdot) \|^2_{L_2^\gamma(\mathbb{R}_+)} \\ &\quad = \sum_{j=1}^{p} \widehat{\lambda}_j \| P_{t_j} u_0(\cdot) - y_j(\cdot) \|^2_{L_2^\gamma(\mathbb{R}_+)} - \sum_{j=1}^{p} \widehat{\lambda}_j \| P_{t_j} \widehat{u}_0(\cdot) - y_j(\cdot) \|^2_{L_2^\gamma(\mathbb{R}_+)} \\ &\quad \leq \sum_{j=1}^{p} \widehat{\lambda}_j \| P_{t_j} u_0(\cdot) - y_j(\cdot) \|^2_{L_2^\gamma(\mathbb{R}_+)} \leq \sum_{j=1}^{p} \widehat{\lambda}_j \sigma_j . \end{aligned}$$

It means that the function $u_0(\cdot) - \widehat{u}_0(\cdot)$ is valid for the problem (14)–(15). The value of the functional (12) at the function $u_0(\cdot)$ is equal to the value of the functional (14). The proof is completed. □

Lemma 3 *The values of problems* (2)–(3) *and* (14)–(15) *with* $\sigma_j = \delta_j$, $j = 1, \ldots, p$, *are coincide.*

Proof Using the Parseval–Plancherel equality, let's move from the problem (14)–(15) to the problem

$$\int_{\mathbb{R}_+} e^{-2|\xi|^2 \tau} \, \mathrm{d}\mu(\xi) \longrightarrow \max, \tag{17}$$

$$\sum_{j=1}^{p} \widehat{\lambda}_j \int\limits_{\mathbb{R}_+} e^{-2|\xi|^2 t_j} \mathrm{d}\mu(\xi) \le \sum_{j=1}^{p} \widehat{\lambda}_j \delta_j^2, \tag{18}$$

where

$$\mathrm{d}\mu(\xi) = \frac{1}{2^{2\nu}\Gamma^2(\nu+1)} |F_\gamma u_0(\xi)|^2 \xi^{2\nu+1}\, \mathrm{d}\xi \ge 0.$$

The Lagrange function of this problem has the form

$$\begin{aligned}\mathcal{L}_1(\mathrm{d}\mu(\cdot), \nu) = \nu_0 \int\limits_{\mathbb{R}_+} e^{-2|\xi|^2\tau}\, \mathrm{d}\mu(\xi) \\ +\nu_1 \left(\sum_{j=1}^{p} \widehat{\lambda}_j \int\limits_{\mathbb{R}_+} e^{-2|\xi|^2 t_j} \mathrm{d}\mu(\xi) - \sum_{j=1}^{p} \widehat{\lambda}_j \delta_j^2 \right),\end{aligned}$$

where the set ν of the Lagrange multipliers now has the form $\nu = (\nu_0, \nu_1)$. From the fact that the measure $\mathrm{d}\widehat{\mu}(\xi)$, which is the solution to the problem (14)–(15), is valid in this problem, it follows, that it is also valid in the problem (17)–(18). Let $\nu_0 = \widehat{\nu}_0 = -1$, $\nu_1 = \widehat{\nu}_1 = 1$. Then

$$\begin{aligned}\min_{\mathrm{d}\mu(\cdot)\ge 0} \mathcal{L}_1(\mathrm{d}\mu(\cdot), \widehat{\nu}) = \mathcal{L}_1(\mathrm{d}\widehat{\mu}(\cdot), \widehat{\nu}) \\ = \mathcal{L}(\mathrm{d}\widehat{\mu}(\cdot), \widehat{\lambda}) = \min_{\mathrm{d}\mu(\cdot)\ge 0} \mathcal{L}(\mathrm{d}\mu(\cdot), \widehat{\lambda}),\end{aligned} \tag{19}$$

where $\widehat{\nu} = (\widehat{\nu}_0, \widehat{\nu}_1)$ and with respect to (10), we have

$$\widehat{\nu}_1 \left(\sum_{j=1}^{p} \widehat{\lambda}_j \int\limits_{\mathbb{R}_+} e^{-2|\xi|^2 t_j} \mathrm{d}\widehat{\mu}(\xi) - \sum_{j=1}^{p} \widehat{\lambda}_j \delta_j^2 \right) = 0. \tag{20}$$

It means that $\mathrm{d}\widehat{\mu}(\xi)$ is the solution to the problem (17)–(18). Therefore the value of this problem is equal to the value of the problem (17)–(18). It follows that the squared value of the problem (6)– (7) is equal to the value of the problem (14)–(15). Therefore the values of the problems (6)–(7) and (14)–(15) are coincide. The lemma is proved. □

The main result is the next

Theorem 1 *For any* $\tau > 0$ *the equality*

$$E(\tau, \overline{\delta}) = e^{-\theta(\tau)}$$

take place.

1. *If* $0 \le \tau < t_1$, *then* $\theta(\tau) = -\infty$.
2. *If* $\tau = t_{s_j}$, $j = 1, \dots, \varrho$ *then the method* $\widehat{m}$, *defined by the formula* $\widehat{m}(\overline{y}(\cdot))(\cdot) = y_{s_j}(\cdot)$, *is optimal.*
3. *If* $\varrho \ge 2$, $\tau \in (t_{s_j}, t_{s_{j+1}})$, *then the method* $\widehat{m}$, *defined by the formula*

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = (\Psi_{s_j} * y_{s_j})_\gamma(\cdot) + (\Psi_{s_{j+1}} * y_{s_{j+1}})_\gamma(\cdot), \tag{21}$$

where $\Psi_{s_j}(\cdot)$, $\Psi_{s_{j+1}}(\cdot)$ *are functions whose Fourier–Bessel images have the form*

$$F_\gamma \Psi_{s_j}(\xi) = \frac{(t_{s_{j+1}} - \tau)\delta^2_{s_{j+1}} e^{-|\xi|^2(\tau - t_{s_j})}}{(t_{s_{j+1}} - \tau)\delta^2_{s_{j+1}} + (\tau - t_{s_j})\delta^2_{s_j} e^{-2|\xi|^2(t_{s_{j+1}} - t_{s_j})}}, \tag{22}$$

$$F_\gamma \Psi_{s_{j+1}}(\xi) = \frac{(\tau - t_{s_j})\delta^2_{s_j} e^{-|\xi|^2(\tau + t_{s_{j+1}} - 2t_{s_j})}}{(t_{s_{j+1}} - \tau)\delta^2_{s_{j+1}} + (\tau - t_{s_j})\delta^2_{s_j} e^{-2|\xi|^2(t_{s_{j+1}} - t_{s_j})}}, \tag{23}$$

is optimal.

4. *If* $\tau > t_{s_\varrho}$, *then the method* $\widehat{m}$, *defined by the formula* $\widehat{m}(\overline{y}(\cdot))(\cdot) = P_{\tau - t_{s_\varrho}} y_{s_\varrho}(\cdot)$, *is optimal.*

Proof Let $\tau \in [t_{s_j}, t_{s_{j+1}})$. It was shown above that it could be possible to select the set of Lagrange's multipliers in which only the multipliers $\widehat{\lambda}_{s_j}$ and $\widehat{\lambda}_{s_{j+1}}$ are not equal to zero. Therefore the problem (11) takes the form

$$\widehat{\lambda}_{s_j} \| P_{t_{s_j}} u_0(\cdot) - y_{s_j}(\cdot) \|_{L_2^\gamma(\mathbb{R}_+)} + \widehat{\lambda}_{s_{j+1}} \| P_{t_{s_{j+1}}} u_0(\cdot) - y_{s_{j+1}}(\cdot) \|_{L_2^\gamma(\mathbb{R}_+)} \longrightarrow \min,$$
$$u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+).$$

Let $\widehat{u}_0(\cdot) = \widehat{u}_0(\cdot, y(\cdot))$ be the solution to this problem. Then condition (16) is fulfilled. In the Fourier–Bessel images, this condition can be written as

$$\sum_{\kappa=j}^{j+1} \int_{\mathbb{R}_+} \xi^\gamma (e^{-|\xi|^2 t_{s_\kappa}} F_\gamma \widehat{u}_0(\xi) - F_\gamma y_{s_\kappa}(\xi)) e^{-|\xi|^2 t_{s_\kappa}} F_\gamma u_0(\xi)\, \mathrm{d}\xi = 0. \tag{24}$$

Let

$$F_\gamma \widehat{u}_0(\xi) = \frac{\widehat{\lambda}_{s_j} e^{-|\xi|^2 t_{s_j}} F_\gamma y_{s_j} + \widehat{\lambda}_{s_{j+1}} e^{-|\xi|^2 t_{s_{j+1}}} F_\gamma y_{s_{j+1}}}{\widehat{\lambda}_{s_j} e^{-2|\xi|^2 t_{s_j}} + \widehat{\lambda}_{s_{j+1}} e^{-2|\xi|^2 t_{s_{j+1}}}}. \tag{25}$$

Then equality (24) holds for all $u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+)$. Let for a set $\overline{y}(\cdot) = (y_1(\cdot), \dots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}_+))^p$ the functions $F_\gamma y_j(\cdot)$, $j = 1, \dots, p$, finitely supported. Then the function (25) belongs to the space $L_2^\gamma(\mathbb{R}_+)$. Then function $\widehat{u}_0(\cdot) = \widehat{u}_0(\cdot, y(\cdot))$, defined by the formula (25), is also belonging to the space $L_2^\gamma(\mathbb{R}_+)$ and is the solution to the problem (11). Finite functions are dense in $L_2^\gamma(\mathbb{R}_+)$. Therefore, functions with finite Fourier–Bessel's images are dense in $L_2^\gamma(\mathbb{R}_+)$.

Let functions $\widetilde{u}_0(\cdot) \in L_2^\gamma(\mathbb{R}_+)$, $\overline{y}(\cdot) = (y_1(\cdot), \dots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}_+))^p$ satisfy the inequalities

$$\|P_{t_{s_j}}\widetilde{u}_0(\cdot) - y_{s_j}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \le \delta_j, \ \ j = 1, \dots, p.$$

Let's choose a sequence $\overline{y}^{(k)}(\cdot) = (y_1^{(k)}(\cdot), \dots, y_p^{(k)}(\cdot)) \in (L_2^\gamma(\mathbb{R}_+))^p$, $k \in \mathbb{N}$ for which functions $F_\gamma y_j^{(k)}(\cdot)$, $j = 1, \dots, p$, are compactly supported and $\|y_j(\cdot) - y_j^{(k)}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \le 1/k$, $j = 1, \dots, p$, $k \in \mathbb{N}$. Let's fix the number $k \in \mathbb{N}$. There exists the solution $\widehat{u}_0(\cdot, y^{(k)}(\cdot))$ to the problem (11). Due to inequalities

$$\begin{aligned}&\|P_{t_j}\widetilde{u}_0(\cdot) - y_j^{(k)}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)}\\&\quad\le \|P_{t_j}\widetilde{u}_0(\cdot) - y_j(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} + \|y_j(\cdot) - y_j^{(k)}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \le \delta_j + 1/k, \ \ j=1, \dots, p,\end{aligned}$$

the function $\widetilde{u}_0(\cdot)$ is valid in problem (12)–(13) with $\sigma_j = \sigma_j(k) = \delta_j + 1/k$. Let

$$a(k) = \sqrt{\sum_{j=1}^p \widehat{\lambda}_j \sigma_j^2(k) \Big/ \sum_{j=1}^p \widehat{\lambda}_j \delta_j^2}.$$

Due to Lemma 2 the value of the problem (12)–(13) does not exceed the value of the problem (14)–(15).

Let's make the replacing of the function $u_0(\cdot) = a(k)v_0(\cdot)$ for the problem (14)–(15). This problem will take the form

$$a(k)\|P_\tau v_0(\cdot) - P_\tau \widehat{u}_0(\cdot)\|^2_{L_2^\gamma(\mathbb{R}_+)} \longrightarrow \max, \ \ u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+), \tag{26}$$

$$\sum_{j=1}^p \widehat{\lambda}_j \|P_{t_j} v_0(\cdot)\|^2_{L_2^\gamma(\mathbb{R}_+)} \le \sum_{j=1}^p \widehat{\lambda}_j \sigma_j^2. \tag{27}$$

The value of the problem (26)–(27) coincides with the value of the problem (2)–(3), multiplied by $a(k)$, and it is equal to $a(k)e^{-\theta(\tau)}$. Since the function $\widetilde{u}_0(\cdot)$ is valid in the problem (12)–(13), we have

$$\|P_\tau \widetilde{u}_0(\cdot) - P_\tau \widehat{u}_0(\cdot, y^{(k)}(\cdot))\|_{L_2^\gamma(\mathbb{R}_+)} \le a(k)e^{-\theta(\tau)}. \tag{28}$$

Let $\Psi_{s_j}(\cdot)$, $\Phi_{s_{j+1}}(\cdot)$ be functions whose Fourier–Bessel images have the form according to (22)–(23):

$$F_\gamma \Psi_{s_j}(\xi) = \frac{(t_{s_{j+1}} - \tau)\delta^2_{s_{j+1}} e^{-|\xi|^2(\tau - t_{s_j})}}{(t_{s_{j+1}} - \tau)\delta^2_{s_{j+1}} + (\tau - t_{s_j})\delta^2_{s_j} e^{-2|\xi|^2(t_{s_{j+1}} - t_{s_j})}},$$

$$F_\gamma \Psi_{s_{j+1}}(\xi) = \frac{(\tau - t_{s_j})\delta_{s_j}^2 e^{-|\xi|^2(\tau + t_{s_{j+1}} - 2t_{s_j})}}{(t_{s_{j+1}} - \tau)\delta_{s_{j+1}}^2 + (\tau - t_{s_j})\delta_{s_j}^2 e^{-2|\xi|^2(t_{s_{j+1}} - t_{s_j})}}.$$

Let $\tau \in (t_{s_j}, t_{s_{j+1}})$. Fourier–Bessel images (22) and (23) of functions $\Psi_{s_j}(\cdot)$ and $\Psi_{s_{j+1}}(\cdot)$ belong to space of even infinitely differentiable rapidly decreasing functions. Therefore, the functions $\Psi_{s_j}(\cdot)$ and $\Psi_{s_{j+1}}(\cdot)$ belong to this space. In the case under consideration, we define a recovery method using generalized convolution according to (21):

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = (\Psi_{s_j} * y_{s_j})_\gamma(\cdot) + (\Psi_{s_{j+1}} * y_{s_{j+1}})_\gamma(\cdot).$$

Then

$$\begin{aligned} F_\gamma \widehat{m}(\overline{y}^{(k)}(\cdot))(\xi) &= F_\gamma \Psi_{s_j}(\xi) F_\gamma y_{s_j}^{(k)}(\xi) + F_\gamma \Psi_{s_{j+1}}(\xi) F_\gamma y_{s_{j+1}}^{(k)}(\xi) \\ &= e^{-|\xi|^2\tau} F_\gamma \widetilde{u}_0(\cdot, \overline{y}^{(k)}(\cdot))(\xi). \end{aligned} \tag{29}$$

It means that

$$\widehat{m}(\overline{y}^{(k)}(\cdot))(\cdot) = P_\tau \widetilde{u}_0(\cdot, \overline{y}^{(k)}(\cdot))(\cdot). \tag{30}$$

If $\tau = t_{s_j}$, including the case of $\tau = t_{s_\varrho}$, then

$$\begin{aligned} F_\gamma \widehat{m}(\overline{y}^{(k)}(\cdot))(\xi) &= F_\gamma y_{s_j}^{(k)}(\xi) \\ &= e^{-|\xi|^2\tau} F_\gamma \widetilde{u}_0(\cdot, \overline{y}^{(k)}(\cdot))(\xi) = F_\gamma (P_\tau \widetilde{u}_0(\cdot, \overline{y}^{(k)}(\cdot)))(\xi), \end{aligned}$$

so, in this case (30) is also true.

Let again the functions $\widetilde{u}_0(\cdot) \in L_2^\gamma(\mathbb{R}_+)$, $\overline{y}(\cdot) = (y_1(\cdot), \ldots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}_+))^p$ satisfy the inequalities

$$\|P_{t_{s_j}} \widetilde{u}_0(\cdot) - y_{s_j}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \le \delta_j, \ \ j = 1, \ldots, p.$$

Then for any $k \in \mathbb{N}$

$$\begin{aligned} &\|P_\tau \widetilde{u}_0(\cdot) - \widehat{m}\,(\overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \\ &\quad \le \|P_\tau \widetilde{u}_0(\cdot) - \widehat{m}\,(\overline{y}^{(k)}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \\ &\qquad + \|\widehat{m}\,(\overline{y}^{(k)}(\cdot))(\cdot) - \widehat{m}\,(\overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \\ &\quad \le \|P_\tau \widetilde{u}_0(\cdot) - P_\tau \widetilde{u}_0(\cdot, \overline{y}^{(k)}(\cdot))\|_{L_2^\gamma(\mathbb{R}_+)} \\ &\qquad + \|\widehat{m}\,(\overline{y}^{(k)}(\cdot))(\cdot) - \widehat{m}\,(\overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \\ &\quad \le a(k)e^{-\theta(\tau)} + \|\widehat{m}\,(\overline{y}^{(k)}(\cdot) - \overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)}. \end{aligned}$$

Passing in this inequality to the limit at $k \to \infty$, we get

$$\| P_\tau \widetilde{u}_0(\cdot) - \widehat{m}\,(\overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \le e^{-\theta(\tau)}.$$

In this inequality, let's move to the upper edge over all $\widetilde{u}_0(\cdot) \in L_2^\gamma(\mathbb{R}_+)$ and $\overline{y}(\cdot) = (y_1(\cdot), \dots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}_+))^p$, for which $\| P_{t_{s_j}} \widetilde{u}_0(\cdot) - y_{s_j}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+)} \le \delta_j$, $j = 1, \dots, p$. Then we get $e(\tau, \overline{\delta}, \widehat{m}) \le e^{-\theta(\tau)}$. Given the lower bound proved earlier, we obtain

$$e^{-\theta(\tau)} \le E(\tau, \overline{\delta}) \le e(\tau, \overline{\delta}, \widehat{m}) \le e^{-\theta(\tau)},$$

from which it follows that $E(\tau, \overline{\delta}) = e^{-\theta(\tau)}$ and that $\widehat{m}$ is the optimal method.

Let $\tau > t_{s_\varrho}$. Then $\widehat{\lambda}_{s_\varrho} = 1$, the remaining Lagrange multipliers are equal to zero. The problem (11) will take a form

$$\| P_{t_{s_\varrho}} \widetilde{u}_0(\cdot) - y_{s_\varrho}(\cdot)\|^2_{L_2^\gamma(\mathbb{R}_+)} \Longrightarrow \min .$$

Let for a given set $\overline{y}(\cdot) = (y_1(\cdot), \dots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}_+))^p$ functions $F_\gamma y_j$, $j = 1, \dots, p$, be finitely supported. Then solution $\widetilde{u}_0(\cdot) = \widetilde{u}_0(\cdot, \overline{y}(\cdot))$ to this problem exists and $F_\gamma \widetilde{u}_0(\xi) = e^{|\xi|^2 t_{s_\varrho}} F_\gamma y_{s_\varrho}$. The inequality (28) in this case is proved as before. Now we define the method $\widehat{m}$ by equality

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = P_{\tau - t_{s_\varrho}}. \tag{31}$$

Then

$$F_\gamma \widehat{m}(\overline{y}^{(k)}(\cdot))(\xi) = e^{-|\xi|^2(\tau - t_{s_\varrho})} F_\gamma y_{s_\varrho}(\xi) = e^{-|\xi|^2 \tau} F_\gamma \widehat{u}_0(\cdot, \overline{y}^{(k)}(\cdot)).$$

It means again that

$$\widehat{m}(\overline{y}^{(k)}(\cdot))(\cdot) = P_\tau \widehat{u}_0(\cdot, \overline{y}^{(k)}(\cdot)).$$

Further reasoning repeats the reasoning in the previous case. The proof is finished. □

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Appendix A: Basic concepts, designations and initial facts

We need to use some notations and facts from the weighted functional Kipriyanov spaces theory [5]. We will present the facts we need, as in the work [14, 15]

Let

$$\mathbf{R}_+^N = \{x = (x', x''),\ x' = (x_1, \dots, x_n),\ x'' = (x_{n+1}, \dots, x_N),\ x_1 \ge 0, \dots, x_n \ge 0\},$$

where $1 \leq n \leq N-1$.

Let's denote by Ω^+ a domain adjacent to the hyper-planes $x_1 = 0, \dots, x_n = 0$. The boundary of Ω^+ consists of two parts: Γ^+ in $\mathbf{R}^N_+$ and Γ_0 in the hyper-planes $x_1 = 0, \dots, x_n = 0$.

Let Ω^+_δ be an interior sub-domain of Ω^+ such that all its points are located at a distance at least δ from the part of the boundary Γ^+ of the domain Ω^+. Then Ω^+_δ is called a *symmetrically interior (s-interior)* sub-domain of the domain Ω^+.

Let $\Omega \subseteq \mathbf{R}^N$ be a union of Ω^+ and Ω^- obtained from Ω^+ by symmetry with respect to $x' = 0$. We denote by $C^l_{ev}(\Omega^+)$ the linear space of functions possessing the following properties.

1. Every function $\varphi \in C^l_{ev}(\Omega^+)$, and all its partial derivatives of order up to l, are continuous in Ω^+. If a function φ has continuous partial derivatives of any order in Ω^+, we set $l = \infty$.
2. Even continuations of a function $\varphi \in C^l_{ev}(\Omega^+)$ with respect to x' remain in the class $C^l(\Omega)$.

Following [5], we say that functions admitting smooth even continuation with respect to the corresponding variables are *even* with respect to these variables.

We denote by $C^l_{ev,0}(\Omega^+)$ the linear space of functions $\varphi \in C^l_{ev}(\Omega^+)$ vanishing outside some s-interior sub-domain of Ω^+. Let $\gamma = (\gamma_1, \dots, \gamma_n)$, $(x')^\gamma = \prod_{i=1}^n x_i^{\gamma_i}$, where $\gamma_i > 0$ are fixed real numbers. We denote by $L^\gamma_p(\Omega^+)$ a closure of $C_{ev}(\Omega^+)$ by the norm

$$\|f\|_{L^\gamma_p(\Omega^+)} = \left[\int\limits_{\Omega^+} |f(x)|^p \ (x')^\gamma \, \mathrm{d}x \right]^{1/p}.$$

If Ω^+ and R^N_+ coincide, we can omit the symbol R^N_+ and write L^γ_p.

We denote by $L^\gamma_{p,\mathrm{loc}}(\Omega^+)$ the linear space of functions such that

$$\int\limits_{\Omega^+_\delta} |f(x)|^p \ (x')^\gamma \, \mathrm{d}x < +\infty$$

for any s-interior sub-domain Ω^+_δ of the domain Ω^+.

Let $\mathcal{D}_{ev}(\Omega^+)$ ($\mathcal{E}_{ev}(\Omega^+)$) be the set of all restrictions of even functions with respect to x' in the space $\mathcal{D}(\Omega)$ ($\mathcal{E}(\Omega)$) to the set Ω^+. The topology in $\mathcal{D}_{ev}(\Omega^+)$ is induced by the topology in $\mathcal{D}(\Omega)$ ($\mathcal{E}(\Omega)$). By definition, $\mathcal{D}_{ev} = \mathcal{D}_{ev}(\mathbf{R}^N_+)$. We denote by $\mathcal{S}_{ev}$ the linear space of functions $\varphi(x) \in C^\infty_{ev}(\mathbf{R}^N_+)$ that, together with all their derivatives, decrease faster than any power of $|x|^{-1}$ as $|x| \to \infty$. The topology in $\mathcal{S}_{ev}$ is introduced in the same way as in the space $\mathcal{S}$ (see [4–8, 10, 12, 13, 16, 17]). The dual of $\mathcal{D}_{ev}(\Omega^+)$ ($\mathcal{E}_{ev}(\Omega^+)$, $\mathcal{S}_{ev}$) equipped with the weak topology is denoted by $\mathcal{D}'_{ev}(\Omega^+)$ ($\mathcal{E}'_{ev}(\Omega^+)$, $\mathcal{S}'_{ev}$). The following relations hold: $\mathcal{D}_{ev} \subset \mathcal{S}_{ev} \subset \mathcal{S}'_{ev} \subset \mathcal{D}'_{ev}$.

In all three cases, the action of a distribution f on a test function φ is denoted by

$$\langle f(x), \varphi(x) \rangle_\gamma = \langle f(x), \varphi(x) \rangle \,. \tag{A1}$$

We identify each function $f(x) \in L^{\gamma}_{1,loc}(\Omega^+)$ with the functional $f \in \mathcal{D}'_{ev}(\Omega^+)$ called *regular*, acting by the formula

$$\langle f(x), \varphi(x)\rangle = \int\limits_{\Omega^+} f(x)\varphi(x)\ (x')^{\gamma}\, \mathrm{d}x. \tag{A2}$$

The remaining functionals in $\mathcal{D}'_{ev}(\Omega^+)$ are said to be *singular*. Although (A2) does not spread to singular functionals, following [1], we use designation (A2) in addition to (A1) both for regular and singular functionals.

As an example of a singular functional in $\mathcal{D}'_{ev}(\Omega^+)$ we can recall the weighted δ–function $\delta_\gamma(x)$ that is the functional defined by the equality $\langle \delta_\gamma(x)\,,\ \varphi\rangle_\gamma{=}\varphi(0)$

A *mixed generalized shift* is defined by

$$f \to (T^y_x f)(x) = \prod_{i=1}^{n} T^{y_i}_{x_i} f(x', x'' - y''),$$

where each of the generalized shifts $T^{y_i}_{x_i}$ is defined by (see [9])

$$\begin{aligned}(T^{y_i}_{x_i} f)(x) = &\frac{\Gamma(\frac{\gamma_i+1}{2})}{\sqrt{\pi}\ \Gamma\left(\frac{\gamma_i}{2}\right)}\\ &\times \int_0^{\pi} f\left(x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos\alpha}, x_{i+1}, \ldots, x_N\right) \sin^{\gamma_i - 1}\alpha\ \mathrm{d}\alpha,\\ &i = 1, \ldots, n,\end{aligned}$$

and $\prod_{k=1}^{n} T^{y_k}_{x_k}$ is understood as the superposition of operators.

The *generalized convolution* of functions $f,\ g \in L^{\gamma}_2(R^N_+)$ is defined by

$$(f * g)_\gamma(x) = \int\limits_{R^N_+} f(y) T^y_x g(x)(y')^{\gamma}\mathrm{d}y.$$

If $f \in \mathcal{D}'_{ev},\ g \in \mathcal{E}'_{ev}$, then the generalized convolution $(f * g)_\gamma$ of such distributions is defined by

$$\langle (f * g)_\gamma(x), \varphi(x)\rangle_\gamma = \langle f(y), \langle g(x), T^y_x\varphi(x)\rangle_\gamma\rangle_\gamma,\ \ \varphi(x) \in \mathcal{D}_{ev}.$$

The direct and inverse *mixed Fourier–Bessel transforms* are introduced by

$$\begin{aligned}F_\gamma[\varphi(x', x'')](\xi) &= \int\limits_{R^N_+} \varphi(x) \prod_{k=1}^{n} j_{\nu_k}(\xi_k x_k) e^{-ix''\cdot\xi''}(x')^{\gamma}\, \mathrm{d}x\\ &= (2\pi)^{N-n} 2^{2|\nu|} \prod_{k=1}^{n} \Gamma^2(\nu_k + 1) F^{-1}_\gamma[\psi(x', -x'')](\xi),\end{aligned}$$

$$F^{-1}_\gamma[\psi](x) = \frac{1}{(2\pi)^{N-n} 2^{2|\nu|} \prod_{k=1}^{n} \Gamma^2(\nu_k + 1)}$$

$$\times \int_{R_+^N} \psi(\xi) \prod_{k=1}^{n} j_{\nu_k}(\xi_k x_k) e^{ix''\cdot\xi''} (\xi')^{\gamma}\, \mathrm{d}\xi$$
$$= \frac{1}{(2\pi)^{N-n} 2^{2|\nu|} \prod_{k=1}^{n} \Gamma^2(\nu_k+1)} F_{\gamma}[\psi(\xi', -\xi'')](x),$$

where

$$\begin{aligned} x'\cdot\xi' &= x_1\xi_1 + \ldots + x_n\xi_n, \quad x''\cdot\xi'' = x_{n+1}\xi_{n+1} + \ldots + x_N\xi_N, \\ |\nu| &= \nu_1 + \ldots + \nu_n, \\ j_{\nu_k}(z_k) &= \frac{2^{\nu_k}\Gamma(\nu_k+1)}{z_k^{\nu_k}} J_{\nu_k}(z_k) \\ &= \Gamma(\nu_k+1) \sum_{m=1}^{\infty} \frac{(-1)^m z_k^{2m}}{2^{2m} m! \Gamma(m+\nu_k+1)}, \end{aligned}$$

$\Gamma(\cdot)$ is the Euler gamma-function, $J_{\nu_k}(\cdot)$ is the Bessel function of the first kind, $\nu_k = (\gamma_k - 1)/2,\ k = 1, \ldots, n$.

Theorem 2 [6] *The following Parseval–Plancherel formula holds for the Fourier–Bessel transform:*

$$\|\varphi\|_{L_2^{\gamma}} = (2\pi)^{N-n} 2^{2|\nu|} \prod_{k=1}^{n} \Gamma^2(\nu_k+1) \|\widehat{\varphi}\|_{L_2^{\gamma}}, \quad \widehat{\varphi} = F_{\gamma}[\varphi].$$

The Fourier–Bessel transform of a distribution f is defined by the formula

$$\langle F_{\gamma}[f], \varphi \rangle_{\gamma} = \langle f, F_{\gamma}[\varphi] \rangle_{\gamma},$$

where $\varphi \in S$.

The Paley–Wiener–Schwartz theorem for the Fourier transform is well known (see [3]). A generalization of this theorem to the case of the Fourier–Bessel transform can be found in [5] (also see [8]). We formulate this result in a convenient form.

Theorem 3 *(The Paley–Wiener–Schwartz theorem for the Fourier–Bessel transform).*

1. *(a counterpart of the Paley–Wiener–Schwartz theorem). An entire analytic function $\Psi(\zeta) = \Psi(\zeta', \zeta'') = \Psi(\zeta_1, \ldots, \zeta_n, \zeta_{n+1}, \ldots, \zeta_N)$ in $\mathbf{C}^N$, even with respect to $\zeta' = (\zeta_1, \ldots, \zeta_n)$, is the Fourier–Bessel transform of some compactly supported distribution with support in the set $G_{C,a} = \{x \in R^N : |x_k| \le R_k,\ k = 1, \ldots, N\}$ if and only if*

$$|\Psi(\zeta)| \le C\,(1+|\zeta|)^{\aleph} \exp\left(\sum_{l=1}^{N} a_l |Im\, \zeta_l|\right),\ \zeta \in \mathbf{C}^N, \tag{A3}$$

where $C, \aleph$ are some positive constants.

2. *(a counterpart of the Paley–Wiener theorem). An entire analytic function* $\Psi(\zeta)$ *in* $\mathbf{C}^N$ *is the Fourier–Bessel transform of some function* $u \in C_0^\infty(R^N)$ *with support in the set* $G_{C,a} = \{x \in R^N : |x_k| \le R_k,\ k = 1, \dots, N\}$ *if and only if for every* $\aleph = 0, 1, 2, \dots,$ *there is a constant* $C_\aleph$ *such that*

$$|\Psi(\zeta)| \le C_\aleph (1 + |\zeta|)^{-\aleph} \exp\left(\sum_{l=1}^{N} a_l |Im\, \zeta_l|\right),\ \zeta \in \mathbf{C}^N. \tag{A4}$$

Let $\beta = (\beta', \beta'')$ be a multi-index with nonnegative integer components $\beta' = (\beta_1, \beta_2, \dots, \beta_n),\ \beta'' = (\beta_{n+1}, \dots, \beta_N)$. We denote by $B_{x'}^{\beta'}$ the operator defined by

$$B_{x'}^{\beta'} u = B_{x_1}^{\beta_1} B_{x_2}^{\beta_2} \dots B_{x_n}^{\beta_n} u,$$

where $B_{x_i} = B_{x_i,\gamma_i}$ is the Bessel operator acting with respect to x_i by the formula

$$B_{x_i} u = B_{x_i,\gamma_i} u = \frac{\partial^2 u}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial u}{\partial x_i} = x_i^{-\gamma_i} \frac{\partial}{\partial x_i}\left(x_i^{\gamma_i} \frac{\partial u}{\partial x_i}\right).$$

We denote by $D_{x''}^{\beta''}$ the operator acting by

$$D_{x''}^{\beta''} f(x', x'') = \frac{\partial^{|\beta''|} f(x', x'')}{\partial x_{n+1}^{\beta_{n+1}} \dots x_N^{\beta_N}},$$

where $|\beta''| = \beta_{n+1} + \dots + \beta_N$.

A function of the form $B_{x'}^{\beta'} D_{x''}^{\beta''} f(x', x'')$ is referred to as the *mixed B-derivative* of $f(x', x'')$.

We consider the polynomial

$$Q(\zeta) = P(-\zeta_1^2, \dots, -\zeta_n^2, -i\zeta_{n+1}, \dots, -i\zeta_N) = P(-\zeta_1^2, \dots, -\zeta_n^2, (-i\zeta''))$$
$$= \sum_{2|\beta'|+|\beta''| \le m} b_\beta (-\zeta_1^2)^{\beta_1} \dots (-\zeta_n^2)^{\beta_n} (-i\zeta'')^{\beta''}.$$

We define the operator $P = P(B_{x'}, D_{x''})$ with constant coefficients and the symbol $P(-\zeta_1^2, \dots, -\zeta_n^2, -i\zeta_{n+1}, \dots, -i\zeta_N)$ by the formula

$$Pu = \sum_{2|\beta'|+|\beta''| \le m} b_\beta B_{x'}^{\beta'} D_{x''}^{\beta''} u. \tag{A5}$$

In particular, the B-elliptic operator Δ_B is defined by the formula (see [5])

$$\Delta_B u = \sum_{k=1}^{n} \left(\frac{\partial^2 u}{\partial x_k^2} + \frac{\gamma_k}{x_k}\frac{\partial u}{\partial x_k}\right) + \sum_{k=n+1}^{N} \frac{\partial^2 u}{\partial x_k^2}.$$

Taking the Fourier–Bessel transform on both sides of (A5), we obtain the identity

$$F_\gamma[Pu] = \sum_{2|\beta'|+|\beta''|\leq m} b_\beta F_\gamma[B_{x'}^{\beta'} D_{x''}^{\beta''} u(x', x'')](\zeta', \zeta'')$$
$$= P(-\zeta_1^2, \dots, -\zeta_n^2, -i\zeta'') F_\gamma[u](\zeta', \zeta'').$$

Theorem 4 [15] *Let* $f \in \mathcal{E}'_{ev}(\mathbf{R}_+^N)$. *The equation*

$$Pu = f \tag{A6}$$

has a solution $u(x', x'') \in \mathcal{E}'_{ev}(\mathbf{R}_+^N)$ *if and only if*

$$F_\gamma[f](\zeta) = P(-\zeta_1^2, \dots, -\zeta_n^2, -i\zeta'')\widehat{\psi}(\zeta),$$

where $\widehat{\psi}(\zeta)$ *is an entire analytic function in* $\mathbf{C}^N$, *even with respect to variables* ζ', *in other words, when the Fourier–Bessel transform of the right side of Eq. (A6) is divided "entirely" by the operator symbol* P.

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