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ORIGINAL ARTICLE

About mean value theorems for the singular parabolic equation

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Abstract

A huge number of physical, geometric, and probabilistic problems lead to the construction and study of parabolic partial differential equations. The emergence of new problems of information propagation and processes with memory leads to the need to consider parabolic type equations with various operators acting on spatial variables. In this article, mean value theorems for the singular parabolic equation were obtained. The singularity is due to the presence of the Laplace–Bessel operator.

Keywords Singular parabolic equation · Bessel operator · Mean value theorem

Mathematics Subject Classification 35K67 · 11H60

1 Introduction

Despite the large number of papers related to singular differential equations, some issues remain unexplored. The generalized divergence theorem and the second Green's formula for the Laplace–Bessel operator, published in [1], made possible to obtain significant progress in questions of existence and uniqueness theorems for solutions of singular differential equations, as well as mean value theorems. This article devoted to study of singular parabolic equation. Parabolic equations appeared in the study of the phenomena of heat propagation and diffusion by means of mathematics. The simplest but most important representative of parabolic equations is the heat equation $u_t = a^2 \Delta u$.

Book [2] presents a modern qualitative theory of partial differential equations, including parabolic equations second order. In [3], the Cauchy problem for one-dimensional parabolic equations involving Bessel operator was considered. Stabilization of solutions of certain singular quasilinear parabolic equation was obtained

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in [4]. In [5–7], the Cauchy problem for differential–difference equations of parabolic type was studied.

2 Definitions

Let $\mathbb{R}^n$ be n-dimensional Euclidean space, $x \in \mathbb{R}^n$. In the theory of weighted harmonic analysis, we use the weight measure of the form $x^\gamma \mathrm{d}x$, where $x = (x_1, \ldots, x_n)$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\gamma_1 > 0, ..., \gamma_n > 0$, $x^\gamma = x_1^{\gamma_1} \cdot ... \cdot x_n^{\gamma_n}$. If at least one of the variables x_i in the weight x^γ is negative, then raising it to a real power gives a multi-valued mapping, which is not convenient for work. On the other hand, by their nature, the functions that appear when working with the Bessel operator are usually even. Therefore, we will consider the orthant

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n, \ \ x_1{>}0, \ldots, x_n{>}0\},$$

and

$$\overline{\mathbb{R}}^n_+ = \{x \in \mathbb{R}^n, \ \ x_1{\geq}0, \ldots, x_n{\geq}0\}.$$

On negative semiaxes, all functions will be continued in an even way.

Spaces $L_p^\gamma(\mathbb{R}^n_+) = L_p^\gamma$, $1{\leq}p{<}\infty$ are spaces of measurable functions, even with respect to each variable x_i, $i = 1, ..., n$, such that their pth power is integrable with weight x^γ by $\mathbb{R}^n_+$. The L_p^γ–norm of f is given by

$$\|f\|_{p,\gamma} = \left(\int\limits_{\mathbb{R}^n_+} |f(x)|^p x^\gamma \mathrm{d}x \right)^{1/p}.$$

In [8], it was shown that L_p^γ is a Banach space.

In weighed harmonic analysis, the Hankel transform, or Fourier–Bessel transform, expresses a given function $f(x)$, $x \in \mathbb{R}^n_+$ as a weighted integral of a product of the normalized Bessel functions of the first type

$$\mathbf{F}_\gamma[f](\xi) = \mathbf{F}_\gamma[f(x)](\xi) = \widetilde{f}(\xi) = \int\limits_{\mathbb{R}^n_+} f(x)\,\mathbf{j}_\gamma(x;\xi) x^\gamma \mathrm{d}x.$$

In this formula

$$\mathbf{j}_\gamma(x;\xi) = \prod_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\xi_i), \qquad \gamma_1 > 0, ..., \gamma_n > 0,$$

the symbol j_ν is used for the normalized Bessel function of the first kind $j_\nu(x)=\dfrac{2^\nu\Gamma(\nu+1)}{x^\nu}\,J_\nu(x)$, where J_ν is Bessel function of the first kind [9]. All normalized Bessel functions $j_{\frac{\gamma_i-1}{2}}(x_i\xi_i)$ in the product $\prod\limits_{i=1}^{n} j_{\frac{\gamma_i-1}{2}}(x_i\xi_i)$ differ by an indices $\frac{\gamma_i-1}{2}$. The Hankel transform is defined on functions from $L_1^\gamma(\mathbb{R}_+^n)$.

The inversion formula is

$$\mathbf{F}_\gamma^{-1}[\widetilde{f}(\xi)](x) = f(x) = \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}_+^n}\mathbf{j}_\gamma(x,\xi)\widetilde{f}(\xi)\xi^\gamma\,d\xi.$$

Similar to the Fourier transform, the Hankel transform reduces the Laplace-Bessel operator to multiplication by the minus square of module of variables (see [8] for the one-dimensional case)

$$\mathbf{F}_\gamma[\Delta_\gamma f](\xi) = -\mid\xi\mid^2\,\widetilde{f}(\xi), \tag{1}$$

where

$$\Delta_\gamma = \sum_{k=1}^{n}(B_{\gamma_k})_{x_k} \tag{2}$$

is the Laplace–Bessel operator

$$(B_{\gamma_k})_{x_k} = \frac{\partial^2}{\partial x_k^2} + \frac{\gamma_k}{x_k}\frac{\partial}{\partial x_k}$$

is the Bessel operator and $k = 1, ..., n$.

On the space of functions summable with a weight x^γ on $\mathbb{R}_+^n$, consider the generalized translation operator

$$({}^\gamma\mathbf{T}_x^y f)(x) = {}^\gamma\mathbf{T}_x^y f(x) = ({}^{\gamma_1}T_{x_1}^{y_1}\ldots{}^{\gamma_n}T_{x_n}^{y_n} f)(x), \tag{3}$$

where each of one-dimensional generalized translation ${}^{\gamma_i}T_{x_i}^{y_i}$ acts for $i=1, ..., n$ according to

$$({}^{\gamma_i}T_{x_i}^{y_i} f)(x)=\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma_i}{2}\right)}\times$$

$$\times\int\limits_0^\pi f(x_1, ..., x_{i-1}, \sqrt{x_i^2+\tau_i^2-2x_iy_i\cos\varphi_i}, x_{i+1}, ..., x_n)\ \sin^{\gamma_i-1}\varphi_i\,d\varphi_i.$$

Next, we will use notation

$$C(\gamma) = \pi^{-\frac{n}{2}} \prod_{i=1}^{n} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

The generalized convolution constructed by (3) is

$$(f * g)_\gamma(x) = (f * g)_\gamma = \int\limits_{\mathbb{R}^n_+} f(y)({}^\gamma\mathbf{T}^y_x g)(x) y^\gamma \, dy. \tag{4}$$

The modified Bessel function of the first kind $I_\alpha(x)$ of a non-integer order α is defined as the sum of the series (see [9])

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}. \tag{5}$$

For integer values of α, the function (5) is defined by taking the limit in the expression presented above. Let

$$\mathbf{i}_\gamma(x, \xi) = \prod_{i=1}^{n} i_{\frac{\gamma_i-1}{2}}(x_i \xi_i), \tag{6}$$

where $i_\nu(x) = \frac{2^\nu \Gamma(\nu+1)}{x^\nu} I_\nu(x)$ is normalized modified Bessel function of the first kind.

3 Maximum and minimum principle for the singular parabolic equation

In this section, we consider a maximum and minimum principle for a solution to the singular parabolic equation of the form

$$u_t = a^2 \Delta_\gamma u, \qquad u = u(x, t). \tag{7}$$

Classical results concerning to the topics of maximum and minimum principle for the heat equation $u_t = a^2 \Delta u$ can be found in [2].

Let $t \in [0, T]$, $x \in \Omega^+$, where Ω^+ is a bounded simply connected domain in $\mathbb{R}^n_+$, $S^+ = \partial\Omega^+$. We deal with a cylinder C^+_T in the space $\overline{\mathbb{R}^{n+1}_+} = \{(x, t) : x \in \overline{\mathbb{R}^n_+}, t \geq 0\}$ of the form

$$C^+_T = \{(x, t) : x \in \Omega^+, 0 < t < T\}.$$

Its closure is

$$\overline{C^+_T} = \{(x, t) : x \in \overline{\Omega^+}, 0 \leq t \leq T\}.$$

The part of the boundary of the cylinder C_T^+, consisting of its lower base where $t = 0$ and surface, will be denoted by Γ^+.

Theorem 1 *The function $u = u(x,t)$ that satisfies Eq. (7) in the cylinder C_T^+ and is continuous up to its boundary takes the maximal and minimal values on Γ^+.*

Proof Since the theorem for minimum reduces to the theorem for maximum by reversing the sign of $u = u(x,t)$, we prove only the maximum theorem.

Denote by the maximum value of the function $u = u(x,t)$ in the cylinder $\overline{C_T^+}$ by M and the maximum value of the function $u = u(x,t)$ on Γ^+ by m. We have $M \geq m$ and we should prove that $M = m$. Assume the opposite. Suppose that there is a solution $u = u(x,t)$ of (7) for which $M > m$. Let this function takes the value M at the point (x^0, t_0) where $x^0 \in \Omega^+$, $0 < t_0 \leq T$. Consider the function

$$v(x,t) = u(x,t) + a^2 \frac{M-m}{2(n+\mid \gamma \mid)d^2} \mid x - x^0 \mid^2,$$

where d is the diameter of the region Ω^+. On Γ^+ we have $\mid x - x^0 \mid \leq d$, $v(x,t) < M$ and $v(x^0, t_0) = M$. Therefore, $v(x,t)$ does not attain its maximum on Γ^+. Let $v(x,t)$ has a maximum at (x^*, t_*), $x^* \in \Omega^+$, $0 < t_* \leq T$. That means

$$\frac{\partial v}{\partial x_i} = 0, \qquad \frac{\partial^2 v}{\partial x_i^2} \leq 0$$

for all $i = 1, ..., n$ and $\frac{\partial v}{\partial t} = 0$ for $0 < t_* < T$ and $\frac{\partial v}{\partial t} \geq 0$ for $t_* = T$, therefore at (x^*, t_*) should be

$$v_t - a^2 \Delta_\gamma v \geq 0.$$

From the other side

$$v_t - a^2 \Delta_\gamma v = u_t - a^2 \Delta_\gamma u - a^2 \frac{M-m}{d^2} = -a^2 \frac{M-m}{d^2} < 0.$$

Therefore, we obtain a contradiction. That means that $M = m$. □

Let functions $\varphi(x)$ and $\psi(x,t)$ are continuous in $\overline{\Omega^+}$ and $\overline{C_T^+}$, respectively, and the values of $\psi(x,t)$ coincide with the values of $\varphi(x)$ at the S^+ for $t = 0$. From Theorem 1, it immediately follows that the solution to the problem

$$\begin{cases} u_t = a^2 (\Delta_\gamma)_x u, \\ u(x,0) = \varphi(x), \\ \frac{\partial u}{\partial x_i} \mid_{x_i = 0} = 0, \qquad i = 1, \dots, n, \\ u \mid_{S^+} = \psi(x,t), \qquad x \in S^+ \end{cases} \tag{8}$$

is unique.

4 Fundamental solution for the multidimensional singular heat equation

In this section, we will find the fundamental solution of a linear differential operator. The construction of this solution is an important task, since all other solutions can be constructed from it.

Let $\varphi(x)$ is a continuous and bounded function in $\overline{\Omega^+}$. We consider the problem for the multidimensional singular heat equation

$$\begin{cases} u_t = a^2(\Delta_\gamma)_x u, \\ u(x,0) = \varphi(x), \end{cases} \tag{9}$$

where $| u(x,t) | < \infty$.

Applying multidimensional Hankel transform $\mathbf{F}_\gamma$ by $x \in \mathbb{R}^n_+$ to (9), we obtain by (1)

$$\begin{cases} \widetilde{u}_t + a^2\xi^2\widetilde{u} = 0, \\ \widetilde{u}(\xi,0) = \widetilde{\varphi}(\xi). \end{cases} \tag{10}$$

Multiplying the first equality of (10) by $e^{a^2|\xi|^2t}$, we can write

$$e^{a^2|\xi|^2t}\widetilde{u}_t + a^2\xi^2 e^{a^2|\xi|^2t}\widetilde{u} = 0 \Rightarrow \frac{\partial}{\partial t}(e^{a^2|\xi|^2t}\widetilde{u}) = 0.$$

That means that

$$e^{a^2|\xi|^2t}\widetilde{u}(\xi,t) = f(\xi),$$

where $f(\xi)$ is an arbitrary function and

$$\widetilde{u}(\xi,t) = e^{-a^2|\xi|^2t} f(\xi).$$

Using the second equality of (10), we get

$$\widetilde{u}(\xi,0) = \widetilde{\varphi}(\xi) \Rightarrow f(\xi) = \widetilde{\varphi}(\xi).$$

Therefore, we can write the multidimensional Hankel transform $\mathbf{F}_\gamma$ of $u(x,t)$ in the form of generalized convolution

$$\widetilde{u}(\xi,t) = e^{-a^2|\xi|^2t}\widetilde{\varphi}(\xi) = (\mathbf{F}_\gamma)_x((\mathbf{F}_\gamma^{-1})_\xi\{e^{-a^2|\xi|^2t}\}(x) * \varphi(x))_\gamma.$$

Using formula (17), we get

$$u(x,t) = \left(S_\gamma(\cdot,t) * \varphi(\cdot)\right)_\gamma(x) =$$
$$= \frac{1}{2^{|\gamma|}a^{n+|\gamma|}t^{\frac{n+|\gamma|}{2}}\Gamma\left(\frac{n+|\gamma|}{2}\right)\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\varphi(y)\left({}^\gamma\mathbf{T}_x^y e^{-\frac{|x|^2}{4a^2t}}\right)y^\gamma\,dy,$$

where

$$\mathbf{S}_\gamma(x,t) = (\mathbf{F}_\gamma^{-1})_\xi\{e^{-a^2\xi^2 t}\}(x) = C(n,\gamma,a)\frac{e^{-\frac{|x|^2}{4a^2t}}}{t^{\frac{n+|\gamma|}{2}}},$$

$$C(n,\gamma,a) = \frac{1}{2^{|\gamma|}a^{n+|\gamma|}t^{\frac{n+|\gamma|}{2}}\Gamma\left(\frac{n+|\gamma|}{2}\right)\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}.$$

This representation is the generalized Poisson formula. Denote

$$\mathbf{G}_\gamma(x,y,t) = C(n,\gamma,a)\frac{1}{t^{\frac{n+|\gamma|}{2}}}\,{}^\gamma\mathbf{T}_x^y e^{-\frac{|x|^2}{4a^2t}}.$$

Then

$$u(x,t) = \int\limits_{\mathbb{R}_+^n}\mathbf{G}_\gamma(x,y,t)\varphi(y)y^\gamma\,\mathrm{d}y.$$

The function $\mathbf{G}_\gamma(x,y,t)$ is called the **fundamental solution of the multidimensional singular heat equation**.

5 Mean value theorems for the singular heat equation

In the theory of boundary value problems for elliptic, parabolic, and hyperbolic equations, mean value theorems play an important role in questions of uniqueness and qualitative study of the solutions' behavior. An approach related to accompanying distributions, which allows, from a general point of view, to consider mean value formulas for solutions of linear partial differential equations was proposed in [10]. Mean value theorems for weighted spherical means connected with singular hyperbolic and ultrahyperbolic equations were given in [11, 12]. Here, we consider mean value theorems connected with singular parabolic equation.

It is known that with the help of the Laplace–Bessel operator, integration over volume can be replaced by integration over the surface. The second Green's formula for the Laplace–Bessel operator of the form (2) was given in [1] in the form

$$\int\limits_{G^+}(v\Delta_\gamma u - u\Delta_\gamma v)\,x^\gamma\,\mathrm{d}x = \int\limits_{S^+}\left(v\frac{\partial u}{\partial^\circ} - u\frac{\partial v}{\partial^\circ}\right)x^\gamma\,\mathrm{d}S, \tag{11}$$

where $u, v \in C^2(G^+)$, such that $\frac{\partial u}{\partial x_i}\ |_{x_i=0}= 0$, $\frac{\partial v}{\partial x_i}\ |_{x_i=0}= 0$, for $i = 1,\ldots,n$, the domain $G^+ \in \overline{\mathbb{R}}_+^n$ and $S^+ = \partial G^+$, $^\circ = \mathbf{e}_1\cos\eta_1 + \cdots + \mathbf{e}_n\cos\eta_n$ is an outer surface normal vector for S^+. If u, v are B-harmonic: $\Delta_\gamma u = 0$, $\Delta_\gamma v = 0$ on G^+, then (11) reduces to

$$\int\limits_{S^+} \left(v \frac{\partial u}{\partial^\circ} - u \frac{\partial v}{\partial^\circ} \right) x^\gamma \, \mathrm{d}S = 0. \tag{12}$$

If, additionally, $v = 1$, then (12) can be written in the form

$$\int\limits_{S^+} \frac{\partial u}{\partial^\circ} x^\gamma \, \mathrm{d}S = 0.$$

We are interesting in the obtaining analogy of this property for the singular parabolic operator.

Let $D_t = \frac{\partial}{\partial t}$, L_γ is the singular parabolic operator

$$L_\gamma = (\Delta_\gamma)_x - D_t,$$

and L_γ^* is its adjoint

$$L_\gamma^* = (\Delta_\gamma)_x + D_t.$$

We can write

$$uL_\gamma^* v - vL_\gamma u = u\Delta_\gamma v - v\Delta_\gamma u + (uv)_t = ((\nabla'_\gamma)_x \cdot (u(\nabla''_\gamma)_x v - v(\nabla''_\gamma)_x u)) + (uv)_t,$$

where (see [1])

$$(\nabla'_\gamma)_x = \left(\frac{1}{x_1^{\gamma_1}} \frac{\partial}{\partial x_1}, \ldots, \frac{1}{x_n^{\gamma_n}} \frac{\partial}{\partial x_n} \right), \qquad (\nabla''_\gamma)_x = \left(x_1^{\gamma_1} \frac{\partial}{\partial x_1}, \ldots, x_n^{\gamma_n} \frac{\partial}{\partial x_n} \right).$$

The domain $G^+ \subset \overline{\mathbb{R}}_+^n$ is called **Green-suitable** if G^+ is a union of domains $G_1^+, \ldots, G_m^+$ without common interior points. Each domain $G_j^+ \subset \overline{\mathbb{R}}_+^n$ be such that each line perpendicular to the plane $x_i = 0$, $i = 1, \ldots, n$, either does not intersect G_j^+ or has only one common segment with G_j^+ (possibly degenerating into a point) of the form

$$\alpha_i^j(x') \le x_i \le \beta_i^j(x'), \qquad x' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \qquad i = 1, \ldots, n,$$

where α_i, β_i are smooth for i=1, ..., n.

Theorem 2 *Let $D^+ \subset \mathbb{R}_+^{n+1}$ be Green-suitable domain. A function $u = u(x, t)$, such that $u \in C^2(D^+)$, $u_{x_i}|_{x_i=0} = 0$, $i = 1, .., n$, be a solution of the equation*

$$(\Delta_\gamma)_x u - u_t = 0$$

if and only if the equality

$$\int\limits_{\partial D^+} \left(-\frac{\partial u}{\partial^{\circ\prime}} + u\nu_{n+1} \right) x^\gamma \, \mathrm{d}S = 0 \tag{13}$$

holds. In (13) $^\circ = (^{\circ\prime}, \nu_{n+1})$ *is the exterior unit normal to* ∂D^+, $^{\circ\prime} = (\nu_1, ..., \nu_n)$ *is vector consisting of the first n components of* $^\circ$, $\frac{\partial u}{\partial^{\circ\prime}} = \frac{\partial u}{\partial x_1}\nu_1 + ... + \frac{\partial u}{\partial x_n}\nu_n$.

Proof Let first $u = u(x,t) \in C^2(D^+)$, $u_{x_i}\ |_{x_i=0} = 0$, $i = 1, .., n$, be a solution of $L_\gamma u = 0$. We take a vector

$$\begin{aligned}\mathbf{Q} &= \left(u\nabla''_\gamma v - v\nabla''_\gamma u, uv\right)\\ &= \left(u \cdot x_1^{\gamma_1}\frac{\partial v}{\partial x_1} - v \cdot x_1^{\gamma_1}\frac{\partial u}{\partial x_1}, \ldots, u \cdot x_n^{\gamma_n}\frac{\partial v}{\partial x_n} - v \cdot x_n^{\gamma_n}\frac{\partial u}{\partial x_n}, uv\right)\\ &= \left(x_1^{\gamma_1}\left(u\frac{\partial v}{\partial x_1} - v\frac{\partial u}{\partial x_1}\right), \ldots, x_n^{\gamma_n}\left(u\frac{\partial v}{\partial x_n} - v\frac{\partial u}{\partial x_n}\right), uv\right),\end{aligned}$$

and a vector operator $\mathbf{H}_\gamma = ((\nabla'_\gamma)_x, D_t)$. Let

$$\mathbf{h} = \left(u\frac{\partial v}{\partial x_1} - v\frac{\partial u}{\partial x_1}, \ldots, u\frac{\partial v}{\partial x_n} - v\frac{\partial u}{\partial x_n}, uv\right).$$

Then, the scalar product $(\mathbf{H}_\gamma \cdot \mathbf{Q})$ is

$$\begin{aligned}(\mathbf{H}_\gamma \cdot \mathbf{Q}) &= (\nabla'_\gamma \cdot (u\nabla''_\gamma v - v\nabla''_\gamma u)) + (uv)_t\\ &= \sum_{i=1}^n\left(\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}\left(u \cdot x_i^{\gamma_i}\frac{\partial v}{\partial x_i}\right) - \frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}\left(v \cdot x_i^{\gamma_i}\frac{\partial u}{\partial x_i}\right)\right) + (uv)_t\\ &= \sum_{i=1}^n\left(\frac{1}{x_i^{\gamma_i}}\frac{\partial u}{\partial x_i}\cdot x_i^{\gamma_i}\frac{\partial v}{\partial x_i} + u \cdot \frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial v}{\partial x_i} - \right.\\ &\left. -\frac{1}{x_i^{\gamma_i}}\frac{\partial v}{\partial x_i}\cdot x_i^{\gamma_i}\frac{\partial u}{\partial x_i} - v \cdot \frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial u}{\partial x_i}\right) + (uv)_t\\ &= \sum_{i=1}^n\left(u \cdot \frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial v}{\partial x_i} - v \cdot \frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial u}{\partial x_i}\right) + (uv)_t\\ &= \sum_{i=1}^n\left(uB_{\gamma_i}v - vB_{\gamma_i}u\right) + (uv)_t = v\Delta_\gamma u - u\Delta_\gamma v + (uv)_t.\end{aligned}$$

Then, combine formula (11) from [1] of the form

$$\int\limits_{G^+}(\nabla'_\gamma \cdot \mathbf{F})\, x^\gamma \mathrm{d}x = \int\limits_{\partial G^+}(\mathbf{g} \cdot {}^\circ)\ x^\gamma\ \mathrm{d}S,$$

where $G^+ \subset \mathbb{R}^n_+$ is Green-suitable domain. Here, $\vec{F}$ is a vector-function $\vec{F} = (F_1(x), \ldots, F_n(x))$, $F_1(x) = x_1^{\gamma_1}g_1(x)$, ..., $F_n(x) = x_n^{\gamma_n}g_n(x)$, $\vec{g} =$

$(g_1(x), \ldots, g_n(x))$ is a vector-function continuously differentiable in G^+ and the classical divergence theorem. We obtain

$$\int\limits_{D^+} (\mathbf{H}_\gamma \cdot \mathbf{F})\, x^\gamma \mathrm{d}x\mathrm{d}t = \int\limits_{D^+} (uL^*_\gamma v - vL_\gamma u)\, x^\gamma \mathrm{d}x\mathrm{d}t$$
$$= \int\limits_{D^+} (u\Delta_\gamma v - v\Delta_\gamma u + (uv)_t)\, x^\gamma \mathrm{d}x\mathrm{d}t$$
$$= \int\limits_{\partial D^+} \left(u\frac{\partial v}{\partial ^\circ{}'} - v\frac{\partial u}{\partial ^\circ{}'} + uv\nu_{n+1} \right) x^\gamma\, \mathrm{d}S,$$

where $\vec{\nu} = (^\circ{}', \nu_{n+1})$ is the exterior unit normal to ∂D^+, $\vec{\nu}\,' = (\nu_1, ..., \nu_n)$ is vector consisting of the first n components of $^\circ$, $\frac{\partial u}{\partial ^\circ{}'} = \frac{\partial u}{\partial x_1}\nu_1 + ... + \frac{\partial u}{\partial x_n}\nu_n$. If $L_\gamma u = L^*_\gamma v = 0$ in $\overline{D^+}$, then we get

$$\int\limits_{\partial D^+} \left(u\frac{\partial v}{\partial ^\circ{}'} - v\frac{\partial u}{\partial ^\circ{}'} + uv\nu_{n+1} \right) x^\gamma\, \mathrm{d}S = 0. \tag{14}$$

If $v = 1$, this equality reduces to (13).

Sufficiency is proved by reproducing the proof of necessity in reverse order. The theorem has been proven. □

Now, let consider

$$k_\gamma(x^0, x, t_0, t) = \begin{cases} \frac{1}{(t_0-t)^{\frac{n+|\gamma|}{2}}}\, {}^\gamma\mathbf{T}_x^{x^0} e^{-\frac{|x|^2}{4a^2(t_0-t)}}, & t > 0, \\ 0, \quad t \le 0, \end{cases}$$

where (see (18))

$${}^\gamma\mathbf{T}_x^{x^0} e^{-\frac{|x|^2}{4a^2(t_0-t)}} = e^{-\frac{|x|^2+|x^0|^2}{4a^2(t_0-t)}}\, \mathbf{i}_\gamma \left(x, \frac{x^0}{2a^2(t_0-t)} \right).$$

In particular, for $n = 1$

$${}^\gamma T_x^{x_0} e^{-\frac{x^2}{4a^2t}} =$$
$$= 2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)\left(\frac{2a^2(t_0-t)}{xx_0}\right)^{\frac{\gamma-1}{2}} e^{-\frac{x^2+x_0^2}{4a^2(t_0-t)}} I_{\frac{\gamma-1}{2}}\left(\frac{xx_0}{2a^2(t_0-t)}\right).$$

For a fixed point $(x^0, t_0) \in \mathbb{R}^{n+1}_+$ and for small enough $r > 0$, we define the so-called **generalized parabolic sphere**

$$\Gamma_\gamma(x^0, t_0, r) = \overline{\left\{(x, t) \in \mathbb{R}^{n+1}_+ : k_\gamma(x^0, x, t_0, t) = r^{-(n+|\gamma|)}\right\}}$$

and **generalized parabolic ball**

$$\Omega_\gamma(x^0, t_0, r) = \{(x, t) \in \mathbb{R}^{n+1}_+ : k_\gamma(x^0, x, t_0, t) < r^{-(n+|\gamma|)}\}$$

with "center" at (x^0, t_0) and radius r. It is easy to see that when $r \to 0$ the generalized parabolic balls $\Omega_\gamma(x^0, t_0, r)$ shrink to the center (x^0, t_0).

Theorem 3 *Let $u = u(x, t) \in C^2(D^+)$ $u_{x_i}\ |_{x_i=0}= 0$, $i = 1, .., n$, be a solution to the equation*

$$(\Delta_\gamma)_x u - u_t = 0$$

in a Green-suitable domain $D^+ \subset \mathbb{R}^{n+1}_+$. Then, for $(x^0, t_0) \in D^+$ and for almost every sufficiently small r, the next formula is valid

$$u(x^0, t_0) = - \int\limits_{\Gamma_\gamma(x^0,t_0,r)} u \frac{\partial k_\gamma(x^0, x, t_0, t)}{\partial \circ\,'} x^\gamma \, \mathrm{d}S. \tag{15}$$

$\mathrm{d}S$ is the n-dimensional measure on surface $\Gamma_\gamma(x^0, t_0, r)$, $\circ = (\circ\,', \nu_{n+1})$ is the exterior unit normal to $\Gamma_\gamma(x^0, t_0, r)$, $\circ\,' = (\nu_1, ..., \nu_n)$ is vector consisting of the first n components of $\circ$, $\frac{\partial k_\gamma(x^0,x,t_0,t)}{\partial \circ\,'} = \frac{\partial k_\gamma(x^0,x,t_0,t)}{\partial x_1}\nu_1 + ... + \frac{\partial k_\gamma(x^0,x,t_0,t)}{\partial x_n}\nu_n$.

Proof Assume that r is fixed and define for each $\tau \in (t_0 - r^2, t_0)$ the next sets

$$\Omega^s_\gamma(x^0, t_0, r) = \{(x, t) \in \Omega_\gamma(x^0, t_0, r) : t < s\},$$
$$T^s_\gamma(x^0, t_0, r) = \{(x, t) \in \overline{\Omega_\gamma(x^0, t_0, r)} : t = s\},$$
$$\Gamma^s_\gamma(x^0, t_0, r) = \{(x, t) \in \Gamma_\gamma(x^0, t_0, r) : t < s\}.$$

Let u such that $(\Delta_\gamma)_x u - u_t = 0$. Applying (14) to functions u and $v = k_\gamma(x^0, x, t_0, t)$, we obtain

$$\int\limits_{\Gamma^s_\gamma(x^0,t_0,r)\cup T^s_\gamma(x^0,t_0,r)} \left(u \frac{\partial k_\gamma(x^0, x, t_0, t)}{\partial \circ\,'} - \right.$$
$$\left. -k_\gamma(x^0, x, t_0, t) \frac{\partial u}{\partial \circ\,'} + u k_\gamma(x^0, x, t_0, t)\nu_{n+1} \right) x^\gamma \, \mathrm{d}S = 0.$$

On $\Gamma^s_\gamma(x^0, t_0, r)$ function $k_\gamma(x^0, x, t_0, t) = r^{-(n+|\gamma|)}$. Also, on $\Gamma^s_\gamma(x^0, t_0, r)$, we have $\circ\,' = 0$ and $\nu_{n+1} = 1$; therefore

$$\int\limits_{\Gamma^s_\gamma(x^0,t_0,r)} u\frac{\partial k_\gamma(x^0,x,t_0,t)}{\partial^\circ{}'}\, x^\gamma\, \mathrm{d}S +$$
$$+r^{-(n+|\gamma|)} \int\limits_{\Gamma^s_\gamma(x^0,t_0,r)} \left(-\frac{\partial u}{\partial^\circ{}'} + u\nu_{n+1}\right) x^\gamma\, \mathrm{d}S$$
$$+ \int\limits_{T^s_\gamma(x^0,t_0,r)} uk_\gamma(x^0,x,t_0,t)\, x^\gamma\, \mathrm{d}S = 0.$$

Passing to the limit in the last term, we get

$$\lim_{s\to t_0-0} \int\limits_{T^s_\gamma(x^0,t_0,r)} uk_\gamma(x^0,x,t_0,t)\, x^\gamma\, \mathrm{d}S = u(x^0,t_0).$$

Therefore, taking into account (13), we obtain (15). □

Data Availability Not applicable.

Declarations

Conflict of interest The authors declare that there are no conflicts of interest.

6 Appendix

In this section, we give some formulas and calculations used in the article.

Proposition 1 *[13] Integral* $\int\limits_{S_1^+(n)} \mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\, \mathrm{d}S$ *is calculated by the formula*

$$\int\limits_{S_1^+(n)} \mathbf{j}_\gamma(x,r\theta)\theta^\gamma\, \mathrm{d}S = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\, j_{\frac{n+|\gamma|}{2}-1}(r\mid x\mid), \tag{16}$$

where

$$S_1^+(n) = \{x\in\overline{\mathbb{R}}^n_+ :\mid x\mid= 1\}\cup\{x\in\overline{\mathbb{R}}^n_+ : x_i = 0, \mid x\mid \le r, i = 1,\dots,n\}$$

is the part of the unit sphere belonging to $\mathbb{R}^n_+$.

Proposition 2 *The next formula is valid*

$$(\mathbf{F}_\gamma^{-1})_\xi\{e^{-a^2|\xi|^2 t}\}(x)=\frac{2^{-|\gamma|}}{a^{n+|\gamma|}t^{\frac{n+|\gamma|}{2}}\Gamma\left(\frac{n+|\gamma|}{2}\right)\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}e^{-\frac{|x|^2}{4a^2t}}. \tag{17}$$

Proof We have

$$(\mathbf{F}_\gamma^{-1})_\xi\{e^{-a^2|\xi|^2 t}\}(x)=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}_+^n}\mathbf{j}_\gamma(x,\xi)e^{-a^2|\xi|^2 t}\xi^\gamma\,\mathrm{d}\xi=\{\xi=r\theta\}=$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_0^\infty e^{-a^2r^2t}r^{n+|\gamma|-1}dr\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(x,r\theta)\theta^\gamma\,\mathrm{d}S.$$

Using formula (16), we obtain

$$\begin{aligned}
&(\mathbf{F}_\gamma^{-1})_\xi\{e^{-a^2|\xi|^2 t}\}(x)\\
&=\frac{2^{1-|\gamma|}}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\int\limits_0^\infty e^{-a^2r^2t}j_{\frac{n+|\gamma|}{2}-1}(r\mid x\mid)r^{n+|\gamma|-1}dr\\
&=\frac{2^{\frac{n-|\gamma|}{2}}}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\frac{1}{\mid x\mid^{\frac{n+|\gamma|}{2}-1}}\int\limits_0^\infty e^{-a^2r^2t}J_{\frac{n+|\gamma|}{2}-1}(r\mid x\mid)r^{\frac{n+|\gamma|}{2}}dr\\
&=\frac{2^{-|\gamma|}}{a^{n+|\gamma|}t^{\frac{n+|\gamma|}{2}}\Gamma\left(\frac{n+|\gamma|}{2}\right)\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}e^{-\frac{|x|^2}{4a^2t}}.
\end{aligned}$$

□

Proposition 3 *The next formula is valid*

$${}^\gamma\mathbf{T}_x^y e^{-\frac{|x|^2}{4a^2t}}=e^{-\frac{|x|^2+|y|^2}{4a^2t}}\mathbf{i}_\gamma\left(x,\frac{y}{2a^2t}\right), \tag{18}$$

where $\mathbf{i}_\gamma$ *is defined by* (6).

Proof We have

$${}^\gamma\mathbf{T}_x^y e^{-\frac{|x|^2}{4a^2t}}=\prod_{k=1}^{n}{}^\gamma T_{x_k}^{y_k}e^{-\frac{1}{4a^2t}x_k^2}.$$

Using the formula 3.154 from [13], we obtain

$$
{}^{\gamma}T_{x_k}^{y_k} e^{-\frac{1}{4a^2t}x_k^2} =
$$

$$
= \frac{2^{\gamma_k} C(\gamma_k)}{(4x_k y_k)^{\gamma_k - 1}} \int\limits_{|x_k-y_k|}^{x_k+y_k} z e^{-\frac{1}{4a^2t}z^2} [(z^2-(x_k-y_k)^2)((x_k+y_k)^2-z^2)]^{\frac{\gamma_k}{2}-1} dz.
$$

Find the integral

$$
I = \int\limits_{|x_k-y_k|}^{x_k+y_k} z e^{-\frac{1}{4a^2t}z^2} [(z^2-(x_k-y_k)^2)((x_k+y_k)^2-z^2)]^{\frac{\gamma_k}{2}-1} dz = \{z^2 = \zeta\} =
$$

$$
= \frac{1}{2} \int\limits_{(x_k-y_k)^2}^{(x_k+y_k)^2} e^{-\frac{1}{4a^2t}\zeta} [(\zeta-(x_k-y_k)^2)((x_k+y_k)^2-\zeta)]^{\frac{\gamma_k}{2}-1} d\zeta
$$

$$
= \{\zeta - (x_k-y_k)^2 = w\} =
$$

$$
= \frac{1}{2} e^{-\frac{(x_k-y_k)^2}{4a^2t}} \int\limits_0^{4x_k y_k} e^{-\frac{1}{4a^2t}w} [w(4x_k y_k - w)]^{\frac{\gamma_k}{2}-1} dw.
$$

Applying formula 2.3.6.2 from [14] of the form

$$
\int\limits_0^a x^{\alpha-1}(a-x)^{\alpha-1} e^{-px} \mathrm{d}x = \sqrt{\pi}\,\Gamma(\alpha) \left(\frac{a}{p}\right)^{\alpha-1/2} e^{-ap/2} I_{\alpha-1/2}(ap/2),
$$

$$
\operatorname{Re}\alpha > 0, \tag{19}
$$

we get

$$
\int\limits_0^{4x_k y_k} e^{-\frac{1}{4a^2t}w} [w(4x_k y_k - w)]^{\frac{\gamma_k}{2}-1} dw =
$$

$$
= (4a)^{\gamma_k-1} \sqrt{\pi}\,\Gamma\left(\frac{\gamma_k}{2}\right) e^{-\frac{x_k y_k}{2a^2t}} (t x_k y_k)^{\frac{\gamma_k-1}{2}} I_{\frac{\gamma_k-1}{2}}\left(\frac{x_k y_k}{2a^2t}\right)
$$

and

$$
I = 2^{2\gamma_k-3} a^{\gamma_k-1} \sqrt{\pi}\,\Gamma\left(\frac{\gamma_k}{2}\right) e^{-\frac{x_k^2+y_k^2}{4a^2t}} (t x_k y_k)^{\frac{\gamma_k-1}{2}} I_{\frac{\gamma_k-1}{2}}\left(\frac{x_k y_k}{2a^2t}\right).
$$

Then, substituting the resulting formula into the product $\prod\limits_{k=1}^{n} {}^{\gamma}T_{x_k}^{y_k} e^{-\frac{1}{4a^2t}x_k^2}$ after simplification, we get (18). □

7 Conclusion

Parabolic equations are the main sources of diffusion problems and the theory of stochastic processes. If diffusion is slowed down or accelerated, or if the process has memory, then instead of the Laplace operator, other operators appear in the heat equation. The main attention in the article was given to a detailed study of the properties of a parabolic equation with a Bessel operator acting on all spatial variables. Maximum and minimum principle for the singular parabolic equation as well as the uniqueness of its solution were given. Using the form of fundamental solution of the singular parabolic equation, mean value theorems were obtained.

References

1. Ekincioglu, I., Guliyev, V.S., Shishkina, E.L.: Fractional weighted spherical mean and maximal inequality for the weighted spherical mean and its application to singular PDE. J. Math. Sci. 1–21 (2023)
2. Karapetyants, A.N., Kravchenko, V.V.: Methods of Mathematical Physics: Classical and Modern. Birkhäuser, Cham, Switzerland (2022)
3. Betancor, J.J., de León-Contreras, M.: Parabolic equations involving Bessel operators and singular integrals. Integr. Eqn. Oper. Theory **90**(18), 1–43 (2018)
4. Muravnik, A.B.: Stabilization of solutions of certain singular quasilinear parabolic equations. Math. Notes **74**(6), 812–818 (2003)
5. Muravnik, A.B.: The Cauchy problem for certain inhomogeneous difference-differential parabolic equations. Math. Notes **74**(4), 510–519 (2003)
6. Muravnik, A.B.: On the unique solvability of the Cauchy problem for some difference-differential parabolic equations. Differ. Equ. **40**(5), 742–752 (2004)
7. Muravnik, A.B.: Uniqueness of the solution of the Cauchy problem for some differential-difference parabolic equations. Differ. Equ. **40**(10), 1461–1466 (2004)
8. Kipriyanov, I.A.: Singular Elliptic Boundary Value Problems. Nauka, Moscow (1997)
9. G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1922
10. I.P. Polovinkin, Mean value theorems for linear partial differential equations. J. Math. Sci. (N.Y.) **197**(3), 399–403 (2014)
11. Lyakhov, L.N., Polovinkin, I.P., Shishkina, E.L.: On a Kipriyanov problem for a singular ultrahyperbolic equation. Differ. Equ. **50**(4), 513–525 (2014)
12. Lyakhov, L.N., Polovinkin, I.P., Shishkina, E.L.: Formulas for the solution of the Cauchy problem for a singular wave equation with Bessel time operator. Dokl. Math. **90**(3), 737–742 (2014)
13. Shishkina, E.L., Sitnik, S.M.: Transmutations, singular and fractional differential equations with applications to mathematical physics. Elsevier, Amsterdam (2020)
14. Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I.: Integrals and series. 1, Elementary Functions. Gordon & Breach Sci. Publ., New York (1992)