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ON A SINGULAR HEAT EQUATION AND PARABOLIC BESSEL POTENTIAL

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Abstract
The main goal of this paper is to analyze the solution to the singular heat equation. A singular heat equation is the parabolic equation, where the Laplace-Bessel operator acts by spatial variables. We study its fundamental solution as well as the solution to the Cauchy problem. Also, we give semigroup properties for singular thermal potential, construct fractional powers of the Laplace-Bessel operator using Balakrishnan formulas, and consider parabolic Bessel potential. Finally, we prove the boundedness of the parabolic Bessel potential.

Keywords Bessel operator · Singular heat equation · Parabolic Bessel potential

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Introduction

This paper consists of seven sections. In the first section, we give a brief overview of the results on non-classical parabolic equations and describe the content of the article. In the second section of this paper, a brief background is presented. We define weighted L_p^γ spaces, generalized convolution, Hankel transform, Fourier-Bessel transform, and others. The third section contains properties of fundamental solution to the singular parabolic differential equation $(\Delta_\gamma)_x u(x,t) = u_t(x,t)$, where

$$\Delta_\gamma = \sum_{i=1}^{n} B_{\gamma_i} \tag{1}$$

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is Laplace-Bessel operator, $B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$, $i = 1, ..., n$. In the fourth section, we prove theorem concerning properties of the solution to the Cauchy problem for the multidimensional singular heat equation. The fifth section contains theorem about semigroup properties of singular thermal potential. As a conclusion, we can construct fractional powers of Laplace-Bessel operator using Balakrishnan formulas. Finally, in the sixth section, we consider parabolic Bessel potential, obtain its boundedness, and provide a connection with an iterated singular parabolic differential equation. The seventh section is the conclusion.

Operator Eq. 1 and singular elliptic problems were considered in [1, 2]. The classical theory of parabolic equation can be found in [3]. Non-classic parabolic equations arise from different practical problems. For example, generalizations of diffusion equation have attracted a growing interest due to its widespread applications in anomalous diffusion processes [4], and non-classic parabolic equation is used to describe physical phenomena such as non-Newtonian flow [5]. The connection of parabolic equation and its generalizations with Markov processes with continuous paths, called diffusion processes, has been studied extensively, for example, in the [6–8]. Quasilinear parabolic equations arise in soil mechanics [9]. A stochastic analogue of the linear Oskolkov equation, which is a non-classical equation connected with a parabolic equation, was studied in [10]. Non-classical boundary value problems for the parabolic equation emerge in heat conduction [11]. Pure mathematical interest to the asymptotic behavior of the solution to non-classic parabolic equation was demonstrated in [12], where the Cauchy problem for a degenerate parabolic equation with inhomogeneous density was studied. The paper [13] was devoted to properties of sets of initial-value functions providing the stabilization of Cauchy problem solutions for parabolic equations. Solutions of a direct problem for a stochastic pseudo-parabolic equation with fractional Caputo derivative are investigated in [14]. The classical parabolic potential was considered in [15].

Preliminaries

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space,

$$\mathbb{R}^n_+=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n,\ \ x_1>0,\dots,x_n>0\},$$

$\gamma=(\gamma_1,...,\gamma_n)$ is a multi-index consisting of positive fixed real numbers γ_i, i=1, ..., n, and $|\gamma|=\gamma_1+\ldots+\gamma_n$. We deal with Laplace-Bessel operator $\Delta_\gamma=\sum\limits_{i=1}^n B_{\gamma_i}$, where $B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$, $i = 1, ..., n$.

Next, we give some definitions (see [16]).

Let $L_p^\gamma(\mathbb{R}^n_+) = L_p^\gamma$, $1\leq p<\infty$ be the space of all measurable in $\mathbb{R}^n_+$ functions even with respect to each variable x_i, $i = 1, ..., n$ such that

$$\int\limits_{\mathbb{R}^n_+} |f(x)|^p x^\gamma\,dx < \infty,$$

where and further$x^\gamma=\prod\limits_{i=1}^n x_i^{\gamma_i}$. For a real number $p \geq 1$, the L_p^γ–norm of f is defined by

$$\|f\|_{L_p^\gamma(\mathbb{R}^n_+)} = \|f\|_{p,\gamma} = \left(\int\limits_{\mathbb{R}^n_+} |f(x)|^p x^\gamma\,dx\right)^{1/p}.$$

It is known (see [1]) that L_p^γ is a Banach space.

Let $\Omega \subset \mathbb{R}^n_+ \cup \{x_i = 0, i = 1, ..., n\}$ and $\mathrm{mes}_\gamma(\Omega)$ be *weighed measure* of Ω:

$$\mathrm{mes}_\gamma(\Omega) = \int\limits_\Omega x^\gamma\,dx.$$

For every measurable function $f(x)$ defined on $\mathbb{R}^n_+$, we consider

$$\mu_\gamma(f,t) = \mathrm{mes}_\gamma\{x \in \mathbb{R}^n_+ : |f(x)| > t\} = \int\limits_{\{x:\ |f(x)|>t\}^+} x^\gamma dx,$$

where $\{x : |f(x)| > t\}^+$=$\{x \in \mathbb{R}^n_+ : |f(x)| > t\}$. We will name the function $\mu_\gamma = \mu_\gamma(f,t)$ as *weighted distribution function* $|f(x)|$.

Space $L^\gamma_\infty(\mathbb{R}^n_+)$=$L^\gamma_\infty$ is defined as a set of measurable on $\mathbb{R}^+_n$ and even with respect to each variable function *f*(*x*) such as

$$\|f\|_{L^\gamma_\infty(\mathbb{R}^+_n)} = \|f\|_{\infty,\gamma} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_+}{}_\gamma |f(x)| = \inf_{a\in\mathbb{R}}\{\mu_\gamma(f,a)=0\} < \infty.$$

By the Chebyshev inequality, we have

$$\mu_\gamma(f,t) \le \frac{\|f\|^p_{p,\gamma}}{t^p}. \tag{2}$$

Norms of the spaces L^γ_p and L^γ_∞ are connected by the following equality:

$$\|f\|_{\infty,\gamma} = \lim_{p\to\infty}\|f\|_{p,\gamma}.$$

Let $L^\gamma_{1,loc}(X)$, $X \subset \mathbb{R}^n_+$ be the space of all functions integrable with the weight x^γ on compact subsets in X:

$$f \in L^\gamma_{1,loc}(X) \quad \Leftrightarrow \quad \int\limits_X |f(x)|x^\gamma dx < \infty.$$

Let Ω be finite or infinite open set in $\mathbb{R}^n$ symmetric with respect to each hyperplane x_i=0, $i = 1,...,n$, $\Omega_+ = \Omega \cap \mathbb{R}^n_+$ and $\overline{\Omega}_+ = \Omega \cap \overline{\mathbb{R}}{}^n_+$ where $\overline{\mathbb{R}}{}^n_+$=$\{x$=$(x_1,\dots,x_n)$$\in$$\mathbb{R}^n, x_1$$\ge$$0,\dots,x_n$$\ge$$0\}$. We deal with the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+ functions and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions with respect to x_i for any $i = 1,...,n$ are continuous up to x_i=0. Class $C^m_{ev}(\overline{\Omega}_+)$ consists of all functions from $C^m(\overline{\Omega}_+)$ such that $\left.\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}\right|_{x_i=0} = 0$ for all non-negative integer $k \le \frac{m-1}{2}$. In the following, we will denote $C^m_{ev}(\overline{\mathbb{R}}{}^n_+)$ by C^m_{ev}. We set

$$C^\infty_{ev}(\overline{\Omega}_+) = \cap C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m and $C^\infty_{ev}(\overline{\mathbb{R}}_+) = C^\infty_{ev}$. Let $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$ be the space of all functions $f$$\in$$C^\infty_{ev}(\overline{\Omega}_+)$ with a compact support. We will use the notation $\overset{\circ}{C}{}^\infty_{ev}(\Omega_+)$=$\mathcal{D}_{ev}(\Omega_+)$.

The space of weighted generalized functions $\mathcal{D}'_{ev}(\mathbb{R}^n_+) = \mathcal{D}'_{ev}$ is a class of continuous linear functionals that map a set of test functions $f \in \mathcal{D}_{ev}$ into the set of real numbers. Each function $u(x) \in L^\gamma_{1,loc}$ will be identified with the functional $u \in \mathcal{D}'_{ev}(\mathbb{R}^n_+) = \mathcal{D}'_{ev}$ acting according to the formula

$$(u,f)_\gamma = \int\limits_{\mathbb{R}^n_+} u(x)f(x)\,x^\gamma\,dx, \qquad f \in \mathcal{D}_{ev}. \tag{3}$$

Generalized functions $u \in \mathcal{D}'_{ev}$ acting by the formula Eq. 3 will be called regular weighted generalized functions. All other generalized functions $u \in \mathcal{D}'_{ev}$ will be called singular weighted generalized functions.

Weighted delta-function $\delta_\gamma \in \mathcal{D}'_{ev}$ is defined by the equality

$$(\delta_\gamma,\varphi)_\gamma = \varphi(0), \qquad \varphi(x) \in \mathcal{D}_{ev}.$$

The generalized convolution has the form

$$(f * g)_\gamma(x) = (f * g)_\gamma = \int\limits_{\mathbb{R}^n_+} f(y)({}^\gamma\mathbf{T}^y_x g)(x) y^\gamma\,dy. \tag{4}$$

Here, ${}^{\gamma}\mathbf{T}_x^y$ is the multidimensional generalized translation

$$({}^{\gamma}\mathbf{T}_x^y f)(x) = {}^{\gamma}\mathbf{T}_x^y f(x) = ({}^{\gamma_1}T_{x_1}^{y_1} \dots {}^{\gamma_n}T_{x_n}^{y_n} f)(x),$$

${}^{\gamma_i}T_{x_i}^{y_i}$ is the one-dimensional generalized translation

$$({}^{\gamma_i}T_{x_i}^{y_i} f)(x) = \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma_i}{2}\right)} \times$$

$$\times \int_0^{\pi} f(x_1, \dots, x_{i-1}, \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos\varphi_i}, x_{i+1}, \dots, x_n) \ \sin^{\gamma_i - 1}\varphi_i \, d\varphi_i.$$

Here, $\gamma_i > 0$, i=1, …, n. For $\gamma_i = 0$ the one-dimensional generalized translation ${}^{\gamma_i}T_{x_i}^{y_i}$ is given by ${}^{0}T_{x_i}^{y_i} = \frac{f(x+y)+f(x-y)}{2}$. Convolution Eq. 4 is adapted for operator Δ_γ. We put $C(\gamma) = \pi^{-\frac{n}{2}} \prod_{i=1}^{n} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}$.

Let $p, q, r \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L_p^\gamma$, $g \in L_q^\gamma$, then $(f * g)_\gamma$ and

$$\|(f * g)_\gamma\|_{r,\gamma} \le \|f\|_{p,\gamma} \|g\|_{q,\gamma}. \tag{5}$$

Normalized function of the first kind j_ν is $j_\nu(x) = \frac{2^\nu \Gamma(\nu+1)}{x^\nu} J_\nu(x)$, where J_ν is the Bessel function of the first kind (see [17]).
For $x \in \mathbb{R}^n$, we set $\mathbf{j}_\gamma(x, \xi) = \prod_{i=1}^{n} j_{\frac{\gamma_i - 1}{2}}(x_i \xi_i)$, $\mathbf{j}_\gamma(0, \xi) = 1$.
Multivariate Hankel transform of function $f \in L_1^\gamma(\mathbb{R}_+^n)$ is

$$\mathbf{F}_\gamma[f(x)](\xi) = \mathbf{F}_\gamma[f(x)](\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}_+^n} f(x)\, \mathbf{j}_\gamma(x; \xi) x^\gamma \, dx. \tag{6}$$

Fourier-Bessel transform of function $f \in L_1^\gamma(\mathbb{R}_+^n \times (-\infty, \infty))$ is

$$\mathbf{F}_B[f(x,t)](\xi, \tau) = \int_{-\infty}^{\infty} \int_{\mathbb{R}_+^n} \mathbf{j}_\gamma(x, \xi) \cdot e^{it\tau} f(x, t)\, x^\gamma \, dx\, dt. \tag{7}$$

We also need formulas

$$|S_1^+(n)|_\gamma = \int_{S_1^+(n)} x^\gamma dS = \frac{\prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}, \tag{8}$$

$$\int_{S_1^+(n)} \mathbf{j}_\gamma(x, r\sigma)\sigma^\gamma \, dS = \frac{\prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)} j_{\frac{n+|\gamma|}{2}-1}(r|x|), \tag{9}$$

where $S_1^+(n)$ is a part of a unit sphere with a center at the origin belonging to $\mathbb{R}_+^n$:

$$S_1^+(n) = \{x \in \overline{\mathbb{R}}_+^n : |x| = 1\} \cup \{x \in \overline{\mathbb{R}}_+^n : x_i = 0, |x| \le 1, i = 1, ..., n\}.$$

We will use the notation $f_m \rightrightarrows f$ to denote $\{f_m\}$ uniformly converging to f.

Fundamental solution to the singular heat equation

Consider the following function

$$E_\gamma(x,t) = \frac{1}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\begin{cases} t^{-\frac{n+|\gamma|}{2}}e^{-\frac{|x|^2}{4t}} & \text{if } t>0;\\ 0 & \text{if } t\leq 0.\end{cases} \tag{10}$$

A number of remarkable properties of Eq. 10 are gathered in [18]. Among them are the following:

1. For all $x \in \mathbb{R}^n_+$, $t > 0$ function $E_\gamma(x,t)$ can be estimated using the power function:
$$0 < E_\gamma(x,t) < \frac{C_{n,\gamma}}{t^{\frac{n+|\gamma|}{2}}},$$
where $C_{n,\gamma}$ is some constant;
2. For all $x \in \mathbb{R}^n_+$, $t > 0$ function $E_\gamma(x,t)$ is a solution of the singular heat equation
$$(\Delta_\gamma)_x u(x,t) = u_t(x,t). \tag{11}$$

The function $E_\gamma(x,t)$ is called a **fundamental solution** of Eq. 11. In addition, the following properties of Eq. 10 are valid.

Lemma 1

1. *The Hankel transform Eq. 6 of the function $E_\gamma(x,t)$ by $x \in \mathbb{R}^n_+$, $t > 0$ is*
$$(\mathbf{F}_\gamma)_x[E_\gamma(x,t)](\xi,t) = e^{-t|\xi|^2}. \tag{12}$$
2. *Fourier-Bessel transform Eq. 7 of the function $E_\gamma(x,t)$ by (x,t), where $x \in \mathbb{R}^n_+$, $t \in \mathbb{R}$ is*
$$(\mathbf{F}_B)_{x,t}[E_\gamma(x,t)](\xi,\tau) = (|\xi|^2 - i\tau)^{-1}. \tag{13}$$
3. *Let $s > 0$, $t > 0$. Generalized convolution Eq. 4 of $E_\gamma(x,t)$ with $E_\gamma(x,t)$ by $x \in \mathbb{R}^n_+$ is*
$$\int\limits_{\mathbb{R}^n_+} E_\gamma(y,s)({}^\gamma\mathbf{T}^y_x E_\gamma(x,t))\, y^\gamma\, dy = E_\gamma(x,t+s). \tag{14}$$
4. *For all $\varepsilon > 0$, we obtain integral estimate*
$$\left|\int\limits_{\mathbb{R}^n_+}({}^\gamma\mathbf{T}^y_x E_\gamma(x,t))\varphi(y)y^\gamma\, dy\right| \leq$$
$$\leq \frac{1}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\frac{e^{\frac{1}{4t}\left(\frac{1}{\varepsilon}-1\right)|x|^2}}{t^{\frac{n+|\gamma|}{2}}}\int\limits_{\mathbb{R}^n_+} e^{-\frac{1-\varepsilon}{4t}|y|^2}|\varphi(y)|y^\gamma\, dy. \tag{15}$$

Proof

1. Let (r,σ) be the usual spherical coordinates, then, using formula Eq. 9 and integral 2.12.9.3 from [20], we get

$$(\mathbf{F}_\gamma)_x[E_\gamma(x,t)](\xi,t)=\int\limits_{\mathbb{R}^n_+}E_\gamma(x,t)\,\mathbf{j}_\gamma(x;\xi)x^\gamma\,dx=$$

$$=\frac{C_{n,\gamma}}{t^{\frac{n+|\gamma|}{2}}}\int\limits_{\mathbb{R}^n_+}e^{-\frac{|x|^2}{4t}}\,\mathbf{j}_\gamma(x;\xi)x^\gamma\,dx=\{x=r\sigma\}=$$

$$=\frac{C_{n,\gamma}}{t^{\frac{n+|\gamma|}{2}}}\int\limits_0^\infty e^{-\frac{r^2}{4t}}r^{n+|\gamma|-1}dr\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(r\sigma,\xi)\sigma^\gamma\,dS=$$

$$=\frac{1}{t^{\frac{n+|\gamma|}{2}}}\frac{2^{1-n-|\gamma|}}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty e^{-\frac{r^2}{4t}}j_{\frac{n+|\gamma|}{2}-1}(r|\xi|)r^{n+|\gamma|-1}dr=$$

$$=\frac{1}{(2t)^{\frac{n+|\gamma|}{2}}|\xi|^{\frac{n+|\gamma|}{2}-1}}\int\limits_0^\infty e^{-\frac{r^2}{4t}}J_{\frac{n+|\gamma|}{2}-1}(r|\xi|)r^{\frac{n+|\gamma|}{2}}dr=$$

$$=\frac{1}{(2t)^{\frac{n+|\gamma|}{2}}|\xi|^{\frac{n+|\gamma|}{2}-1}}2^{\frac{n+|\gamma|}{2}}e^{-t|\xi|^2}|\xi|^{\frac{n+|\gamma|-2}{2}}t^{\frac{n+|\gamma|}{2}}=e^{-t|\xi|^2}.$$

2. Using spherical coordinates (r,σ) and formula (12), we obtain

$$(\mathbf{F}_B)_{x,t}[E_\gamma(x,t)](\xi,\tau)=\int\limits_{-\infty}^\infty\int\limits_{\mathbb{R}^n_+}\mathbf{j}_\gamma(x,\xi)\cdot e^{-it\tau}E_\gamma(x,t)\,x^\gamma\,dx\,dt=$$

$$=\int\limits_0^\infty e^{-(|\xi|^2-i\tau)t}dt=(|\xi|^2-i\tau)^{-1}.$$

3. For $t>0$, we have

$$(\mathbf{F}_\gamma)_x\int\limits_{\mathbb{R}^n_+}E_\gamma(y,s)({}^\gamma\mathbf{T}^y_xE_\gamma(x,t))\,y^\gamma\,dy=$$

$$=(\mathbf{F}_\gamma)_x[E_\gamma(x,t)](\xi,t)(\mathbf{F}_\gamma)_x[E_\gamma(x,s)](\xi,s)=$$

$$=e^{-t|\xi|^2}e^{-s|\xi|^2}=e^{-(t+s)|\xi|^2}.$$

Therefore,

$$\int\limits_{\mathbb{R}^n_+} E_\gamma(y,s)({}^\gamma\mathbf{T}^y_x E_\gamma(x,t))\, y^\gamma\, dy = (\mathbf{F}_\gamma)^{-1}_\xi e^{-(t+s)|\xi|^2} = E_\gamma(x,t+s).$$

4. Let $t>0$, $\beta=(\beta_1,...,\beta_n)$, $\langle xy\cos\beta\rangle = x_1y_1\cos\beta_1+...+x_ny_n\cos\beta_n$. For integral $\mathbf{I}=\int\limits_{\mathbb{R}^n_+}({}^\gamma\mathbf{T}^y_x E_\gamma(x,t))\varphi(y)y^\gamma\,\mathrm{d}y$, we obtain

$$\mathbf{I} = \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i}{2}\right)} t^{-\frac{n+|\gamma|}{2}} \int\limits_{\mathbb{R}^n_+}\int\limits_0^\pi ... \int\limits_0^\pi \exp\left(-\frac{1}{4t}\left[|x|^2+|y|^2-2\langle xy\cos\beta\rangle\right]\right)\times$$

$$\times\varphi(y)\prod_{i=1}^n \sin^{\gamma_i-1}\beta_i\, d\beta_i y^\gamma\, dy.$$

Changing variables of integration by

$$\begin{aligned} \widetilde{y}_1 &= y_1\cos\beta_1, & \widetilde{y}_2 &= y_1\sin\beta_1,\\ \widetilde{y}_3 &= y_2\cos\beta_2, & \widetilde{y}_4 &= y_2\sin\beta_2, \dots,\\ \widetilde{y}_{2n-1} &= y_n\cos\beta_n, & \widetilde{y}_{2n} &= y_n\sin\beta_n, \end{aligned} \tag{16}$$

gives

$$\mathbf{I} = \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i}{2}\right)} t^{-\frac{n+|\gamma|}{2}} \int\limits_{\widetilde{\mathbb{R}}^{2n}_+} \exp\left(-\frac{1}{4t}\left(|x-\widetilde{y}'|^2+|\widetilde{y}''|^2\right)\right)\widetilde{\varphi}(\widetilde{y})\prod_{i=1}^n y_{2i}^{\gamma_i-1} d\widetilde{y},$$

where

$$\begin{aligned} \widetilde{y} &= (\widetilde{y}_1,...,\widetilde{y}_{2n}) = (\widetilde{y}',\widetilde{y}''),\\ \widetilde{y}' &= (\widetilde{y}_1,\widetilde{y}_3,...,\widetilde{y}_{2n-1}),\\ \widetilde{y}'' &= (\widetilde{y}_2,\widetilde{y}_4,...,\widetilde{y}_{2n}),\\ \widetilde{\varphi}(\widetilde{y}) &= \varphi\left(\sqrt{\widetilde{y}_1^2+\widetilde{y}_2^2},...,\sqrt{\widetilde{y}_{2n-1}^2+\widetilde{y}_{2n}^2}\right), \end{aligned}$$

and $\widetilde{\mathbb{R}}^{2n}_+ = \{\widetilde{y}\in\mathbb{R}^{2n} : \widetilde{y}_{2i}>0, i=1,...,n\}$. Since $2|\langle x,\widetilde{y}'\rangle| \le \frac{1}{\varepsilon}|x|^2+\varepsilon|\widetilde{y}'|^2$, then for all $\varepsilon>0$, we obtain

$$|x-\widetilde{y}'|^2 = |x|^2 - 2\langle x,\widetilde{y}'\rangle + |\widetilde{y}'|^2 \ge \left(1-\frac{1}{\varepsilon}\right)|x|^2 + (1-\varepsilon)|\widetilde{y}'|^2, \tag{17}$$

$$\exp\left(-\frac{1}{4t}\left(|x-\widetilde{y}'|^2+|\widetilde{y}''|^2\right)\right) \le$$

$$\le \exp\left(-\frac{1}{4t}\left(\left(1-\frac{1}{\varepsilon}\right)|x|^2+(1-\varepsilon)|\widetilde{y}'|^2+|\widetilde{y}''|^2\right)\right),$$

and

$$|\mathbf{I}| \le \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i}{2}\right)} \frac{e^{-\frac{1}{4t}\left(1-\frac{1}{\varepsilon}\right)|x|^2}}{t^{\frac{n+|\gamma|}{2}}}\times$$

$$\times \int\limits_{\widetilde{\mathbb{R}}^{2n}_+} \exp\left(-\frac{1}{4t}\left((1-\varepsilon)|\widetilde{y}'|^2 + |\widetilde{y}''|^2\right)\right)|\widetilde{\varphi}(\widetilde{y})| \prod_{i=1}^{n} y_{2i}^{\gamma_i - 1} d\widetilde{y} \leq$$

$$\leq \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)} \frac{e^{\frac{1}{4t}\left(\frac{1}{\varepsilon}-1\right)|x|^2}}{t^{\frac{n+|\gamma|}{2}}} \times$$

$$\times \int\limits_{\widetilde{\mathbb{R}}^{2n}_+} \exp\left(-\frac{1-\varepsilon}{4t}\left(|\widetilde{y}'|^2 + |\widetilde{y}''|^2\right)\right)|\widetilde{\varphi}(\widetilde{y})| \prod_{i=1}^{n} y_{2i}^{\gamma_i - 1} d\widetilde{y}.$$

Returning back to the old variables $y=(y_1, ..., y_n)$, we get

$$|\mathbf{I}| \leq \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)} \frac{e^{\frac{1}{4t}\left(\frac{1}{\varepsilon}-1\right)|x|^2}}{t^{\frac{n+|\gamma|}{2}}} \times$$

$$\times \int\limits_{\mathbb{R}^n_+} \int\limits_0^{\pi} ... \int\limits_0^{\pi} e^{-\frac{1-\varepsilon}{4t}|y|^2} |\varphi(y)| \prod_{i=1}^{n} \sin^{\gamma_i - 1}\beta_i \, d\beta_i y^{\gamma} dy =$$

$$= \frac{1}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)} \frac{e^{\frac{1}{4t}\left(\frac{1}{\varepsilon}-1\right)|x|^2}}{t^{\frac{n+|\gamma|}{2}}} \int\limits_{\mathbb{R}^n_+} e^{-\frac{1-\varepsilon}{4t}|y|^2} |\varphi(y)| y^{\gamma} dy.$$

Here, we took into account that

$$\int\limits_0^{\pi} ... \int\limits_0^{\pi} \prod_{i=1}^{n} \sin^{\gamma_i - 1}\beta_i \, d\beta_i = \pi^{\frac{n}{2}} \prod_{i=1}^{n} \frac{\Gamma\left(\frac{\gamma_i}{2}\right)}{\Gamma\left(\frac{\gamma_i+1}{2}\right)}$$

(see [16]). This completes the proof. □

Function $E_\gamma(x,t)$ is singular at the origin; thus, this fundamental solution for the heat equation is a generalized solution in $\mathcal{D}'_{ev}(\mathbb{R}^n_+)$.
The function $E_\gamma(x,t)$ is non-negative, vanishes for $t<0$, is infinitely differentiable for $(x,t) \neq (0,0)$, and is locally integrable in $\mathbb{R}^{n+1}_+$.
Let us consider spherical coordinates $x = r\sigma$ in $\mathbb{R}^n_+$, where $r = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$, $\sigma = (\sigma_1, ..., \sigma_n)$,

$$\begin{aligned}
\sigma_1 &= \cos\varphi_1\\
\sigma_2 &= \sin\varphi_1 \cos\varphi_2\\
\sigma_3 &= \sin\varphi_1 \sin\varphi_2 \cos\varphi_3\\
&\vdots\\
\sigma_{n-1} &= \sin\varphi_1 \cdots \sin\varphi_{n-2} \cos\varphi_{n-1}\\
\sigma_n &= \sin\varphi_1 \cdots \sin\varphi_{n-2} \sin\varphi_{n-1}.
\end{aligned}$$

When $x^\gamma dx = r^{n+|\gamma|-1}\sigma^\gamma dS$.Writing the integral $\int\limits_{\mathbb{R}^n_+} E_\gamma(x,t)x^\gamma dx$ in spherical coordinates:$\{x = r\sigma\}$ and using Eq. 8, for $t > 0$, we obtain the following:

$$\int\limits_{\mathbb{R}^n_+} E_\gamma(x,t)x^\gamma dx = \frac{t^{-\frac{n+|\gamma|}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} e^{-\frac{|x|^2}{4t}} x^\gamma dx =$$

$$= \frac{t^{-\frac{n+|\gamma|}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)} \int\limits_{0}^{\infty} e^{-\frac{r^2}{4t}} r^{n+|\gamma|-1} dr \int\limits_{S_1^+(n)} \sigma^\gamma dS =$$

$$= \frac{t^{-\frac{n+|\gamma|}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)} \cdot 2^{n+|\gamma|-1}\Gamma\left(\frac{n+|\gamma|}{2}\right) t^{\frac{\gamma+n}{2}} \cdot \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)} = 1.$$

So we get the averaging property of the kernel $E_\gamma(x,t)$:

$$\int\limits_{\mathbb{R}^n_+} E_\gamma(x,t)x^\gamma dx = 1.$$

Also, we have

$$E_\gamma(x,t) \to \delta_\gamma(x), \qquad t \to +0 \qquad \text{in} \qquad \mathcal{D}'_{ev}(\mathbb{R}^n_+).$$

Generalized convolution and non-homogeneous singular heat equation

The solution to the Cauchy problem for the singular heat equation is obtained by convolving in the sense of Eq. 4 the fundamental solution $E_\gamma(x,t)$ with the initial data by variable x. The solution to the Cauchy problem for the inhomogeneous singular heat equation is constructed by the Hankel transform method.
Using properties of E_γ from "Fundamental solution to the singular heat equation," we obtain the following result.

Theorem 1 *If $\varphi(x)$ is a continuous function on $\mathbb{R}^n_+$ satisfying for some constant $a > 0$ to the inequality*

$$A = \int\limits_{\mathbb{R}^n_+} e^{-a|x|^2}|\varphi(x)|\, x^\gamma dx < \infty, \qquad x \in \mathbb{R}^n_+, \tag{18}$$

then the generalized convolution Eq. 4, taking by x:

$$(G_t^\gamma \varphi)(x) = (E_\gamma * \varphi)_\gamma(x) = \int\limits_{\mathbb{R}^n_+} E_\gamma(y,t)({}^\gamma\mathbf{T}_x^y \varphi)(x) y^\gamma dy = \int\limits_{\mathbb{R}^n_+} ({}^\gamma\mathbf{T}_x^y E_\gamma(x,t))\varphi(y) y^\gamma dy \tag{19}$$

is defined for all $(x,t) \in \mathbb{R}^n_+ \times [0,T]$, $T = \frac{1}{4a}$, indefinitely differentiable on $\mathbb{R}^n_+ \times (0,T]$, and has an even continuation by each variable $x_1, ..., x_n$.

The following properties are valid:

1. $|(G_t^\gamma \varphi)(x)| \leq C_{n,\gamma}(A)\exp(2a|x|^2)$ *for* $0 < t < \frac{T}{2} = \frac{1}{8a}$, $C_{n,\gamma}(A)$ *is some constant*;
2. $(\Delta_\gamma)_x(G_t^\gamma \varphi)(x)=\frac{\partial}{\partial t}(G_t^\gamma \varphi)(x)$ *for* $0 < t < T$;
3. $(G_t^\gamma \varphi)(x)$ *tends to* $\varphi(x)$ *uniformly on each compact subset of* $\mathbb{R}_+^n$ *as t tends to zero from the right*: $(G_t^\gamma \varphi)(x) \rightrightarrows \varphi(x)$, $t \to +0$.

Proof Let us notice that generalized translation ${}^\gamma\mathbf{T}_x^y$ is self-adjoint with weight x^γ (see [16]), so

$$(G_t^\gamma \varphi)(x) = (E_\gamma * \varphi)_\gamma(x) = \int\limits_{\mathbb{R}_+^n} ({}^\gamma\mathbf{T}_x^y E_\gamma(x,t))\varphi(y)y^\gamma\,dy$$

and since $t < \frac{a}{4}$ we can take $\varepsilon > 0$ in Eq. 15 such that

$$\frac{1-\varepsilon}{4t} = a, \Rightarrow \varepsilon = 1 - 4at.$$

That gives

$$\left|\int\limits_{\mathbb{R}_+^n} ({}^\gamma\mathbf{T}_x^y E_\gamma(x,t))\varphi(y)y^\gamma\,dy\right| \leq \frac{1}{2^{|\gamma|}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)} \frac{e^{\frac{a}{1-4at}|x|^2}}{t^{\frac{n+|\gamma|}{2}}} \int\limits_{\mathbb{R}_+^n} e^{-a|y|^2}\varphi(y)y^\gamma\,dy.$$

Whereby Eq. 18 yields

$$|(G_t^\gamma \varphi)(x)| = \left|\int\limits_{\mathbb{R}_+^n} ({}^\gamma\mathbf{T}_x^y E_\gamma(x,t))\varphi(y)y^\gamma\,dy\right| \leq \frac{A}{2^{|\gamma|}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)} \frac{e^{\frac{a}{1-4at}|x|^2}}{t^{\frac{n+|\gamma|}{2}}} =$$

$$= \frac{A}{2^{|\gamma|}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)} \frac{e^{\frac{1}{4(T-t)}|x|^2}}{t^{\frac{n+|\gamma|}{2}}}. \tag{20}$$

It was taken into account here that $T = \frac{1}{4a}$. This estimate and the form of $E_\gamma(x,t)$ give that $(G_t^\gamma \varphi)(x)$ is defined for all $(x,t) \in \mathbb{R}_+^n \times [0,T]$, $T = \frac{1}{4a}$, and indefinitely differentiable on $\mathbb{R}_+^n \times [0,T]$, and has an even continuation by each variable $x_1, ..., x_n$.

For $0 < t < \frac{1}{8a}$ from Eq. 20, we get property 1. Property 2 follows from the fact that function $E_\gamma(x,t)$ is a fundamental solution of Eq. 11.

Applying change of variables Eq. 16 and using the same notations as in Lemma 1, we obtain

$$(G_t^\gamma \varphi)(x) =$$

$$= \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i}{2}\right)} t^{-\frac{n+|\gamma|}{2}} \int\limits_{\widetilde{\mathbb{R}}_+^{2n}} \exp\left(-\frac{1}{4t}\left(|x-\widetilde{y}'|^2 + |\widetilde{y}''|^2\right)\right)\widetilde{\varphi}(\widetilde{y}) \prod_{i=1}^n y_{2i}^{\gamma_i-1}\,d\widetilde{y},$$

where $\widetilde{\mathbb{R}}_+^{2n} = \{\widetilde{y} \in \mathbb{R}^{2n} : \widetilde{y}_{2i} > 0, i = 1, ..., n\}$. In order to show property 3, we split the integral defining $(G_t^\gamma \varphi)(x)$ into two parts

$$(G_t^\gamma \varphi)(x) = \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)}(I_1 + I_2),$$

where

$$I_1 = \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)} t^{-\frac{n+|\gamma|}{2}} \int\limits_{A_+} \exp\left(-\frac{1}{4t}\left(|x-\widetilde{y}'|^2 + |\widetilde{y}''|^2\right)\right)\widetilde{\varphi}(\widetilde{y}) \prod_{i=1}^{n} y_{2i}^{\gamma_i - 1} d\widetilde{y},$$

$$I_2 = \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)} t^{-\frac{n+|\gamma|}{2}} \int\limits_{B_+} \exp\left(-\frac{1}{4t}\left(|x-\widetilde{y}'|^2 + |\widetilde{y}''|^2\right)\right)\widetilde{\varphi}(\widetilde{y}) \prod_{i=1}^{n} y_{2i}^{\gamma_i - 1} d\widetilde{y},$$

$$A_+ = \{\widetilde{y} \in \widetilde{\mathbb{R}}_+^{2n} : |x-\widetilde{y}'|^2 + |\widetilde{y}''|^2 \le 1\}, B_+ = \{\widetilde{y} \in \widetilde{\mathbb{R}}_+^{2n} : |x-\widetilde{y}'|^2 + |\widetilde{y}''|^2 \ge 1\}.$$

We show that $I_1 \rightrightarrows \varphi$ and $I_2 \rightrightarrows 0$ for $t \to +0$ on each compact subset of $\mathbb{R}_+^n$.

Substituting $x - \widetilde{y}' = z'\sqrt{t}$, $\widetilde{y}'' = z''\sqrt{t}$, $z = (z', z'') \in \mathbb{R}_+^{2n}$, $z' = (z_1, z_3, ..., z_{2n-1})$, $z'' = (z_2, z_4, ..., z_{2n})$ in I_1, we obtain

$$I_1 = \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)} \int\limits_{\left\{z\in\widetilde{\mathbb{R}}_+^{2n} : |z| \le \frac{1}{\sqrt{t}}\right\}} e^{-\frac{|z|^2}{4}} \widetilde{\varphi}(x - z'\sqrt{t}, z''\sqrt{t}) \prod_{i=1}^{n} z_{2i}^{\gamma_i - 1} dz.$$

For $|z| \le \frac{1}{\sqrt{t}}$, the function $\widetilde{\varphi}(x - z'\sqrt{t}, z''\sqrt{t})$ is uniformly bounded. Returning back to the old variables y=$(y_1, ..., y_n)$, we get

$$I_1 = \frac{1}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)} \int\limits_{\left\{y\in\mathbb{R}_+^n : |y| \le \frac{1}{\sqrt{t}}\right\}} e^{-\frac{|y|^2}{4}} \, {}^\gamma\mathbf{T}_x^{\sqrt{t}y}\varphi(x) y^\gamma dy.$$

The properties of the generalized translation (see [21]) give ${}^\gamma\mathbf{T}_x^{\sqrt{t}y}\varphi(x) \rightrightarrows \varphi(x)$ for $t \to +0$. We should calculate

$$\int\limits_{\left\{y\in\mathbb{R}_+^n : |y| \le \frac{1}{\sqrt{t}}\right\}} e^{-\frac{|y|^2}{4}} y^\gamma dy = \{y = r\theta\} =$$

$$= \int\limits_0^{\frac{1}{\sqrt{t}}} e^{-\frac{r^2}{4}} r^{n+|\gamma|-1} dr \int\limits_{S_1^+(n)} \theta^\gamma dS = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)} \int\limits_0^{\frac{1}{\sqrt{t}}} e^{-\frac{r^2}{4}} r^{n+|\gamma|-1} dr;$$

here, we use Eq. 8. Since

$$\lim_{t\to+0} \int\limits_0^{\frac{1}{\sqrt{t}}} e^{-\frac{r^2}{4}} r^{n+|\gamma|-1} dr = 2^{|\gamma|+n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right),$$

after elementary calculations, we get

$$I_1 = \frac{1}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)} \int\limits_{\left\{y\in\mathbb{R}_+^n : |y| \le \frac{1}{\sqrt{t}}\right\}} e^{-\frac{|y|^2}{4}} \, {}^\gamma\mathbf{T}_x^{\sqrt{t}y}\varphi(x) y^\gamma dy \rightrightarrows \varphi(x), \qquad t \to +0.$$

Now, let us consider I_2. Splitting the exponential into two equal parts, taking into account that $|x-\widetilde{y}'|^2+|\widetilde{y}''|^2 \geq 1$, we obtain

$$|I_2| \leq \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)} t^{-\frac{n+|\gamma|}{2}} \times$$

$$\times \int\limits_{B_+} e^{-\frac{1}{8t}\left(|x-\widetilde{y}'|^2+|\widetilde{y}''|^2\right)} e^{-\frac{1}{8t}\left(|x-\widetilde{y}'|^2+|\widetilde{y}''|^2\right)} |\widetilde{\varphi}(\widetilde{y})| \prod_{i=1}^{n} y_{2i}^{\gamma_i-1} d\widetilde{y} \leq$$

$$\leq \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)} \frac{e^{-\frac{1}{8t}}}{t^{\frac{n+|\gamma|}{2}}} \int\limits_{B_+} e^{-\frac{1}{8t}\left(|x-\widetilde{y}'|^2+|\widetilde{y}''|^2\right)} |\widetilde{\varphi}(\widetilde{y})| \prod_{i=1}^{n} y_{2i}^{\gamma_i-1} d\widetilde{y}.$$

By Eq. 17 we get for all $\varepsilon > 0$

$$\exp\left(-\frac{1}{8t}\left(|x-\widetilde{y}'|^2+|\widetilde{y}''|^2\right)\right) \leq \exp\left(-\frac{1}{8t}\left(\left(1-\frac{1}{\varepsilon}\right)|x|^2+(1-\varepsilon)|\widetilde{y}'|^2+|\widetilde{y}''|^2\right)\right).$$

Putting $\varepsilon = 1+8at$, we can write

$$|I_2| \leq \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)} \frac{e^{-\frac{1}{8t}} e^{-\frac{1}{8t}\left(1-\frac{1}{\varepsilon}\right)|x|^2}}{t^{\frac{n+|\gamma|}{2}}} \int\limits_{B_+} e^{-\frac{1}{8t}\left((1-\varepsilon)|\widetilde{y}'|^2+|\widetilde{y}''|^2\right)} |\widetilde{\varphi}(\widetilde{y})| \prod_{i=1}^{n} y_{2i}^{\gamma_i-1} d\widetilde{y} \leq$$

$$\leq \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)} \frac{e^{-\frac{1}{8t}} e^{-\frac{1}{8t}\left(1-\frac{1}{\varepsilon}\right)|x|^2}}{t^{\frac{n+|\gamma|}{2}}} \int\limits_{B_+} e^{-\frac{1-\varepsilon}{8t}\left(|\widetilde{y}'|^2+|\widetilde{y}''|^2\right)} |\widetilde{\varphi}(\widetilde{y})| \prod_{i=1}^{n} y_{2i}^{\gamma_i-1} d\widetilde{y} =$$

$$= \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)} \frac{e^{-\frac{1}{8t}} e^{-\frac{a}{1+8at}|x|^2}}{t^{\frac{n+|\gamma|}{2}}} \int\limits_{B_+} e^{-a|\widetilde{y}|^2} |\widetilde{\varphi}(\widetilde{y})| \prod_{i=1}^{n} y_{2i}^{\gamma_i-1} d\widetilde{y} \leq$$

$$\leq \frac{\pi^{-\frac{n}{2}}}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)} \frac{e^{-\frac{1}{8t}} e^{-\frac{a}{1+8at}|x|^2}}{t^{\frac{n+|\gamma|}{2}}} \int\limits_{\widetilde{\mathbb{R}}^{2n}_+} e^{-a|\widetilde{y}|^2} |\widetilde{\varphi}(\widetilde{y})| \prod_{i=1}^{n} y_{2i}^{\gamma_i-1} d\widetilde{y} =$$

$$\leq \frac{1}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)} \frac{e^{-\frac{1}{8t}} e^{-\frac{a}{1+8at}|x|^2}}{t^{\frac{n+|\gamma|}{2}}} \int\limits_{\mathbb{R}^{n}_+} e^{-a|x|^2} |\varphi(x)|\, x^{\gamma}\, dx =$$

$$= \frac{1}{2^{|\gamma|}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)} \frac{A e^{-\frac{1}{8t}} e^{-\frac{a}{1+8at}|x|^2}}{t^{\frac{n+|\gamma|}{2}}} \rightrightarrows 0, \qquad t \to +0.$$

This completes the proof. □

Operator G_t^{γ} is called **singular thermal potential** with density $\varphi = \varphi(x)$.

Corollary 1 *Let Ω be a bounded simply connected domain in $\mathbb{R}^n_+$ and φ be a continuous and bounded function in $\overline{\Omega}$, then*

$$u(x,t) = (G_t^\gamma \varphi)(x) = (E_\gamma * \varphi)_\gamma(x) = \int\limits_{\mathbb{R}^n_+} E_\gamma(y,t)({}^\gamma\mathbf{T}_x^y \varphi)(x) y^\gamma dy$$

is a solution to the Cauchy problem for the multidimensional singular heat equation

$$\begin{cases} (\Delta_\gamma)_x u(x,t) = u_t(x,t), \\ u(x,0) = \varphi(x). \end{cases} \tag{21}$$

Cauchy problem for the multidimensional singular heat equation was studied in [22].
Using the fundamental solution $E_\gamma(x,t)$ of the operator $(\Delta_\gamma)_x - \frac{\partial}{\partial t}$, one can construct a solution to the equation

$$(\Delta_\gamma)_x u(x,t) - u_t(x,t) = f(x) \tag{22}$$

with an arbitrary right-hand side f. More precisely, if $f \in \mathcal{D}'_{ev}$ is such that the generalized convolution $(E_\gamma(x,t) * f(x))_\gamma$ exists in $\mathcal{D}'_{ev}$, then the solution to equation Eq. 22 exists in $\mathcal{D}'_{ev}$ and is given by the formula $(E_\gamma(x,t) * f(x))_\gamma$. This solution is unique in the class of weighted generalized functions from $\mathcal{D}'_{ev}$ for which this generalized convolution exists.

Semigroup view to singular thermal potential and Balakrishnan formulas

Let us consider the operator G_t^γ defined by Eq. 19:

$$(G_t^\gamma \varphi)(x) = \int\limits_{\mathbb{R}^n_+} E_\gamma(y,t)({}^\gamma\mathbf{T}_x^y \varphi)(x) y^\gamma dy = \int\limits_{\mathbb{R}^n_+} ({}^\gamma\mathbf{T}_x^y E_\gamma(x,t))\varphi(y) y^\gamma dy.$$

For $t > 0$, this operator acts

$$G_t^\gamma \,:\, C_b(\mathbb{R}^n_+) \to C_b(\mathbb{R}^n_+),$$

where $C_b(\mathbb{R}^n_+)$ is the class of bounded continuous functions, admitting even continuation by each of the variables $x_1, ..., x_n$.

Theorem 2 *Let Ω be a bounded simply connected domain in $\mathbb{R}^n_+$, $\varphi \in C_b(\mathbb{R}^n_+)$, then $G_t^\gamma \varphi$ satisfies semigroup properties*

1. $G_0^\gamma \varphi = \varphi$,
2. $G_{t_1+t_2}^\gamma \varphi = G_{t_1}^\gamma \circ G_{t_2}^\gamma \varphi$.

Proof By Corollary 1, we get $G_0^\gamma \varphi = \lim\limits_{t\to+0} G_t^\gamma \varphi = \varphi$, i.e., $G_0^\gamma = I$ is the identity operator.

Using properties (7.1), (7.3), and (7.5) from [21], we get

$$\int\limits_{\mathbb{R}^n_+} ({}^\gamma\mathbf{T}_y^z E_\gamma(y,t_1))({}^\gamma\mathbf{T}_x^z E_\gamma(x,t_2))\, z^\gamma dz = \int\limits_{\mathbb{R}^n_+} ({}^\gamma\mathbf{T}_z^y E_\gamma(z,t_1))({}^\gamma\mathbf{T}_x^z E_\gamma(x,t_2))\, z^\gamma dz =$$

$$= \int\limits_{\mathbb{R}^n_+} E_\gamma(z,t_1)({}^\gamma\mathbf{T}_z^y\, {}^\gamma\mathbf{T}_x^z E_\gamma(x,t_2))\, z^\gamma dz = \int\limits_{\mathbb{R}^n_+} E_\gamma(z,t_1)({}^\gamma\mathbf{T}_x^y\, {}^\gamma\mathbf{T}_x^z E_\gamma(x,t_2))\, z^\gamma dz =$$

$$= {}^\gamma\mathbf{T}_x^y \int\limits_{\mathbb{R}^n_+} E_\gamma(z,t_1)({}^\gamma\mathbf{T}_x^z E_\gamma(x,t_2))\, z^\gamma dz = ({}^\gamma\mathbf{T}_x^y E_\gamma(x,t_1+t_2)).$$

Now, we compute $G_{t_1+t_2}^\gamma$ directly using Eq. 14

$$(G^\gamma_{t_1+t_2}\varphi)(x)=\int\limits_{\mathbb{R}^n_+}E_\gamma(y,t_1+t_2)({}^\gamma\mathbf{T}^y_x\varphi)(x)y^\gamma\,dy=\int\limits_{\mathbb{R}^n_+}({}^\gamma\mathbf{T}^y_xE_\gamma(x,t_1+t_2))\varphi(y)y^\gamma\,dy=$$

$$=\int\limits_{\mathbb{R}^n_+}\varphi(y)\left(\int\limits_{\mathbb{R}^n_+}({}^\gamma\mathbf{T}^z_xE_\gamma(x,t_1))({}^\gamma\mathbf{T}^z_yE_\gamma(y,t_2))\,z^\gamma\,dz\right)y^\gamma\,dy=$$

$$=\int\limits_{\mathbb{R}^n_+}({}^\gamma\mathbf{T}^z_xE_\gamma(x,t_1))\left(\int\limits_{\mathbb{R}^n_+}({}^\gamma\mathbf{T}^z_yE_\gamma(y,t_2))\varphi(y)y^\gamma\,dy\right)z^\gamma\,dz=(G_{t_1}\circ G_{t_2})\varphi(x).$$

That gives semigroup property 2 for G^γ_t, which completes the proof. □

So G^γ_t is a semigroup $\mathbb{R}^n_+$. Let function $u(x,t)$ satisfy $(\Delta_\gamma)_xu(x,t)=u_t(x,t)$. If the initial values $u(x,0)=\varphi(x)$ imply that

$$\lim_{t\to+0}\frac{u(x,t)-u(x,0)}{t}=u_t(x,0)=(\Delta_\gamma)_xu(x,0)=(\Delta_\gamma)_x\varphi(x).$$

Since solution $u(x,t)=(G^\gamma_t\varphi)(x)$ we have

$$\lim_{t\to+0}\frac{1}{t}(G^\gamma_t-I)\varphi=(\Delta_\gamma)_x\varphi(x).$$

So Δ_γ is the infinitesimal generator of the semigroup G^γ_t and $G^\gamma_t\Delta_\gamma\varphi=\Delta_\gamma G^\gamma_t\varphi$, $\varphi\in D((\Delta_\gamma)_x)$.
So taking into account the general semigroup theory, we obtain a contracting semigroup

$$J^\gamma_\lambda\varphi=\int\limits_0^\infty\lambda e^{-\lambda s}G^\gamma_s\varphi\,ds,\qquad\lambda>0$$

with the property $\lim\limits_{\lambda\to\infty}J^\gamma_\lambda\varphi=\varphi$.
Now, we can construct the fractional powers of $(-\Delta_\gamma)^{\frac{\alpha}{2}}$ in terms of the so-called Balakrishnan integral formula [23]. Namely, for representation the positive fractional power $(-\Delta_\gamma)^\alpha$, $\alpha\in(0,1)$, in case of the infinitesimal generator Δ_γ of a semigroup G^γ_t, $t\geq 0$, we can use formula

$$(-\Delta_\gamma)^{\frac{\alpha}{2}}\varphi=\frac{1}{\Gamma(-\alpha)}\int\limits_0^\infty t^{-\frac{\alpha}{2}-1}(G^\gamma_t-I)\varphi\,dt.\tag{23}$$

In the case $\alpha>1$, this formula can be written with the usage of "finite differences" $(I-G^\gamma_t)^\ell$, $\ell=[\alpha]+1$:

$$(-\Delta_\gamma)^{\frac{\alpha}{2}}\varphi=\frac{1}{\Gamma(-\alpha)A_\alpha(l)}\int\limits_0^\infty t^{-\frac{\alpha}{2}-1}(I-G^\gamma_t)^\ell\varphi\,dt,\tag{24}$$

where $A_\alpha(\ell)=\sum\limits_{k=0}^{\ell}(-1)^{k-1}\begin{pmatrix}\ell\\k\end{pmatrix}=\sum\limits_{k=0}^{\ell}(-1)^{k-1}\frac{\ell!}{k!(\ell-k)!}$.
The negative power of the operator $(-\Delta_\gamma)$ for $\alpha\in(0,1)$ can be defined by equality

$$(-\Delta_\gamma)^{-\frac{\alpha}{2}}\varphi=\frac{1}{\Gamma(\alpha)}\int\limits_0^\infty t^{\frac{\alpha}{2}-1}G^\gamma_t\varphi\,dt.\tag{25}$$

Parabolic Bessel potential

In this section, we consider the singular heat conduction operator of the form

$$T_\gamma = -(\Delta_\gamma)_x + \frac{\partial}{\partial t}$$

and its negative fractional powers. The action of the operator T_γ in the images of the Hankel transform takes the form

$$\mathbf{F}_B[T_\gamma \varphi(x,t)](\xi,\tau) = (|\xi|^2 - i\tau)\mathbf{F}_B[\varphi(x,t)](\xi,\tau).$$

So we can define the negative real power of T_γ as

$$T_\gamma^\alpha \varphi(x,t) = \mathbf{F}_B^{-1}(|x|^2 - it)^{-\frac{\alpha}{2}}\mathbf{F}_B[\varphi], \qquad \alpha > 0.$$

When $\alpha = 0$, we get $T_\gamma^0\varphi(x,t) = \varphi(x,t)$, so $T_\gamma^0 = I$ is a unit operator.
Let us consider the function

$$E_\gamma^\alpha(x,t) = C_{n,\gamma}(\alpha)\begin{cases} t^{\frac{\alpha-n-|\gamma|}{2}-1}e^{-\frac{|x|^2}{4t}} & \text{if } t > 0; \\ 0 & \text{if } t \le 0, \end{cases} \tag{26}$$

$$C_{n,\gamma}(\alpha) = \frac{1}{2^{|\gamma|}\Gamma\left(\frac{\alpha}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}.$$

It is easy to see (formula Eq. 10) that when $\alpha = 1$, we get a fundamental solution $E_\gamma^1(x,t) = E_\gamma(x,t)$ of Eq. 11.
Using spherical coordinates (r,σ) and Lemma 1, we obtain

$$\begin{aligned}(\mathbf{F}_B)_{x,t}[E_\gamma^\alpha(x,t)](\xi,\tau) &= \int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}_+^n}\mathbf{j}_\gamma(x,\xi)\cdot e^{-it\tau}E_\gamma^\alpha(x,t)\,x^\gamma\,dx\,dt = \\
&= C_{n,\gamma}(\alpha)\int\limits_0^\infty\int\limits_{\mathbb{R}_+^n}\mathbf{j}_\gamma(x,\xi)\frac{e^{-it\tau-\frac{|x|^2}{4t}}}{t^{\frac{n+|\gamma|-\alpha}{2}+1}}\,x^\gamma\,dx\,dt = \{x = r\sigma\} = \\
&= C_{n,\gamma}(\alpha)\int\limits_0^\infty\frac{e^{-it\tau}}{t^{\frac{n+|\gamma|-\alpha}{2}+1}}\left(\int\limits_0^\infty e^{-\frac{r^2}{4t}}r^{n+|\gamma|-1}dr\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(r\sigma,\xi)\sigma^\gamma\,dS\right)dt = \\
&= \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^\infty e^{-(|\xi|^2-i\tau)t}t^{\frac{\alpha}{2}-1}dt = (|\xi|^2 - i\tau)^{-\frac{\alpha}{2}}.\end{aligned}$$

Therefore,

$$(\mathbf{F}_B)_{x,t}[E_\gamma^\alpha(x,t)](\xi,\tau) = (|\xi|^2 - i\tau)^{-\frac{\alpha}{2}}$$

and we can introduce parabolic Bessel potential for the function $\varphi(x,t)$ in the form

$$(T_\gamma^\alpha\varphi)(x,t)=\int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^n_+}E_\gamma^\alpha(y,\tau)({}^\gamma\mathbf{T}_x^y\varphi(x,t-\tau))y^\gamma dyd\tau=$$

$$=\int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^n_+}({}^\gamma\mathbf{T}_x^yE_\gamma^\alpha(x,t-\tau))\varphi(y)y^\gamma dyd\tau.$$

If for $t>0$ we consider generalized Gauss-Weierstrass integral of the form (see [18], formula (15))

$$(\mathscr{W}_t^\gamma\psi)(x)=\frac{2^{-|\gamma|}}{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}\frac{1}{t^{\frac{n+|\gamma|}{2}}}\int\limits_{\mathbb{R}^n_+}e^{-\frac{|y|^2}{4t}}({}^\gamma\mathbf{T}_x^y\psi(x))y^\gamma dy, \tag{27}$$

we can write

$$(T_\gamma^\alpha\varphi)(x,t)=\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^\infty\tau^{\frac{\alpha}{2}-1}((\mathscr{W}_\tau^\gamma)_x\varphi(x,t-\tau))d\tau.$$

It is remarkable that another potential, namely, generalized Bessel potential (see [19]), can also be represented as a one-dimensional integral of the generalized Gauss-Weierstrass integral (see [18], formula (21)).
A linear operator A has *strong type* $(p,q)_\gamma$, $1\le p\le\infty$, $1\le q\le\infty$ it is defined on L_p^γ, has values from L_q^γ and and the following inequality is valid:

$$\|Af\|_{q,\gamma}\le K\|f\|_{p,\gamma},\quad\forall f\in L_p^\gamma, \tag{28}$$

where constant K does not depend on f. We say that a linear operator A is an operator of *weak type* $(p,q)_\gamma$ $(1\le p\le\infty, 1\le q<\infty)$ if

$$\mu_\gamma(Af,\lambda)\le\left(\frac{K\|f\|_{p,\gamma}}{\lambda}\right)^q,\quad\forall f\in L_p^\gamma,$$

where K does not depend on f and λ, $\lambda>0$.
If $q=\infty$, then a quasilinear operator A is an operator of weak type $(p,q)_\gamma$ when it has strong type $(p,q)_\gamma$.
Marcinkiewicz's interpolation theorem was proved in general form in [24]. Here, we give a special case of this theorem, adapted for estimating integrals with power-law weights.

Theorem 3 *Let* $1\le p_i\le q_i<\infty$, $(i=1,2)$, $q_1\ne q_2$, $0<\tau<1$, $\frac{1}{p}=\frac{1-\tau}{p_1}+\frac{\tau}{p_2}$, $\frac{1}{q}=\frac{1-\tau}{q_1}+\frac{\tau}{q_2}$. *If A is a linear operator of weak type* $(p_1,q_1)_\gamma$ *and of weak type* $(p_2,q_2)_\gamma$ *with norms* K_1 *and* K_2, *respectivy, then A is an operator of strong type* $(p,q)_\gamma$ *and*

$$||Af||_{q,\gamma}\le MK_1^{1-\tau}K_2^\tau||f||_{p,\gamma}, \tag{29}$$

where $M=M(\gamma,\tau,p_1,p_2,q_1,q_2)$ *and does not depend on f and A in any other way.*

Theorem 4 *Potential* $T_\gamma^\alpha\varphi$, $\alpha>0$, *where* $\varphi\in L_p^\gamma(\mathbb{R}^{n+1}_+)$ *converges absolutely for* $0<\alpha<n+|\gamma|+2$ *and* $1\le p<\frac{n+|\gamma|+2}{\alpha}$.

Proof Let without loss of generality $\varphi(x,t)\ge0$ and

$$\|\varphi(x,t)\|_{p,\gamma}=\left(\int\limits_{\mathbb{R}^{n+1}_+}|\varphi(x,t)|^px^\gamma dxdt\right)^{1/p}=1.$$

Taking some fixed number $\mu > 0$, we can decompose $E^\alpha_\gamma(x,t)$ as

$$E^\alpha_\gamma(x,t) = {}^1E^\alpha_\gamma(x,t) + {}^2E^\alpha_\gamma(x,t) + {}^3E^\alpha_\gamma(x,t) + {}^3E^\alpha_\gamma(x,t),$$

where

$${}^1E^\alpha_\gamma(x,t) = \begin{cases} E^\alpha_\gamma(x,t) & \text{if } |x| \leq \mu \text{ and } t \in (0,1); \\ 0 & \text{if } |x| > \mu, \end{cases}$$

$${}^2E^\alpha_\gamma(x,t) = \begin{cases} E^\alpha_\gamma(x,t) & \text{if } |x| \leq \mu \text{ and } t \geq 1; \\ 0 & \text{if } |x| > \mu, \end{cases}$$

$${}^3E^\alpha_\gamma(x,t) = \begin{cases} E^\alpha_\gamma(x,t) & \text{if } |x| \leq \mu \text{ and } t \in (0,1); \\ 0 \text{ if } |x| > \mu, \end{cases}$$

$${}^4E^\alpha_\gamma(x,t) = \begin{cases} E^\alpha_\gamma(x,t) & \text{if } |x| > \mu \text{ and } t \geq 1; \\ 0 \text{ if } |x| \leq \mu. \end{cases}$$

We have

$$(\mathcal{T}^\alpha_\gamma\varphi)(x,t) = ({}^1\mathcal{T}^\alpha_\gamma\varphi)(x,t) + ({}^2\mathcal{T}^\alpha_\gamma\varphi)(x,t) + ({}^3\mathcal{T}^\alpha_\gamma\varphi)(x,t) + ({}^4\mathcal{T}^\alpha_\gamma\varphi)(x,t),$$

where

$$({}^j\mathcal{T}^\alpha_\gamma\varphi)(x,t) = \int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^n_+} {}^jE^\alpha_\gamma(y,\tau)({}^\gamma\mathbf{T}^y_x\varphi(x,t-\tau))y^\gamma dy d\tau, \qquad j = 1,2,3,4. \tag{30}$$

The integral $({}^1\mathcal{T}^\alpha_\gamma\varphi)(x,t)$ converges absolutely almost everywhere for $\varphi{\in}L^\gamma_1(\mathbb{R}^{n+1}_+)$ because $0{<}\alpha$:

$$\left|({}^1\mathcal{T}^\alpha_\gamma\varphi)(x,t)\right| \leq$$

$$\leq C_{n,\gamma}(\alpha)\int\limits_0^1\int\limits_{\{y\in\mathbb{R}^n_+:|y|<\mu\}} \tau^{\frac{\alpha-n-|\gamma|}{2}-1}e^{-\frac{|y|^2}{4t}}|({}^\gamma\mathbf{T}^y_x\varphi(x,t-\tau))|y^\gamma dy d\tau \leq$$

$$\leq \text{const}\int\limits_0^1 \tau^{\frac{\alpha-n-|\gamma|}{2}-1}d\tau\int\limits_0^\infty e^{-\frac{r^2}{4t}}r^{n+|\gamma|-1}dr = \text{const}\int\limits_0^1\tau^{\frac{\alpha}{2}-1}d\tau < \infty$$

Here, we used the fact that

$$\int\limits_0^\infty e^{-\frac{r^2}{4t}}r^{n+|\gamma|-1}dr = 2^{n+|\gamma|-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)t^{\frac{n+|\gamma|}{2}}.$$

$$\left|({}^1\mathcal{T}^\alpha_\gamma\varphi)(x,t)\right| \leq$$

$$\leq C_{n,\gamma}(\alpha)\int\limits_0^1\int\limits_{\{y\in\mathbb{R}^n_+:|y|<\mu\}}\tau^{\frac{\alpha-n-|\gamma|}{2}-1}e^{-\frac{|y|^2}{4t}}|({}^\gamma\mathbf{T}^y_x\varphi(x,t-\tau))|y^\gamma dyd\tau\leq$$

$$\leq \mathrm{const}\int\limits_0^1\tau^{\frac{\alpha-n-|\gamma|}{2}-1}d\tau\int\limits_0^\infty e^{-\frac{r^2}{4t}}r^{n+|\gamma|-1}dr=\mathrm{const}\int\limits_0^1\tau^{\frac{\alpha}{2}-1}d\tau<\infty$$

The integral $({}^2\mathcal{T}^\alpha_\gamma\varphi)(x,t)$ converges absolutely almost everywhere for $\varphi\in L^\gamma_1(\mathbb{R}^{n+1}_+)$ because $0<\alpha<n+|\gamma|+2$:

$$\left|({}^2\mathcal{T}^\alpha_\gamma\varphi)(x,t)\right|\leq$$

$$\leq C_{n,\gamma}(\alpha)\int\limits_1^\infty\int\limits_{\{y\in\mathbb{R}^n_+:|y|<\mu\}}\tau^{\frac{\alpha-n-|\gamma|}{2}-1}e^{-\frac{|y|^2}{4t}}|({}^\gamma\mathbf{T}^y_x\varphi(x,t-\tau))|y^\gamma dyd\tau\leq$$

$$\leq \mathrm{const}\int\limits_1^\infty\tau^{\frac{\alpha-n-|\gamma|}{2}-1}d\tau<\infty.$$

The integral $({}^3\mathcal{T}^\alpha_\gamma\varphi)(x,t)$ converges everywhere since for a function $\varphi\in L^\gamma_1(\mathbb{R}^{n+1}_+)$ by the same reason as $({}^1\mathcal{T}^\alpha_\gamma\varphi)(x,t)$.

Let $\varphi\in L^\gamma_p(\mathbb{R}^{n+1}_+)$, $\frac{1}{p}+\frac{1}{p'}=1$. Using Eq. 5, we obtain

$$\|({}^4\mathcal{T}^\alpha_\gamma\varphi)(x,t)\|^{p'}_{p'}\leq\mathrm{const}\int\limits_1^\infty\tau^{\left(\frac{\alpha-n-|\gamma|}{2}-1\right)p'}d\tau<\infty$$

since $1\leq p<\frac{n+|\gamma|+2}{\alpha}$. Thus, theorem is proved. □

Theorem 5 *The operator $\mathcal{T}^\alpha_\gamma$ is bounded from $L^\gamma_p(\mathbb{R}^{n+1}_+)$ to $L^\gamma_q(\mathbb{R}^{n+1}_+)$, where $1<p<\frac{n+|\gamma|+2}{\alpha}$, $q=\frac{(n+|\gamma|+2)p}{n+|\gamma|+2-\alpha p}$:*

$$\|\mathcal{T}^\alpha_\gamma\varphi\|_{q,\gamma}\leq\mathcal{C}_{n,\gamma}\|\varphi\|_{p,\gamma}.$$

Proof To apply Marcinkiewicz's theorem, we prove that the operators ${}^+\mathcal{T}^\alpha_\gamma={}^1\mathcal{T}^\alpha_\gamma+{}^3\mathcal{T}^\alpha_\gamma$ and ${}^-\mathcal{T}^\alpha_\gamma={}^2\mathcal{T}^\alpha_\gamma+{}^4\mathcal{T}^\alpha_\gamma$ (see (30)) have weak types $(p_1,q_1)_\gamma$ and $(p_2,q_2)_\gamma$, where p_1,q_1,p_2,q_2 such that $\frac{1}{p}=\frac{1-\tau}{p_1}+\frac{\tau}{p_2}$, $\frac{1}{q}=\frac{1-\tau}{q_1}+\frac{\tau}{q_2}$, $0<\tau<1$. For this, we obtain an estimate for

$$\sup_{0<\lambda<\infty}\lambda(\mu_\gamma({}^\pm\mathcal{T}^\alpha_\gamma\varphi,\lambda))^{1/p}=$$

$$=\sup_{0<\lambda<\infty}\lambda\Big(\mathrm{mes}_\gamma\{(x,t)\in\mathbb{R}^{n+1}_+\;:\;|({}^\pm\mathcal{T}^\alpha_\gamma\varphi(x,t)|>\lambda\}\Big).$$

For this, it suffices to estimate

$$\mathrm{mes}_\gamma\{(x,t)\in\mathbb{R}^{n+1}_+\;:\;|{}^j\mathcal{T}^\alpha_\gamma\varphi(x,t)|>\lambda\},\qquad j=1,2,3,4 \tag{31}$$

and apply inequality

$$\mathrm{mes}_\gamma\{(x,t)\in\mathbb{R}^{n+1}_+\;:\;|A+B|>2\lambda\}\leq$$

$$\leq \mathrm{mes}_\gamma\{(x,t)\in\mathbb{R}^{n+1}_+ \,:\, |A|>\lambda\} + \mathrm{mes}_\gamma\{(x,t)\in\mathbb{R}^{n+1}_+ \,:\, |B|>\lambda\}.$$

But estimates for Eq. 31 are obtained from Eq. 2 and from theorem 4. Thus, the proof follows by the Marcinkiewicz interpolation theorem 3. □

Parabolic Bessel potential can be used to solve iterated non-homogeneous equation

$$\left((\Delta_\gamma)_x - \frac{\partial}{\partial t}\right)^m u(x,t) = f(x,t), \qquad m\in\mathbb{N}.$$

Conclusion

In this paper, we constructed and studied a solution for the singular parabolic differential equation $(\Delta_\gamma)_x u(x,t) = u_t(x,t)$, where $\Delta_\gamma = \sum\limits_{i=1}^{n} B_{\gamma_i}$ is Laplace-Bessel operator, $B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$, $i = 1, ..., n$. Using fundamental solution, we define singular thermal potential which is a solution to the Cauchy problem for the multidimensional singular heat equation. We obtained semigroup properties of operator G_t^γ and constructed fractional powers of $(-\Delta_\gamma)$ using Balakrishnan formulas. The last section of this paper deals with the description of the negative real power of a singular parabolic operator. This operator is called parabolic Bessel potential. The singularity of this potential is generated by the singular Bessel differential operator. The boundedness of this potential is proved here. The approach to the study of parabolic Bessel potentials involves the Fourier-Bessel transform and the Marcinkiewicz interpolation theorem.

Declarations

Conflict of interest The authors declare no competing interests.

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