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ON RECOVERY OF THE SOLUTION TO THE CAUCHY PROBLEM FOR THE SINGULAR HEAT EQUATION

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Abstract. We present the results related to the solution of the problem of the best recovery of the solution to the Cauchy problem for the heat equation with the B-elliptic Laplace–Bessel operator in spatial variables from an exactly or approximately known finite set of temperature profiles.

Keywords: Laplace–Bessel operator, optimal recovery, Fourier–Bessel transform, heat equation, singular equation.

1. Introduction

It is well known that the temperature distribution in $\mathbb{R}^N$ is described by the equation

$$\frac{\partial u}{\partial t} = \Delta u + f(x,t),$$

where $\Delta = \partial^2/\partial x_1^2 + \ldots \partial^2/\partial x_n^2$ is the Laplace operator in $\mathbb{R}^N$.

The authors of [13] state the following problem. Let temperature distributions $u(\cdot,t_1),\ldots,u(\cdot,t_p)$ be known at moments of time $0 \le t_1 < \cdots < t_p$ that are given approximately. More precisely, functions $y_j(\cdot) \in L_2(\mathbb{R}^N)$ are such that $\|u(\cdot,t_j) - y_j(\cdot)\|_{L_2(\mathbb{R}^N)} \le \varepsilon_j$, where $\varepsilon_j > 0$, $j = 1,\ldots,p$. For each set of such functions, we want to obtain a function in $L_2(\mathbb{R}^N)$ that best approximates the real temperature distribution in $\mathbb{R}^N$ at a fixed time τ in some sense. In [14], the problem of restoring the temperature of a pipe from inaccurate measurements, closely related to the one described above, is considered.

We study a similar problem for a singular heat equation with the Bessel operator [2, 6–12, 16–18, 22, 24]. Features of the above type arise in models of mathematical physics in cases where characteristics of media (e.g., diffusion characteristics or thermal conductivity characteristics) have degenerate power-type inhomogeneities. In addition, such equations lead to situations where isotropic diffusion processes with axial or spherical symmetry are studied.

It should be noted that the problem under consideration is closely related to the problem of restoring powers of the Laplace operator from an incomplete spectrum considered in [15, 23]. These results were transferred to the case of the Laplace–Bessel operator in [19, 21].

2. Prerequisites

Let $\mathbb{R}_+^N = \{x = (x', x''),\ x'=(x_1,\ldots,x_n),\ x''=(x_{n+1},\ldots,x_N),\ x_1>0,\ldots,x_n>0\}$, $\gamma = (\gamma_1,\ldots,\gamma_n)$, $\nu_\kappa = (\gamma_\kappa - 1)/2$, $(x')^\gamma = \prod\limits_{\kappa=1}^{n} x_\kappa^{\gamma_\kappa}$, $\gamma_\kappa > 0$, $\kappa = 1,\ldots,n$. Let Ω^+ denote the domain adjacent to hyperplanes $x_1 = 0,\ldots,x_n = 0$. The boundary of the domain Ω^+ consists of two parts: Γ^+, located in the part of the space $\mathbb{R}_N^+$, and Γ_0, belonging to hyperplanes $x_1 = 0,\ldots,x_n = 0$.

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Let $L_p^\gamma(\Omega^+)$ denote the linear space of functions such that

$$\|f\|_{L_p^\gamma(\Omega^+)} = \left(\int\limits_{\Omega^+} |f(x)|^p \, (x')^\gamma \, dx \right)^{1/p} < +\infty.$$

Let $\Omega \subset \mathbb{R}^N$ be the union of sets Ω^+ and Ω^- obtained from Ω^+ symmetrically with respect to the space $x' = 0$.

Let Ω_ε^+ be an internal subdomain of Ω^+ adjacent to the boundary Γ_0, all points of which are at a distance of more than ε from the part of the boundary Γ^+ of domain Ω^+. Then the domain Ω_ε^+ is called a *symmetrically internal* (*s-internal*) subdomain of Ω^+.

Let $L_{p,loc}^\gamma(\Omega^+)$ denote the linear space of functions such that

$$\int\limits_{\Omega_\varepsilon^+} |f(x)|^p \, (x')^\gamma \, dx < +\infty$$

for any s-internal subdomain Ω_ε^+ of Ω^+.

Let $\mathcal{D}_{ev}(\Omega^+)$ ($\mathcal{E}_{ev}(\Omega^+)$) denote the set of all restrictions of even functions with respect to variables x' from space $\mathcal{D}(\Omega)$ (space $\mathcal{E}(\Omega)$) onto the set Ω^+. Topology in the space $\mathcal{D}_{ev}(\Omega^+)$ (in the space $\mathcal{E}_{ev}(\Omega^+)$) is induced by the topology of the space $\mathcal{D}(\Omega)$ (of the space $\mathcal{E}(\Omega)$). By definition, $\mathcal{D}_{ev} = \mathcal{D}_{ev}(R_+^N)$. Let $\mathcal{S}_{ev}$ denote the linear space of functions $\varphi(x) \in C_{ev}^\infty(R_+^N)$ decreasing as $|x| \to \infty$ together with their derivatives faster than any power of $|x|^{-1}$. Topology in $\mathcal{S}_{ev}$ is introduced in the same way as in the space $\mathcal{S}$ (see [4, 10]). The space dual to $\mathcal{D}_{ev}(\Omega^+)$ ($\mathcal{E}_{ev}(\Omega^+)$, $\mathcal{S}_{ev}$) with its weak topology is denoted as $\mathcal{D}'_{ev}(\Omega^+)$ ($\mathcal{E}'_{ev}(\Omega^+)$, $\mathcal{S}'_{ev}$). The following relations hold: $\mathcal{D}_{ev} \subset \mathcal{S}_{ev} \subset \mathcal{S}'_{ev} \subset \mathcal{D}'_{ev}$.

The action of the functional (distribution) f on the test (main) function φ in all three cases is denoted as

$$\langle f(x), \varphi(x) \rangle_\gamma = \langle f(x), \varphi(x) \rangle . \tag{2.1}$$

The index γ is sometimes omitted if this does not cause confusion.

We identify each function $f(x) \in L_{1,loc}^\gamma(\Omega^+)$ with a functional $f \in \mathcal{D}'_{ev}(\Omega^+)$ using the formula

$$\langle f(x), \varphi(x) \rangle = \int\limits_{\Omega^+} f(x)\varphi(x) \, (x')^\gamma \, dx, \tag{2.2}$$

and we call such functionals *regular*. All other functionals from the space $\mathcal{D}'_{ev}(\Omega^+)$ are called *singular*. However, although Eq. (2.2) cannot be extended to singular functionals similarly to [3], besides from notation (2.1), it is possible for all functionals (including singular ones) to use notation (2.2).

An important example of a singular functional in $\mathcal{D}'_{ev}(\Omega^+)$ is the weight *δ-function* $\delta_\gamma(x)$ defined by the equality

$$\langle \delta_\gamma(x), \, \varphi \rangle_\gamma = \varphi(0).$$

The mixed generalized shift is defined by the formula

$$T^y f(x) = \prod_{i=1}^{n} T_{x_i}^{y_i} f(x', x'' - y''),$$

where each of the generalized shifts $T_{x_i}^{y_i}$ is defined by the formula (see [11])

$$(T_{x_i}^{y_i} f)(x) = \frac{\Gamma(\frac{\gamma_i+1}{2})}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma_i}{2}\right)} \int\limits_0^\pi f\left(x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos\alpha}, x_{i+1}, \ldots, x_N\right) \sin^{\gamma_i - 1}\alpha \, d\alpha, \tag{2.3}$$

$i = 1, \ldots, n,$ and the product $\prod\limits_{k=1}^{n} T_{x_k}^{y_k}$ is understood as a product (superposition) of operators.

The *generalized convolution of functions* $f,\ g \in L_p^\gamma(R_N^+)$ is defined by the formula

$$(f*g)_\gamma(x) = \int\limits_{R_N^+} f(y) T_x^y g(x) (y')^\gamma dy. \tag{2.4}$$

If $f \in \mathcal{D}'_{ev},\ g \in \mathcal{E}'_{ev}$, then the *generalized convolution* $(f*g)_\gamma$ of such distributions is defined by the equality

$$\langle (f*g)_\gamma(x), \varphi(x)\rangle_\gamma = \langle f(y), \langle g(x), T_x^y \varphi(x)\rangle_\gamma\rangle_\gamma,\ \varphi(x) \in \mathcal{D}_{ev}. \tag{2.5}$$

The *j-Bessel function* of order ν is defined by the formula

$$j_\nu(z) = \frac{2^\nu \Gamma(\nu+1)}{z^\nu} J_\nu(z) = \Gamma(\nu+1) \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{2^{2m}\, m! \Gamma(m+\nu+1)},$$

where $\Gamma(\cdot)$ is the *Euler gamma function*, and

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+\nu}}{2^{2m+\nu}\, m! \Gamma(m+\nu+1)},$$

is the *Bessel function* of the first kind of order ν.

Direct $F_{B,\gamma} = F_B = F_\gamma$ and inverse $F_{B,\gamma}^{-1} = F_B^{-1} = F_\gamma^{-1}$ mixed *Fourier–Bessel transforms* are defined by the formulas

$$F_{B,\gamma}[\varphi(x',x'')](\xi) = \int\limits_{R_N^+} \varphi(x) \prod_{k=1}^{n} j_{\nu_k}(\xi_k x_k) e^{-ix''\cdot\xi''} (x')^\gamma\, dx$$
$$= (2\pi)^{N-n} 2^{2|\nu|} \prod_{k=1}^{n} \Gamma^2(\nu_k+1) F_{B,\gamma}^{-1}[\psi(x',-x'')](\xi), \tag{2.6}$$

where

$$x'\cdot\xi' = x_1\xi_1 + \ldots + x_n\xi_n, \quad x''\cdot\xi'' = x_{n+1}\xi_{n+1} + \ldots + x_N\xi_N, |\nu| = \nu_1 + \ldots + \nu_n.$$

The *Parseval–Plancherel formula* is valid for the Fourier–Bessel transform (see [7]):

$$\|\varphi\|_{L_2^\gamma} = (2\pi)^{N-n} 2^{2|\nu|} \prod_{k=1}^{n} \Gamma^2(\nu_k+1) \|\widehat{\varphi}\|_{L_2^\gamma}, \quad \widehat{\varphi} = F_B[\varphi].$$

The Fourier–Bessel transform is defined and invertible for functions of $S_{ev}(R_+^N)$ (see [7]).

Below we use the notation

$$\mathbf{\Pi} = (2\pi)^{N-n} 2^{2|\nu|} \prod_{k=1}^{n} \Gamma^2(\nu_k+1). \tag{2.7}$$

The *B-elliptic operator* Δ_B (the term and notation introduced by Kipriyanov in [8]), also called the *Laplace–Bessel operator*, is defined by the formula

$$\Delta_B u = \sum_{k=1}^{n} \left(\frac{\partial^2 u}{\partial x_k^2} + \frac{\gamma_k}{x_k}\frac{\partial u}{\partial x_k} \right) + \sum_{k=n+1}^{N} \frac{\partial^2 u}{\partial x_k^2} = \sum_{k=1}^{n} B_{x_k} u + \sum_{k=n+1}^{N} \frac{\partial^2 u}{\partial x_k^2}, \tag{2.8}$$

where $B_{x_x} = B_{x_x,\gamma_k}$ is the *Bessel operator* acting on the variable x_k by the formula

$$B_{x_k} u = B_{x_k,\gamma_k} u = \frac{\partial^2 u}{\partial x_k^2} + \frac{\gamma_k}{x_k}\frac{\partial u}{\partial x_k} = x_k^{-\gamma_k} \frac{\partial}{\partial x_k}\left(x_k^{\gamma_k} \frac{\partial u}{\partial x_k} \right). \tag{2.9}$$

We also note useful relations that include the Fourier–Bessel transform and the generalized shift operator (also see [7]).

$$F_B\left[T_x^y\varphi(x)\right](\xi) = \prod_{k=1}^{n} j_{\nu_k}(\xi_k y_k) e^{-iy''\dot{\xi}''} F_B[\varphi(x)](\xi), \tag{2.10}$$

$$T_x^y F_B\left[\psi(\xi)\right](x) = F_B\left[\prod_{k=1}^{n} j_{\nu_k}(\xi_k y_k) e^{iy''\dot{\xi}''}\psi(\xi)\right](x). \tag{2.11}$$

$$T_x^y \delta_\gamma(x) = \delta_\gamma(y), \tag{2.12}$$

$$F_B\left[\Delta_B u(\cdot)\right](\xi) = -|\xi|^2 F_B\left[u(\cdot)\right](\xi). \tag{2.13}$$

3. Problem Statement

Consider the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = \Delta_B u, \quad x \in \mathbb{R}_+^N,\ t > 0, \tag{3.1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}_+^N. \tag{3.2}$$

We assume that $u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+^N)$. The only solution to this problem for the case $N = n = 1$ was obtained in [24]. It is expressed by the following formula, which generalizes the well-known Poisson formula:

$$u(x,t) = P_t\, u_0(\cdot)(x,t) = \frac{1}{2tx^\nu} \int\limits_{\mathbb{R}_+} \eta^{\nu+1}\, u_0(\eta)\, I_\nu\left(\frac{\eta x}{2t}\right) \exp\left(-\frac{\eta^2 + x^2}{4t}\right) d\eta, \tag{3.3}$$

where

$$I_\nu(z) = \sum_{m=1}^{\infty} \frac{z^{2m+\nu}}{2^{2m+\nu} m!\Gamma(m+\nu+1)}$$

is the *modified Bessel function* of the first kind of order ν, $\Gamma(\cdot)$ is the Euler gamma function. For $N \geq n \geq 1$, the explicit representation of the unique solution to problem (3.1)-(3.2) has the form

$$u(x,t) = P_t\, u_0(\cdot)(x,t)$$
$$= \frac{1}{2^N \pi^{(N-n)/2} t^{(N+n)/2} x^\nu} \int\limits_{\mathbb{R}_+^N} \exp\left(-\frac{|x-\eta|^2 - 2x''\cdot\eta''}{4t}\right) \prod_{\kappa=1}^{n}\left(\eta_\kappa^{\nu_\kappa+1}\ I_{\nu_\kappa}\left(\frac{\eta_\kappa x_\kappa}{2t}\right)\right) u_0(\eta)\, d\eta. \tag{3.4}$$

Formula (3.4) can be obtained by applying the Fourier–Bessel transform. However, there is no point in presenting the method for obtaining this formula here since a more general formula for the differential-difference equation was obtained in [18].

Consider the following problem. Let the functions $y_j(\cdot) \in L_2^\gamma(\mathbb{R}_+^N)$ be known at moments $0 \leq t_1 < \cdots < t_p$ and

$$\|u(\cdot, t_j) - y_j(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)} \leq \varepsilon_j,\ j = 1, \ldots, p,$$

where $\varepsilon_j > 0,\ j = 1, \ldots, p$. It is required to assign a function from $L_2^\gamma(\mathbb{R}_+)$ to each such set of functions. The assigned function has to best approximate the true temperature distribution in $\mathbb{R}_+^N$ at a fixed time τ in some sense. For the case $N = n = 1$ this problem is considered in [20]. In this paper, we set $N \geq n \geq 1$.

Following [13], any mapping $m : L_2^\gamma(\mathbb{R}_+^N) \times \cdots \times L_2^\gamma(\mathbb{R}_+^N) \longrightarrow L_2^\gamma(\mathbb{R}_+^N)$ is called the *recovery method* (of temperature in $\mathbb{R}_+^N$ at time τ according to the data). The value

$$e(\tau, \overline{\varepsilon}, m) = \sup_U \|u(\cdot, \tau) - m(\overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)},$$

where $\overline{y}(\cdot)=(y_1(\cdot),\dots,y_p(\cdot))$, $\overline{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_p)$,

$$U=\{(u_0(\cdot),\overline{y}(\cdot)):u_0(\cdot)\in L_2^\gamma(\mathbb{R}_+^N),\ \overline{y}(\cdot)\in(L_2^\gamma(\mathbb{R}_+^N))^p,\ \ \|u(\cdot,t_j)-y_j(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}\le\varepsilon_j,\ j=1,\dots,p\},$$

is called the *error* of this method. The value

$$E(\tau,\overline{\varepsilon})=\inf_{m:(L_2^\gamma(\mathbb{R}_+^N))^p\longrightarrow L_2^\gamma(\mathbb{R}_+^N)}e(\tau,\overline{\varepsilon},m)$$

is called the *optimal recovery error*. The method $\widehat{m}$, for which

$$E(\tau,\overline{\varepsilon})=e(\tau,\overline{\varepsilon},\widehat{m}),$$

is called the *optimal recovery method*.

4. Problem Solution

Let $P_t:L_2^\gamma(\mathbb{R})\longrightarrow L_2^\gamma(\mathbb{R}_+^N)$ be the operator defined by formula (3.4):

$$P_tu_0(\cdot)(x,t)=\frac{1}{2^N\pi^{(N-n)/2}t^{(N+n)/2}x^\nu}\int\limits_{\mathbb{R}_+^N}u_0(\eta)\exp\left(-\frac{|x-\eta|^2-2x''\cdot\eta''}{4t}\right)\prod_{\kappa=1}^n\left(\eta_\kappa^{\nu_\kappa+1}I_{\nu_\kappa}\left(\frac{\eta_\kappa x_\kappa}{2t}\right)\right)d\eta,$$

$t>0$ be a fixed value, P_0 be the identity operator.

Let $\tau\ge0$. Consider the following problem:

$$\|P_\tau u_0(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}\longrightarrow\max,\tag{4.1}$$

$$\|P_{t_j}u_0(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}\le\varepsilon_j,\quad j=1,\dots,p,\ u_0(\cdot)\in L_2^\gamma(\mathbb{R}_+^N).\tag{4.2}$$

A function that satisfies condition (4.2) is called an *admissible function* for problem (4.1)-(4.2).

Let S denote the upper bound of $\|P_\tau u_0(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}$ with condition (4.2).

Lemma 4.1.

$$E(\tau,\overline{\varepsilon})\ge S.$$

Proof. Let $\overline{u}_0(\cdot)$ be an admissible function for problem (4.1)-(4.2). Then $-\overline{u}_0(\cdot)$ is an admissible function for problem (4.1)-(4.2). For any method $m:(L_2^\gamma(\mathbb{R}_+^N))^p\longrightarrow L_2^\gamma(\mathbb{R}_+^N)$ we have:

$$\begin{aligned}2\|P_\tau\overline{u}_0(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}&=\|P_\tau\overline{u}_0(\cdot)-m(0)(\cdot)+m(0)(\cdot)-P_\tau(-\overline{u}_0(\cdot))\|_{L_2^\gamma(\mathbb{R}_+^N)}\\&\le\|P_\tau\overline{u}_0(\cdot)-m(0)(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}+\|m(0)(\cdot)-P_\tau(-\overline{u}_0(\cdot))\|_{L_2^\gamma(\mathbb{R}_+^N)}\\&\le2\sup_{\substack{u_0(\cdot)\in L_2^\gamma(\mathbb{R}_+^N)\\ \|P_{t_j}u_0(\cdot)\|_{L_2^\gamma(\mathbb{R})}\le\varepsilon_j,\ j=1,\dots,p}}\|P_\tau u_0(\cdot)-m(0)(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}\le2\sup_U\|P_\tau u_0(\cdot)-m(\overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}.\end{aligned}$$

On the left-hand side of the resulting inequality we pass to the supremum of admissible functions, and on the right-hand side we pass to the infimum over all methods. This step completes the proof of the lemma. □

Using formula 6.633 (4) from [5], one could easily verify the validity of the equality

$$F_\gamma[P_tu_0(\cdot)](\xi)=\exp(-|\xi|^2t)F_\gamma u_0(\xi).$$

Therefore, according to the Parseval–Plancherel theorem for the Fourier–Bessel transform, the square of the value of problem (4.1)-(4.2) is equal to the value of the following problem

$$:\frac{1}{\mathbf{\Pi}}\int\limits_{\mathbb{R}_+^N}\xi^\gamma e^{-2|\xi|^2\tau}|F_\gamma u_0(\xi)|^2\,d\xi\longrightarrow\max,\quad u_0(\cdot)\in L_2^\gamma(\mathbb{R}_+^N),\tag{4.3}$$

$$\frac{1}{\mathbf{\Pi}}\int\limits_{\mathbb{R}^N_+} \xi^\gamma e^{-2|\xi|^2 t_j}|F_\gamma u_0(\xi)|^2\,d\xi \le \varepsilon_j^2,\ \ j=1,\dots,p. \tag{4.4}$$

Let us move from problem (4.3)-(4.4) to an extended problem (according to the terminology in [13]). In order to do so, replace $\mathbf{\Pi}^{-1}\,|F_\gamma u_0(\xi)|^2\xi^{2\nu+1}\,d\xi$ by a positive measure $d\mu(\xi)$. As a result, we obtain the following problem:

$$\int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2\tau}\,d\mu(\xi) \longrightarrow \max, \tag{4.5}$$

$$\int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2 t_j}\,d\mu(\xi) \le \varepsilon_j^2,\ \ j=1,\dots,p. \tag{4.6}$$

Any measure that satisfies conditions (4.6) is called *admissible* for problem (4.5)-(4.6). An admissible measure $d\widehat{\mu}(\xi)$ such that

$$\int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2\tau}\,d\widehat{\mu}(\xi) = \max\int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2\tau}\,d\mu(\xi), \tag{4.7}$$

where the maximum is taken over all admissible measures, is called a *solution* to problem (4.5)-(4.6).

The Lagrange function for this problem has the form

$$\mathcal{L}(d\mu(\cdot),\lambda) = \lambda_0\int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2\tau}\,d\mu(\xi) + \sum_{j=1}^p \lambda_j\left(\int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2 t_j}\,d\mu(\xi) - \varepsilon_j^2\right),$$

where $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_p)$ is a set of Lagrange multipliers. Extended problem (4.5)-(4.6) is solved in [13]. For the sake of completeness, we rewrite this solution slightly changing the specific values to suit our needs. On the two-dimensional plane (t,y) construct the set

$$M = \mathrm{co}\left\{\left(t_j,\log\left(\frac{1}{\varepsilon_j}\right)\right)\, j=1,\dots,p\right\} + \{(t,0): t\ge 0\},$$

where $\mathrm{co}\,A$ denotes the convex hull of the set A. We introduce the function $\theta(t)$ on the ray $[0,+\infty)$ by the formula

$$\theta(t) = \max\{y : (t,y)\in M\},$$

assuming that $\theta(t)=-\infty$ if $(t,y)\notin M$ for all y. The graph of the function $\theta(t)$ on the ray $[t_1,+\infty)$ is an upward-directed convex (concave) polygonal chain. Let $t_1=t_{s_1}<t_{s_2}<\dots<t_{s_\varrho}$ be its vertices. Obviously, $\{t_{s_1}<t_{s_2}<\dots<t_{s_\varrho}\}\subseteq\{t_1<t_2<\dots<t_p\}$.

There are three cases to consider.

(a) Let $\tau\ge t_1$ and there be an inflection point of the function $\theta(t)$ to the right of τ. Suppose that $\tau\in[t_{s_j},t_{s_{j+1}})$. Let $d\widehat{\mu}(\xi)=x^\gamma T_\xi^{\xi_0}\delta_\gamma$, where parameters A and ξ_0 are determined from the conditions

$$\int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2\tau}\,d\widehat{\mu}(\xi) = Ae^{-2|\xi_0|^2 t_k} = \varepsilon_k^2,\quad k=s_j,s_{j+1}. \tag{4.8}$$

From condition (4.8) we obtain:

$$A = \varepsilon_{s_j}^{2t_{s_{j+1}}/(t_{s_{j+1}}-t_{s_j})}\varepsilon_{s_{j+1}}^{-2t_{s_j}/(t_{s_{j+1}}-t_{s_j})},$$

$$|\xi_0|^2 = \frac{\log\varepsilon_{s_j}/\varepsilon_{s_{j+1}}}{t_{s_{j+1}}-t_{s_j}} = \frac{\log(1/\varepsilon_{s_{j+1}})-\log(1/\varepsilon_{s_j})}{t_{s_{j+1}}-t_{s_j}}.$$

Let $\widehat{\lambda}_0 = -1$, $\widehat{\lambda}_k = 0$, $k \neq s_j, s_{j+1}$. In order to find the values λ_{s_j}, $\lambda_{s_{j+1}}$, some preparations are needed. Let

$$f(v) = \lambda_0 + \sum_{j=1}^{p} \lambda_j e^{-2v(t_j-\tau)}.$$

Suppose that $f(|\xi_0|^2) = f'(|\xi_0|^2) = 0$. Then we obtain a system of linear equations for λ_{s_j}, $\lambda_{s_{j+1}}$:

$$\lambda_{s_j} e^{-2|\xi_0|^2(t_{s_j}-\tau)} + \lambda_{s_{j+1}} e^{-2|\xi_0|^2(t_{s_{j+1}}-\tau)} = 1,$$
$$\lambda_{s_j}(t_{s_j}-\tau) e^{-2|\xi_0|^2(t_{s_j}-\tau)} + \lambda_{s_{j+1}}(t_{s_{j+1}}-\tau) e^{-2|\xi_0|^2(t_{s_{j+1}}-\tau)} = 0,$$

whence we obtain

$$\lambda_{s_j} = \frac{t_{s_{j+1}}-\tau}{t_{s_{j+1}}-t_{s_j}} \left(\frac{\varepsilon_{s_{j+1}}}{\varepsilon_{s_j}}\right)^{2(\tau-t_{s_j})/(t_{s_{j+1}}-t_{s_j})},$$
$$\lambda_{s_{j+1}} = \frac{\tau-t_{s_j}}{t_{s_{j+1}}-t_{s_j}} \left(\frac{\varepsilon_{s_j}}{\varepsilon_{s_{j+1}}}\right)^{2(t_{s_{j+1}}-\tau)/(t_{s_{j+1}}-t_{s_j})}.$$

For a measure $d\widehat{\mu}(\xi)$ we have

$$\min_{d\mu(\cdot)\geq 0} \mathcal{L}(d\mu(\cdot), \widehat{\lambda}) = \mathcal{L}(d\widehat{\mu}(\cdot), \widehat{\lambda}), \tag{4.9}$$

$$\widehat{\lambda}_j \left(\int\limits_{\mathbb{R}_+^N} e^{-2|\xi|^2\tau}\, d\widehat{\mu}(\xi) - \varepsilon_j^2 \right) = 0, \ j = 1, \ldots, p. \tag{4.10}$$

Hence, for any admissible measure $d\mu(\xi)$

$$\widehat{\lambda}_0 \int\limits_{\mathbb{R}_+^N} e^{-2|\xi|^2\tau}\, d\mu \geq \widehat{\lambda}_0 \int\limits_{\mathbb{R}_+^N} e^{-2|\xi|^2\tau}\, d\mu + \widehat{\lambda}_j \left(\int\limits_{\mathbb{R}_+^N} e^{-2|\xi|^2\tau}\, d\mu(\xi) - \varepsilon_j^2 \right)$$

$$\geq \widehat{\lambda}_0 \int\limits_{\mathbb{R}_+^N} e^{-2|\xi|^2\tau}\, d\widehat{\mu} + \widehat{\lambda}_j \left(\int\limits_{\mathbb{R}_+^N} e^{-2|\xi|^2\tau}\, d\widehat{\mu}(\xi) - \varepsilon_j^2 \right) = \widehat{\lambda}_0 \int\limits_{\mathbb{R}_+^N} e^{-2|\xi|^2\tau}\, d\widehat{\mu}.$$

Dividing by $\widehat{\lambda}_0 < 0$, we get

$$\int\limits_{\mathbb{R}_+^N} e^{-2|\xi|^2\tau}\, d\mu \leq \widehat{\lambda}_0 \int\limits_{\mathbb{R}_+^N} e^{-2|\xi|^2\tau}\, d\widehat{\mu}. \tag{4.11}$$

Let

$$\rho(t) = \frac{\log(1/\varepsilon_{s_{j+1}}) - \log(1/\varepsilon_{s_j})}{t_{s_{j+1}}-t_{s_j}}(t-t_{s_j}) + \log(1/\varepsilon_{s_j}).$$

The line $y = \rho(t)$ passes through points $(t_{s_j}, \log(1/\varepsilon_{s_j}))$ and $(t_{s_{j+1}}, \log(1/\varepsilon_{s_{j+1}}))$ and lies at least below the graph of the function $y = \theta(t)$. For the obtained values A and $|\xi_0|^2$ we have:

$$\int\limits_{\mathbb{R}_+^N} e^{-2|\xi|^2 t_i}\, d\widehat{\mu}(\xi) = A e^{-2|\xi_0|^2 t_i} = \varepsilon_{s_j}^{2(t_{s_{j+1}}-t_i)/(t_{s_{j+1}}-t_{s_j})} \varepsilon_{s_{j+1}}^{2(t_i-t_{s_j})/(t_{s_{j+1}}-t_{s_j})}$$
$$= e^{-2\rho(t_i)} \leq e^{-2\log(1/\varepsilon_i)} = \varepsilon_i^2, \quad i = 1, \ldots, p.$$

This means that $d\widehat{\mu}(\xi)$ is an admissible measure for the extended problem (4.5)-(4.6) and is its solution. If we substitute $d\widehat{\mu}(\xi)$ into the functional defined in (4.5), we get the value of problem (4.5)-(4.6), which is also a solution to problem (4.3)-(4.4):

$$\int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2\tau}\, d\widehat{\mu}(\xi) = Ae^{-2|\xi_0|^2\tau} = \varepsilon_{s_j}^{2(t_{s_{j+1}}-\tau)/(t_{s_{j+1}}-t_{s_j})}\varepsilon_{s_{j+1}}^{2(\tau-t_{s_j})/(t_{s_{j+1}}-t_{s_j})} = e^{-2\rho(\tau)} = e^{-2\theta(\tau)}.$$

This means that the value of problem (4.1)-(4.2) is equal to $S = e^{-\theta(\tau)}$.

(b) Let $\tau \geq t_{s_\varrho}$. If the graph of the function $y = \theta(t)$ is a straight line, then $t_{s_\varrho} = t_1$. In this case, we set $\widehat{\lambda}_0 = -1$, $\widehat{\lambda}_{s_\varrho} = 1$, $\widehat{\lambda}_{s_j} = 0$, where $j \neq \varrho$, $d\widehat{\mu}(\xi) = x^\gamma \varepsilon_{s_\varrho}\delta_\gamma(\xi)$. Obviously, condition (4.10) is satisfied. In addition, for all $\xi \in \mathbb{R}^N_+$, the inequality

$$f(|\xi|^2) = -1 + e^{-2|\xi|^2(t_{s_\varrho}-\tau)} \geq 0$$

holds and the equality $f(0) = 0$ is valid. Therefore, condition (4.9) is valid as well. The equality $\theta(t) \equiv \log(1/\varepsilon_{s_\varrho})$ is an identity on the ray $[t_{s_\varrho}, +\infty)$. Therefore, $\log(1/\varepsilon_j) \leq \log(1/\varepsilon_{s_\varrho})$,$j = 1,\dots,p$. Hence,

$$\int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2 t_j}\, d\widehat{\mu}(\xi) = \varepsilon_{s_\varrho}^2 = e^{-2\log(1/\varepsilon_{s_\varrho})}.$$

Therefore, the measure $d\widehat{\mu}(\xi)$ is admissible for problem (4.5)-(4.6) and is its solution. The value of this problem is obtained as follows:

$$\int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2\tau}\, d\widehat{\mu}(\xi) = \varepsilon_{s_\varrho}^2 = e^{-2\log(1/\varepsilon_{s_\varrho})} = e^{-2\theta(t)}.$$

This again means that the solution to problem (4.1)-(4.2) is equal to $S = e^{-\theta(\tau)}$.

(c) Let $\tau < t_1$. For arbitrary $y_0 > 0$ there exists a straight line defined by the equation $y = at + b$, $a > 0$, dividing the point $(\tau, -y_0)$ and the set M. Moreover,

$$-a\tau - y_0 \geq b \geq -at_j + \log(1/\varepsilon_{s_j}),\ j = 1,\dots,p.$$

Let $A = e^{-2b}$. Choose $\xi_0 \in \mathbb{R}^N_+$ such that $|\xi_0|^2 = a$. Then

$$Ae^{-2|\xi_0|^2 t_j} \leq \varepsilon_j^2,\ j = 1,\dots,p.$$

This means that the measure $d\widehat{\mu}(\xi) = x^\gamma T_\xi^{\xi_0}\delta_\gamma(\xi)$ is admissible for problem (4.5)-(4.6) and $Ae^{-2|\xi_0|^2\tau} \geq e^{2y_0}$. Due to arbitrariness of $y_0 > 0$ the value of problem (4.5)-(4.6), as well as the solution to problem (4.1)-(4.2), is equal to $+\infty$.

In all three cases, for all $\tau \geq 0$, the optimal recovery error is estimated from below as $E(\tau, \overline{\varepsilon}) \geq e^{-\theta(\tau)}$.

Let $\tau \geq t_1$ and $\widehat{\lambda}_1, \dots, \widehat{\lambda}_p$ be the Lagrange multipliers from cases (a), (b) for such values of τ.

Lemma 4.2. *Let the problem*

$$\sum_{j=1}^{p} \widehat{\lambda}_j \|P_{t_j}u_0(\cdot) - y_j(\cdot)\|^2_{L_2^\gamma(\mathbb{R}^N_+)} \longrightarrow \min, \quad u_0(\cdot) \in L_2^\gamma(\mathbb{R}^N_+), \tag{4.12}$$

have the solution $\widehat{u}_0(\cdot) = \widehat{u}_0(\cdot, \overline{y}(\cdot))$ for a set of functions $\overline{y}(\cdot) = (y_1(\cdot), \dots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}^N_+))^p$. Then for all $\sigma_1, \dots, \sigma_p$ the value of the problem

$$\|P_\tau u_0(\cdot) - P_\tau \widehat{u}_0(\cdot)\|^2_{L_2^\gamma(\mathbb{R}^N_+)} \longrightarrow \max, \quad u_0(\cdot) \in L_2^\gamma(\mathbb{R}^N_+), \tag{4.13}$$

$$\|P_{t_j}u_0(\cdot) - y_j(\cdot)\|_{L_2^\gamma(\mathbb{R}^N_+)} \leq \sigma_j\ j = 1,\dots,p, \tag{4.14}$$

does not exceed the value of the problem

$$\|P_\tau u_0(\cdot)\|^2_{L_2^\gamma(\mathbb{R}^N_+)} \longrightarrow \max, \quad u_0(\cdot) \in L_2^\gamma(\mathbb{R}^N_+), \tag{4.15}$$

$$\sum_{j=1}^p \widehat{\lambda}_j \|P_{t_j} u_0(\cdot)\|^2_{L_2^\gamma(\mathbb{R}^N_+)} \le \sum_{j=1}^p \widehat{\lambda}_j \sigma_j^2. \tag{4.16}$$

Proof. The zero Fréchet differential of a convex smooth objective functional from (4.12) at the point $\widehat{u}_0(\cdot)$, i.e., the equality

$$2\sum_{j=1}^p \widehat{\lambda}_j \int\limits_{\mathbb{R}^N_+} x^\gamma (P_{t_j}\widehat{u}_0(x) - y_j(x)) P_{t_j} u_0(x) dx = 0, \tag{4.17}$$

is a necessary and sufficient condition for achieving the minimum of this functional on the function $\widehat{u}_0(\cdot)$. Taking this equality into account, one could easily obtain

$$\sum_{j=1}^p \widehat{\lambda}_j \|P_{t_j} u_0(\cdot) - y_j(\cdot)\|^2_{L_2^\gamma(\mathbb{R}^N_+)} = \sum_{j=1}^p \widehat{\lambda}_j \|P_{t_j} u_0(\cdot) - P_{t_j}\widehat{u}_0(\cdot)\|^2_{L_2^\gamma(\mathbb{R}^N_+)} + \sum_{j=1}^p \widehat{\lambda}_j \|P_{t_j}\widehat{u}_0(\cdot) - y_j(\cdot)\|^2_{L_2^\gamma(\mathbb{R}^N_+)}.$$

Let the function $u_0(\cdot)$ be admissible for problem (4.13)-(4.14). Then

$$\sum_{j=1}^p \widehat{\lambda}_j \|P_{t_j}\widehat{u}_0(\cdot) - y_j(\cdot)\|^2_{L_2^\gamma(\mathbb{R}^N_+)} = \sum_{j=1}^p \widehat{\lambda}_j \|P_{t_j} u_0(\cdot) - y_j(\cdot)\|^2_{L_2^\gamma(\mathbb{R}^N_+)} - \sum_{j=1}^p \widehat{\lambda}_j \|P_{t_j}\widehat{u}_0(\cdot) - y_j(\cdot)\|^2_{L_2^\gamma(\mathbb{R}^N_+)}$$

$$\le \sum_{j=1}^p \widehat{\lambda}_j \|P_{t_j} u_0(\cdot) - y_j(\cdot)\|^2_{L_2^\gamma(\mathbb{R}^N_+)} \le \sum_{j=1}^p \widehat{\lambda}_j \sigma_j.$$

This means that the function $u_0(\cdot) - \widehat{u}_0(\cdot)$ is admissible for problem (4.15)-(4.16). The value of functional (4.13) on the function $u_0(\cdot)$ is equal to the value of functional (4.15). □

Lemma 4.3. *Values of problems* (4.1)-(4.2) *and* (4.15)-(4.16) *coincide for* $\sigma_j = \varepsilon_j$, $j = 1, \ldots, p$.

Proof. Using the Parseval–Plancherel equality, we move from problem (4.15)-(4.16) to the problem

$$\int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2\tau}\, d\mu(\xi) \longrightarrow \max, \tag{4.18}$$

$$\sum_{j=1}^p \widehat{\lambda}_j \int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2 t_j} d\mu(\xi) \le \sum_{j=1}^p \widehat{\lambda}_j \varepsilon_j^2, \tag{4.19}$$

where

$$d\mu(\xi) = \frac{1}{2^{2\nu}\Gamma^2(\nu+1)}\, |F_\gamma u_0(\xi)|^2 \xi^{2\nu+1}\, d\xi \ge 0.$$

The Lagrange function of this problem has the form

$$\mathcal{L}_1(d\mu(\cdot), \nu) = \nu_0 \int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2\tau}\, d\mu(\xi) + \nu_1 \left(\sum_{j=1}^p \widehat{\lambda}_j \int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2 t_j} d\mu(\xi) - \sum_{j=1}^p \widehat{\lambda}_j \varepsilon_j^2 \right),$$

where the set ν of Lagrange multipliers has the form $\nu = (\nu_0, \nu_1)$. From the fact that the measure $d\widehat{\mu}(\xi)$ that is a solution to problem (4.15)-(4.16) is admissible for this problem, it follows that it is also admissible for problem (4.18)-(4.19). Let $\nu_0 = \widehat{\nu}_0 = -1$, $\nu_1 = \widehat{\nu}_1 = 1$. Then

$$\min_{d\mu(\cdot)\ge 0} \mathcal{L}_1(d\mu(\cdot), \widehat{\nu}) = \mathcal{L}_1(d\widehat{\mu}(\cdot), \widehat{\nu}) = \mathcal{L}(d\widehat{\mu}(\cdot), \widehat{\lambda}) = \min_{d\mu(\cdot)\ge 0} \mathcal{L}(d\mu(\cdot), \widehat{\lambda}), \tag{4.20}$$

where $\widehat{\nu} = (\widehat{\nu}_0, \widehat{\nu}_1)$. Taking into account (4.10), we have

$$\widehat{\nu}_1 \left(\sum_{j=1}^{p} \widehat{\lambda}_j \int\limits_{\mathbb{R}^N_+} e^{-2|\xi|^2 t_j} d\widehat{\mu}(\xi) - \sum_{j=1}^{p} \widehat{\lambda}_j \varepsilon_j^2 \right) = 0. \tag{4.21}$$

This means that $d\widehat{\mu}(\xi)$ is a solution to problem (4.18)-(4.19). Therefore, the value of this problem is equal to the value of problem (4.18)-(4.19). It follows that the squared value of problem (4.5)-(4.6) is equal to the solution of problem (4.15)-(4.16). Therefore, the values of problems (4.5)-(4.6) and (4.15)-(4.16) coincide. □

Let us now formulate and prove the main result.

Theorem 4.1. *The equality*

$$E(\tau, \overline{\varepsilon}) = e^{-\theta(\tau)}$$

holds for all $\tau > 0$.

(1) *If* $0 \le \tau < t_1$, *then* $\theta(\tau) = -\infty$.
(2) *If* $\tau = t_{s_j}$, $j = 1, \dots, \varrho$, *then the method* $\widehat{m}$ *defined by the formula* $\widehat{m}(\overline{y}(\cdot))(\cdot) = y_{s_j}(\cdot)$ *is optimal.*
(3) *If* $\varrho \ge 2$, $\tau \in (t_{s_j}, t_{s_{j+1}})$, *then the method* $\widehat{m}$ *defined by the formula*

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = (\Psi_{s_j} * y_{s_j})_\gamma(\cdot) + (\Phi_{s_{j+1}} * y_{s_{j+1}})_\gamma(\cdot), \tag{4.22}$$

where $\Psi_{s_j}(\cdot)$, $\Phi_{s_{j1}}(\cdot)$ *are functions, the Fourier–Bessel images of which have the form*

$$F_\gamma \Psi_{s_j}(\xi) = \frac{(t_{s_{j+1}} - \tau)\varepsilon_{s_{j+1}}^2 e^{-|\xi|^2(\tau - t_{s_j})}}{(t_{s_{j+1}} - \tau)\varepsilon_{s_{j+1}}^2 + (\tau - t_{s_j})\varepsilon_{s_j}^2 e^{-2|\xi|^2(t_{s_{j+1}} - t_{s_j})}}, \tag{4.23}$$

$$F_\gamma \Phi_{s_{j+1}}(\xi) = \frac{(\tau - t_{s_j})\varepsilon_{s_j}^2 e^{-|\xi|^2(\tau + t_{s_{j+1}} - 2t_{s_j})}}{(t_{s_{j+1}} - \tau)\varepsilon_{s_{j+1}}^2 + (\tau - t_{s_j})\varepsilon_{s_j}^2 e^{-2|\xi|^2(t_{s_{j+1}} - t_{s_j})}}, \tag{4.24}$$

is optimal.

(4) *If* $\tau > t_{s_\varrho}$, *then the method* $\widehat{m}$ *defined by the formula* $\widehat{m}(\overline{y}(\cdot))(\cdot) = P_{\tau - t_{s_\varrho}} y_{s_\varrho}(\cdot)$ *is optimal.*

Proof. Let $\tau \in [t_{s_j}, t_{s_{j+1}})$. It was shown above that one could choose a set of Lagrange multipliers such that only the multipliers $\widehat{\lambda}_{s_j}$ and $\widehat{\lambda}_{s_{j+1}}$ are nonzero. Therefore, problem (4.12) takes the form

$$\widehat{\lambda}_{s_j} \|P_{t_{s_j}} u_0(\cdot) - y_{s_j}(\cdot)\|_{L_2^\gamma(\mathbb{R}^N_+)} + \widehat{\lambda}_{s_{j+1}} \|P_{t_{s_{j+1}}} u_0(\cdot) - y_{s_{j+1}}(\cdot)\|_{L_2^\gamma(\mathbb{R}^N_+)} \longrightarrow \min,$$

$$u_0(\cdot) \in L_2^\gamma(\mathbb{R}^N_+).$$

Let $\widehat{u}_0(\cdot) = \widehat{u}_0(\cdot, y(\cdot))$ be a solution to this problem. Then condition (4.17) is satisfied. For Fourier–Bessel images this condition can be written in the form

$$\sum_{\kappa=j}^{j+1} \int\limits_{\mathbb{R}^N_+} \xi^\gamma (e^{-|\xi|^2 t_{s_\kappa}} F_\gamma \widehat{u}_0(\xi) - F_\gamma y_{s_\kappa}(\xi)) e^{-|\xi|^2 t_{s_\kappa}} F_\gamma u_0(\xi)\, d\xi = 0. \tag{4.25}$$

Let

$$F_\gamma \widehat{u}_0(\xi) = \frac{\widehat{\lambda}_{s_j} e^{-|\xi|^2 t_{s_j}} F_\gamma y_{s_j} + \widehat{\lambda}_{s_{j+1}} e^{-|\xi|^2 t_{s_{j+1}}} F_\gamma y_{s_{j+1}}}{\widehat{\lambda}_{s_j} e^{-2|\xi|^2 t_{s_j}} + \widehat{\lambda}_{s_{j+1}} e^{-2|\xi|^2 t_{s_{j+1}}}}. \tag{4.26}$$

Then Eq. (4.25) holds for all $u_0(\cdot) \in L_2^\gamma(\mathbb{R}^N_+)$. Let functions $F_\gamma y_j(\cdot)$, $j = 1, \dots, p$ have compact support for the set $\overline{y}(\cdot) = (y_1(\cdot), \dots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}^N_+))^p$. Then function (4.26) belongs to the space $L_2^\gamma(\mathbb{R}^N_+)$. Then the function $\widehat{u}_0(\cdot) = \widehat{u}_0(\cdot, y(\cdot))$ defined by formula (4.26) also belongs to the space $L_2^\gamma(\mathbb{R}^N_+)$ and is a solution to problem (4.12). Compactly supported functions are dense in $L_2^\gamma(\mathbb{R}^N_+)$. Therefore, functions with compactly supported Fourier–Bessel images are dense in $L_2^\gamma(\mathbb{R}^N_+)$.

Let functions $\widetilde{u}_0(\cdot) \in L_2^\gamma(\mathbb{R}_+^N)$, $\overline{y}(\cdot) = (y_1(\cdot), \ldots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}_+^N))^p$ satisfy inequalities

$$\|P_{t_{s_j}}\widetilde{u}_0(\cdot) - y_{s_j}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)} \le \varepsilon_j,\ j = 1, \ldots, p.$$

Choose a sequence $\overline{y}^{(k)}(\cdot) = (y_1^{(k)}(\cdot), \ldots, y_p^{(k)}(\cdot)) \in (L_2^\gamma(\mathbb{R}_+^N))^p$, $k \in \mathbb{N}$, such that functions $F_\gamma y_j^{(k)}(\cdot)$, $j = 1, \ldots, p$, have compact support and $\|y_j(\cdot) - y_j^{(k)}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)} \le 1/k$, $j = 1, \ldots, p$, $k \in \mathbb{N}$. Choose a number $k \in \mathbb{N}$. There exists a solution $\widehat{u}_0(\cdot, y^{(k)}(\cdot))$ to problem (4.12). Due to inequalities

$$\|P_{t_j}\widetilde{u}_0(\cdot) - y_j^{(k)}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)} \le \|P_{t_j}\widetilde{u}_0(\cdot) - y_j(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)} + \|y_j(\cdot) - y_j^{(k)}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)} \le \varepsilon_j + 1/k,\ j = 1, \ldots, p,$$

the function $\widetilde{u}_0(\cdot)$ is admissible for problem (4.13)-(4.14) with $\sigma_j = \sigma_j(k) = \varepsilon_j + 1/k$. Let

$$a(k) = \sqrt{\sum_{j=1}^p \widehat{\lambda}_j \sigma_j^2(k) \Big/ \sum_{j=1}^p \widehat{\lambda}_j \varepsilon_j^2}.$$

By Lemma 4.2 the value of problem (4.13)–(4.14) does not exceed the value of problem (4.15)–(4.16).

Substitute $u_0(\cdot) = a(k)v_0(\cdot)$ into problem (4.15)-(4.16). Then this problem takes the form

$$a(k)\|P_\tau v_0(\cdot) - P_\tau \widehat{u}_0(\cdot)\|^2_{L_2^\gamma(\mathbb{R}_+^N)} \longrightarrow \max, \quad u_0(\cdot) \in L_2^\gamma(\mathbb{R}_+^N), \tag{4.27}$$

$$\sum_{j=1}^p \widehat{\lambda}_j \|P_{t_j} v_0(\cdot)\|^2_{L_2^\gamma(\mathbb{R}_+^N)} \le \sum_{j=1}^p \widehat{\lambda}_j \sigma_j^2. \tag{4.28}$$

The value of problem (4.27)-(4.28) coincides with the value of problem (4.1)-(4.2) multiplied by $a(k)$ and is equal to $a(k)e^{-\theta(\tau)}$. Since the function $\widetilde{u}_0(\cdot)$ is admissible for problem (4.13)-(4.14), we have:

$$\|P_\tau \widetilde{u}_0(\cdot) - P_\tau \widehat{u}_0(\cdot, y^{(k)}(\cdot))\|_{L_2^\gamma(\mathbb{R}_+^N)} \le a(k)e^{-\theta(\tau)}. \tag{4.29}$$

Let $\Psi_{s_j}(\cdot)$, $\Phi_{s_{j+1}}(\cdot)$ be functions, Fourier–Bessel images of which correspond to (4.23)–(4.24):

$$F_\gamma \Psi_{s_j}(\xi) = \frac{(t_{s_{j+1}} - \tau)\varepsilon^2_{s_{j+1}} e^{-|\xi|^2(\tau - t_{s_j})}}{(t_{s_{j+1}} - \tau)\varepsilon^2_{s_{j+1}} + (\tau - t_{s_j})\varepsilon^2_{s_j} e^{-2|\xi|^2(t_{s_{j+1}} - t_{s_j})}},$$

$$F_\gamma \Phi_{s_{j+1}}(\xi) = \frac{(\tau - t_{s_j})\varepsilon^2_{s_j} e^{-|\xi|^2(\tau + t_{s_{j+1}} - 2t_{s_j})}}{(t_{s_{j+1}} - \tau)\varepsilon^2_{s_{j+1}} + (\tau - t_{s_j})\varepsilon^2_{s_j} e^{-2|\xi|^2(t_{s_{j+1}} - t_{s_j})}}.$$

Let $\tau \in (t_{s_j}, t_{s_{j+1}})$. Fourier–Bessel images (4.23) and (4.24) of functions $\Psi_{s_j}(\cdot)$ and $\Phi_{s_{j+1}}(\cdot)$ belong to the space of even infinitely-differentiable rapidly decreasing functions. Consequently, functions $\Psi_{s_j}(\cdot)$ and $\Phi_{s_{j+1}}(\cdot)$ belong to this space. In this case, we define a recovery method using generalized convolution according to (4.22):

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = (\Psi_{s_j} * y_{s_j})_\gamma(\cdot) + (\Phi_{s_{j+1}} * y_{s_{j+1}})_\gamma(\cdot).$$

Then

$$F_\gamma \widehat{m}(\overline{y}^{(k)}(\cdot))(\xi) = F_\gamma \Psi_{s_j}(\xi) F_\gamma y^{(k)}_{s_j}(\xi) + F_\gamma \Phi_{s_{j+1}}(\xi) F_\gamma y^{(k)}_{s_{j+1}}(\xi) = e^{-|\xi|^2\tau} F_\gamma \widetilde{u}_0(\cdot, \overline{y}^{(k)}(\cdot))(\xi). \tag{4.30}$$

Therefore,

$$\widehat{m}(\overline{y}^{(k)}(\cdot))(\cdot) = P_\tau \widetilde{u}_0(\cdot, \overline{y}^{(k)}(\cdot))(\cdot). \tag{4.31}$$

If $\tau = t_{s_j}$, including the case $\tau = t_{s_\varrho}$, then

$$F_\gamma \widehat{m}(\overline{y}^{(k)}(\cdot))(\xi) = F_\gamma y^{(k)}_{s_j}(\xi) = e^{-|\xi|^2\tau} F_\gamma \widetilde{u}_0(\cdot, \overline{y}^{(k)}(\cdot))(\xi) = F_\gamma(P_\tau \widetilde{u}_0(\cdot, \overline{y}^{(k)}(\cdot)))(\xi),$$

and (4.31) holds as well.

Let functions $\widetilde{u}_0(\cdot) \in L_2^\gamma(\mathbb{R}_+^N)$, $\overline{y}(\cdot) = (y_1(\cdot), \ldots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}_+^N))^p$ satisfy inequalities

$$\|P_{t_{s_j}}\widetilde{u}_0(\cdot) - y_{s_j}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)} \le \varepsilon_j,\ j = 1, \ldots, p.$$

Then for all $k \in \mathbb{N}$

$$\|P_\tau \widetilde{u}_0(\cdot) - \widehat{m}\,(\overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}$$

$$\leq \|P_\tau \widetilde{u}_0(\cdot) - \widehat{m}\,(\overline{y}^{(k)}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)} + \|\widehat{m}\,(\overline{y}^{(k)}(\cdot))(\cdot) - \widehat{m}\,(\overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}$$

$$\leq \|P_\tau \widetilde{u}_0(\cdot) - P_\tau \widetilde{u}_0(\cdot, \overline{y}^{(k)}(\cdot))\|_{L_2^\gamma(\mathbb{R}_+^N)} + \|\widehat{m}\,(\overline{y}^{(k)}(\cdot))(\cdot) - \widehat{m}\,(\overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}$$

$$\leq a(k)e^{-\theta(\tau)} + \|\widehat{m}\,(\overline{y}^{(k)}(\cdot) - \overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)}.$$

Passing in this inequality to the limit as $k \to \infty$, we get

$$\|P_\tau \widetilde{u}_0(\cdot) - \widehat{m}\,(\overline{y}(\cdot))(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)} \leq e^{-\theta(\tau)}.$$

In this inequality we pass to the supremum over all $\widetilde{u}_0(\cdot) \in L_2^\gamma(\mathbb{R}_+^N)$ and $\overline{y}(\cdot) = (y_1(\cdot), \ldots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}_+^N))^p$ such that $\|P_{t_{s_j}} \widetilde{u}_0(\cdot) - y_{s_j}(\cdot)\|_{L_2^\gamma(\mathbb{R}_+^N)} \leq \varepsilon_j,\ j = 1, \ldots, p$. Then we get $e(\tau, \overline{\varepsilon}, \widehat{m}) \leq e^{-\theta(\tau)}$. Taking into account the lower estimate proved earlier, we get

$$e^{-\theta(\tau)} \leq E(\tau, \overline{\varepsilon}) \leq e(\tau, \overline{\varepsilon}, \widehat{m}) \leq e^{-\theta(\tau)},$$

whence it follows that $E(\tau, \overline{\varepsilon}) = e^{-\theta(\tau)}$ and $\widehat{m}$ is the optimal method.

Let $\tau > t_{s_\varrho}$. Then $\widehat{\lambda}_{s_\varrho} = 1$, the remaining Lagrange multipliers are equal to zero. Problem (4.12) then takes the form

$$\|P_{t_{s_\varrho}} \widetilde{u}_0(\cdot) - y_{s_\varrho}(\cdot)\|^2_{L_2^\gamma(\mathbb{R}_+^N)} \Longrightarrow \min.$$

Let functions $F_\gamma y_j,\ j = 1, \ldots, p$, have compact support for a given set $\overline{y}(\cdot) = (y_1(\cdot), \ldots, y_p(\cdot)) \in (L_2^\gamma(\mathbb{R}_+^N))^p$. Then a solution $\widetilde{u}_0(\cdot) = \widetilde{u}_0(\cdot, \overline{y}(\cdot))$ to this problem exists, and $F_\gamma \widetilde{u}_0(\xi) = e^{|\xi|^2 t_{s_\varrho}} F_\gamma y_{s_\varrho}$. Inequality (4.29) in this case is proved as before. Now we define the method $\widehat{m}$ by the equality

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = P_{\tau - t_{s_\varrho}}. \tag{4.32}$$

Then

$$F_\gamma \widehat{m}(\overline{y}^{(k)}(\cdot))(\xi) = e^{-|\xi|^2(\tau - t_{s_\varrho})} F_\gamma y_{s_\varrho}(\xi) = e^{-|\xi|^2 \tau} F_\gamma \widehat{u}_0(\cdot, \overline{y}^{(k)}(\cdot)).$$

It means that

$$\widehat{m}(\overline{y}^{(k)}(\cdot))(\cdot) = P_\tau \widehat{u}_0(\cdot, \overline{y}^{(k)}(\cdot)).$$

Further reasoning is similar to the reasoning in the previous case. □

5. Conclusion

In this paper, we transferred the results of [13] onto the case of a singular heat equation using the methods developed in [1, 13–15, 23]. In [1, 14], the method for establishing a lower estimate of the optimal recovery error was modified. It is reasonable to believe that this method can also be transferred to the considered case of a singular heat equation.

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