ISSN 0012-2661, Differential Equations, 2023, Vol. 59, No. 12, pp. 1708–1720. © Pleiades Publishing, Ltd., 2023.

PARTIAL DIFFERENTIAL EQUATIONS

On Singular Heat Equation

E. L. Shishkina[1,2*] **and A. K. Yusupova**[3**]

[1] *Belgorod State University, Belgorod, 308015 Russia*
[2] *Voronezh State University, Voronezh, 394018 Russia*
[3] *Fergana State University, Fergana, 150100 Uzbekistan*
*e-mail: *ilina_dico@mail.ru, **ladyuz@list.ru*

Received October 23, 2023; revised October 23, 2023; accepted November 15, 2023

Abstract—In physics, the singular heat equation with the Bessel operator is used to explain the basic process of heat transport in a substance with spherical or cylinder symmetry. This paper examines the solution of the Cauchy problem for the heat equation with the Bessel operator acting in the space variable. We obtain some properties of the solution and consider the normalized modified Bessel function of the first kind.

DOI: 10.1134/S001226612312011X

INTRODUCTION

Partial differential equations are ubiquitous as mathematical models in the scientific and engineering areas. We consider the singular heat equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\gamma}{x}\frac{\partial}{\partial x}\right)u(x,t)=\frac{\partial}{\partial t}u(x,t),$$

where $\frac{\partial^2}{\partial x^2}+\frac{\gamma}{x}\frac{\partial}{\partial x}$ is called the *Bessel operator*. Many problems of physical interest are described by partial differential equations with the Bessel operator, in particular, the radial component of the Laplacian in all dimensions is described by the Bessel operator.

Terminology referring to partial differential equations with Bessel operators belongs to I.A. Kipriyanov [1, 2]. According to this terminology $\sum_{i=1}^n(B_{\gamma_i})_{x_i}u=f$ is classified as B-elliptic, the equation $\frac{\partial}{\partial x_1}u-\sum_{i=2}^n(B_{\gamma_i})_{x_i}u=f$ is classified as B-parabolic, the equation $(B_{\gamma_1})_{x_1}u-\sum_{k=2}^n(B_{\gamma_i})_{x_i}u=f$ is classified as B-hyperbolic. Such equations were studied mainly by the integral transforms method. B-elliptic equations has been studied in [2–4]. B-hyperbolic equations were investigated in [1, 5–7]. Different problem for B-parabolic equations were solved in [8–10]. We note separately that the general Euler–Poisson–Darboux equation was considered in [11, 12].

In this paper, we consider a solution of the Cauchy problem for the singular heat equation, its connection with the classical case, and its properties. The general behaviour of functions under singular heat flow is the flattening we will see in the fundamental solution, which goes from a sharp spike to a flat line as t tends to infinity. Next, we construct polynomial solutions. Polynomial solutions are frequently used as building blocks in algorithms in order to find closed-form solutions. Also, the property of closeness related to the singular heat equation function i_ν was given. In particular, this function has reproducing kernel property and is an operator function for generalized translation.

1. PRELIMINARY

Different techniques for working with the Bessel operator were developed by I.A. Kipriyanov [2], B.M. Levitan [13], S.S. Platonov [14] and others. Among these techniques, Hankel transform has been recognized as one of the most popular method for solution partial differential equations with the Bessel operator. The special convolution should be used in order to convolution of two functions be the pointwise product of their Hankel transforms. This convolution based on so called *generalized translation*. In this section we give some elements of harmonic analysis with the Bessel operator.

Suppose that $\mathbb{R}_+=(0,\infty)$, Ω be finite or infinite interval in $\mathbb{R}$ symmetric with respect to the origin, $\Omega_+=\Omega\cap\mathbb{R}_+$. We deal with the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+

functions and denote by $C^m(\Omega_+)$ the subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions are continuous up to the origin. Class $C^m_{ev}(\Omega_+)$ consists of all functions from $C^m(\Omega_+)$ such that $f^{(2k+1)}(0)=0$ for all non-negative integer $k\leq\frac{m-1}{2}$. Let $C_{ev,b}(\mathbb{R}_+)$ be the space of all bounded continuous functions, continuously extendable to negative values of the variable x as even functions, on Ω_+.

Let B_γ denote the Bessel operator. Its action on $C^2_{ev}(\mathbb{R}_0)$ is given by

$$B_\gamma=\frac{d^2}{dx^2}+\frac{\gamma}{x}\frac{d}{dx}. \tag{1}$$

If the initial temperature is an power function with even integer positive power, then the convolution with the fundamental solution gives the singular heat polynomial.

Suppose that $\gamma>0$. $L_p^\gamma(\mathbb{R}_+)=L_p^\gamma$, $1\leq p<\infty$ is the space of all measurable in $\mathbb{R}_+$ functions such that

$$\int\limits_0^\infty \bigl|f(x)\bigr|^p x^\gamma dx<\infty.$$

For a real number $p\geq 1$, the L_p^γ–norm of f is defined by

$$\|f\|_{p,\gamma}=\left(\int\limits_{\mathbb{R}^n_+}\bigl|f(x)\bigr|^p x^\gamma\,dx\right)^{1/p}.$$

It is known that L_p^γ is a Banach space [2].

Let consider the space with positive weighted measure $\mathbf{mes}_\gamma$. For a scalar valued measurable function f that takes finite values almost everywhere we define

$$\mu_\gamma(f,t)=\mathbf{mes}_\gamma\Bigl\{x\in(0,\infty):\,\bigl|f(x)\bigr|>t\Bigr\}=\int\limits_{\{x:\,|f(x)|>t\}^+}x^\gamma\,dx,$$

where $\{x{:}|f(x)|{>}t\}^+{=}\{x{\in}(0,\infty){:}|f(x)|{>}t\}$.

Space $L^\gamma_\infty(\mathbb{R}_+){=}L^\gamma_\infty$ is the space of all measurable in $\mathbb{R}_+$ functions, continuously extendable to negative values of the variable x as even functions, for which the norm

$$\|f\|_{\infty,\gamma}=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n_+}{}_\gamma\bigl|f(x)\bigr|=\inf_{a\in\mathbb{R}}\bigl\{\mu_\gamma(f,a)=0\bigr\}$$

is finite.

The *Hankel transform* of a function $f{\in}L_1^\gamma(\mathbb{R}_+)$ is expressed as

$$F_\gamma[f](\xi)=\widehat{f}(\xi)=F_\gamma\bigl[f(x)\bigr](\xi)=f(\xi)=\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\,f(x)\,x^\gamma\,dx, \tag{2}$$

where $\gamma>0$, the symbol j_ν is used for the *normalized Bessel function of the first kind*:

$$j_\nu(x)=\frac{2^\nu\Gamma(\nu+1)}{x^\nu}\;J_\nu(x)=\sum_{m=0}^\infty\frac{(-1)^m\Gamma(\nu+1)}{m!\,\Gamma(m+\nu+1)}\Bigl(\frac{x}{2}\Bigr)^{2m},$$

and J_ν is Bessel function of the first kind. The function $j_{\frac{\gamma-1}{2}}$ is eigenfunction of the operator B_γ:

$$(B_\gamma)_x j_{\frac{\gamma-1}{2}}(x\xi)=-\xi^2 j_{\frac{\gamma-1}{2}}(x\xi),$$

such that $j_{\frac{\gamma-1}{2}}(0)=1$, $j'_{\frac{\gamma-1}{2}}(0)=0$. By definition we put $j_\nu(x)=j_\nu(-x)$.

The inversion formula

$$F_\gamma^{-1}\big[\widehat{f}(\xi)\big](x)=f(x)=\frac{2^{1-\gamma}}{\Gamma^2\left(\dfrac{\gamma+1}{2}\right)}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\,\widehat{f}(\xi)\,\xi^\gamma\,d\xi \tag{3}$$

holds.

The *generalized translation* associated with the Bessel operator is defined by the equality [13]

$$({}^\gamma T_x^y f)(x)={}^\gamma T_x^y f=C(\gamma)\int\limits_0^\pi f\left(\sqrt{x^2+y^2-2xy\cos\varphi}\right)\sin^{\gamma-1}\varphi\,d\varphi,$$

where $C(\gamma)=\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)}$. For $\gamma=0$ generalized translation ${}^\gamma T_x^y$ is $({}^0T_x^y f)(x)=\frac{f(x+y)+f(x-y)}{2}$.

In [13] was shown that $u(x,y)=({}^\gamma T_x^y f)(x)$ is a unique solution of the Cauchy problem

$$(B_\gamma)_x u(x,y)=(B_\gamma)_y u(x,y),$$
$$u(x,0)=f(x),\qquad \frac{\partial}{\partial y}u(x,y)\bigg|_{y=0}=0.$$

For the generalized translation operator ${}^\gamma T_x^y$ the representation [15]

$$({}^\gamma T_x^y f)(x)=\frac{2^\gamma C(\gamma)}{(4xy)^{\gamma-1}}\int\limits_{|x-y|}^{x+y} z f(z)\Big[\big(z^2-(x-y)^2\big)\big((x+y)^2-z^2\big)\Big]^{\frac{\gamma}{2}-1}dz \tag{4}$$

is valid.

Hankel transform from generalized translation of function $f\in L_1^\gamma(\mathbb{R}_+)$ has a form

$$F_\gamma\big[({}^\gamma T_x^y f)(x)\big](\xi)=j_{\frac{\gamma-1}{2}}(y\xi)\,F_\gamma[f](\xi). \tag{5}$$

The generalized convolution, generated by the generalized translation ${}^\gamma T_x^y$ is

$$(f*g)_\gamma(x)=\int\limits_0^\infty f(y)\,{}^\gamma T_x^y g(x)y^\gamma\,dy. \tag{6}$$

Hankel transform applied to generalized convolution is

$$F_\gamma\big[(f*g)_\gamma(x)\big](\xi)=F_\gamma\big[f(x)\big](\xi)F_\gamma\big[g(x)\big](\xi).$$

Let S_{ev} is a class of Schwarz functions on $\mathbb{R}_+$, admitting an even continuation to $(-\infty,0]$. The space of weighted generalized functions S'_{ev} is a class of continuous linear functionals that map a set of test functions $\varphi\in S_{ev}$ into the set of real numbers. Each function $u\in L^\gamma_{1,loc}$ will be identified with the functional $u\in S'_{ev}$ acting according to the formula

$$(u,\varphi)_\gamma=\int\limits_0^\infty u(x)\,\varphi(x)\,x^\gamma\,dx,\quad \varphi\in S_{ev}. \tag{7}$$

Generalized functions $u\in S'_{ev}$ acting by the formula (7) will be called *regular weighted generalized functions*. All other generalized functions $u\in S'_{ev}$ will be called *singular weighted generalized functions*.

Weighted delta-function $\delta_\gamma \in S'_{ev}$ is defined by the equality

$$(\delta_\gamma, \varphi)_\gamma = \varphi(0), \quad \varphi \in S_{ev}. \tag{8}$$

2. GENERALIZED TRANSLATION OF A NONSTATIONARY EXPONENTIAL-SQUARED KERNEL

Nonstationary exponential-squared kernel is given by $e^{\pm\frac{x^2}{4t}}$. In this section we obtain a new kernel applying the generalized translation to $e^{\pm\frac{x^2}{4t}}$ and

The heat kernel

$$H(x,y,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}}, \quad x,y \in \mathbb{R}, \quad t > 0,$$

solves the heat equation $H_t = H_{xx}$ for all $t > 0$ and $x, y \in \mathbb{R}$, with the initial condition

$$\lim_{t\to 0} H(x,y,t) = \delta(x-y). \tag{9}$$

Also for every smooth function φ with compact support we have

$$\lim_{t\to 0} \int\limits_{\mathbb{R}} H(x,y,t)\varphi(y)\,dy = \varphi(x).$$

When we deal with singular heat equation $H_t = B_\gamma H$ we should apply the generalized translation ${}^\gamma T_x^y$ to $e^{-\frac{x^2}{4t}}$ instead of regular shift. So we need to obtain the following statement.

Statement 1. *The generalized translation* ${}^\gamma T_x^y$ *of* $e^{\pm\frac{x^2}{4t}}$, $x, y, t > 0$ *is*

$${}^\gamma T_x^y e^{\pm\frac{x^2}{4t}} = e^{\pm\frac{x^2+y^2}{4t}} i_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right), \tag{10}$$

where i_ν *is the normalized modified Bessel function of the first kind defined by the formula*

$$i_\nu(x) = \frac{2^\nu \Gamma(\nu+1)}{x^\nu}\, I_\nu(x) = \sum_{m=0}^{\infty} \frac{\Gamma(\nu+1)}{m!\,\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m} \tag{11}$$

and I_ν *is modified Bessel function of the first kind.*

Proof. Using the formula (4) we obtain

$$\begin{aligned} {}^\gamma T_x^y e^{\pm\frac{x^2}{4t}} &= \frac{2^\gamma C(\gamma)}{(4xy)^{\gamma-1}} \int\limits_{|x-y|}^{x+y} z e^{\pm\frac{z^2}{4t}} \left[\left(z^2-(x-y)^2\right)\left((x+y)^2-z^2\right)\right]^{\frac{\gamma}{2}-1} dz = \{z^2 = \zeta\} \\ &= \frac{2^{\gamma-1} C(\gamma)}{(4xy)^{\gamma-1}} \int\limits_{(x-y)^2}^{(x+y)^2} e^{\pm\frac{\zeta}{4t}} \left[\left(\zeta-(x-y)^2\right)\left((x+y)^2-\zeta\right)\right]^{\frac{\gamma}{2}-1} d\zeta. \end{aligned}$$

Next, for ${}^\gamma T_x^y e^{-\frac{x^2}{4t}}$ we set $\zeta - (x-y)^2 = w$ and for ${}^\gamma T_x^y e^{\frac{x^2}{4t}}$ we set $(x+y)^2 - \zeta = w$,

$${}^\gamma T_x^y e^{-\frac{x^2}{4t}} = \frac{2^{\gamma-1} C(\gamma)}{(4xy)^{\gamma-1}} e^{-\frac{(x-y)^2}{4t}} \int\limits_0^{4xy} e^{-\frac{w}{4t}} \left[w(4xy-w)\right]^{\frac{\gamma}{2}-1} dw,$$

$${}^\gamma T_x^y e^{\frac{x^2}{4t}} = \frac{2^{\gamma-1} C(\gamma)}{(4xy)^{\gamma-1}} e^{\frac{(x+y)^2}{4t}} \int\limits_0^{4xy} e^{-\frac{w}{4t}} \left[w(4xy-w)\right]^{\frac{\gamma}{2}-1} dw.$$

Applying the formula 2.3.6.2 in [16] we can find the integral

$$\int\limits_0^{4xy} e^{-\frac{w}{4t}}\left[w(4xy-w)\right]^{\frac{\gamma}{2}-1}dw = 2^{2\gamma-2}\sqrt{\pi}e^{-\frac{xy}{4t}}(xyt)^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma}{2}\right)I_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right).$$

So, we obtain

$$\begin{aligned} {}^\gamma T_x^y e^{-\frac{x^2}{4t}} &= \frac{2^{\gamma-1}C(\gamma)}{(4xy)^{\gamma-1}}e^{-\frac{(x-y)^2}{4t}}2^{2\gamma-2}\sqrt{\pi}e^{-\frac{xy}{4t}}(xyt)^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma}{2}\right)I_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right) = e^{-\frac{x^2+y^2}{4t}}i_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right),\\ {}^\gamma T_x^y e^{\frac{x^2}{4t}} &= \frac{2^{\gamma-1}C(\gamma)}{(4xy)^{\gamma-1}}e^{\frac{(x+y)^2}{4t}}2^{2\gamma-2}\sqrt{\pi}e^{-\frac{xy}{4t}}(xyt)^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma}{2}\right)I_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right) = e^{\frac{x^2+y^2}{4t}}i_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right). \end{aligned}$$

The function $i_{\frac{\gamma-1}{2}}$ is an eigenfunction of the Bessel operator B_γ,

$$(B_\gamma)_t\, i_{\frac{\gamma-1}{2}}(\tau t) = \tau^2 i_{\frac{\gamma-1}{2}}(\tau t),$$

such that $i_{\frac{\gamma-1}{2}}(0)=1$, $i'_{\frac{\gamma-1}{2}}(0)=0$. By definition, we put $i_\nu(x)=i_\nu(-x)$.

The generalized translation of $i_{\frac{\gamma-1}{2}}(x)$ is [15]

$${}^\gamma T_x^y i_{\frac{\gamma-1}{2}}(xz) = i_{\frac{\gamma-1}{2}}(xz)i_{\frac{\gamma-1}{2}}(yz).$$

Statement 2. *The reproducing kernel property for* $i_{\frac{\gamma-1}{2}}$ *is valid*:

$$\int\limits_0^\infty i_{\frac{\gamma-1}{2}}(2xy)\, i_{\frac{\gamma-1}{2}}(2xz)\, e^{-x^2}\, d\mu_\gamma(x) = e^{y^2+z^2} i_{\frac{\gamma-1}{2}}(2yz),$$

where the measure $d\mu_\gamma(x)$ *is*

$$d\mu_\gamma(x) = \frac{2^{\frac{3-\gamma}{2}}}{\Gamma\left(\dfrac{\gamma+1}{2}\right)}x^\gamma\, dx.$$

Proof. We have

$$\begin{aligned} &\int\limits_0^\infty i_{\frac{\gamma-1}{2}}(2xy)\, i_{\frac{\gamma-1}{2}}(2xz)\, e^{-x^2}\, d\mu_\gamma(x)\\ &= \frac{2^{\frac{3-\gamma}{2}}}{\Gamma\left(\dfrac{\gamma+1}{2}\right)}\int\limits_0^\infty i_{\frac{\gamma-1}{2}}(2xy)\, i_{\frac{\gamma-1}{2}}(2xz)\, e^{-x^2}\cdot x^\gamma\, dx = \frac{2^{\frac{3-\gamma}{2}}}{\Gamma\left(\dfrac{\gamma+1}{2}\right)}\int\limits_0^\infty {}^\gamma T_y^z i_{\frac{\gamma-1}{2}}(2xy)\cdot e^{-x^2}\cdot x^\gamma\, dx\\ &= \frac{2^{\frac{3-\gamma}{2}}}{\Gamma\left(\dfrac{\gamma+1}{2}\right)}\,{}^\gamma T_y^z\int\limits_0^\infty i_{\frac{\gamma-1}{2}}(2xy)\cdot e^{-x^2}\cdot x^\gamma\, dx = 2\cdot {}^\gamma T_y^z y^{\frac{1-\gamma}{2}}\int\limits_0^\infty I_{\frac{\gamma-1}{2}}(xy)\cdot e^{-x^2}\cdot x^{\frac{\gamma+1}{2}}\, dx. \end{aligned}$$

Since $\int_0^\infty I_{\frac{\gamma-1}{2}}(2xy)\cdot e^{-x^2}\cdot x^{\frac{\gamma+1}{2}}dx = \frac{1}{2}e^{y^2}y^{\frac{\gamma-1}{2}}$, using Statement 1, we get

$$\int\limits_0^\infty i_{\frac{\gamma-1}{2}}(2xy)\, i_{\frac{\gamma-1}{2}}(2xz)\, e^{-x^2}\, d\mu_\gamma(x) = {}^\gamma T_y^z e^{y^2} = e^{y^2+z^2} i_{\frac{\gamma-1}{2}}(2yz).$$

3. SOLUTION OF THE SINGULAR HEAT EQUATION AND ITS PROPERTIES

In this section we consider the singular heat operator

$$\mathcal{H}_\gamma := (B_\gamma)_x - \partial_t, \tag{12}$$

where B_γ is defined by (1) on function spaces $C^2_{ev}(\Omega\times(0,T)$, $T>0$, $\Omega\subset\mathbb{R}_+$.

The direct verification gives that the function

$$E_\gamma(x,t)=\begin{cases}A(\gamma)t^{-\frac{1+\gamma}{2}}e^{-\frac{x^2}{4t}} & \text{if } t>0\\ 0 & \text{if } t\le 0,\end{cases}\qquad A(\gamma)=\frac{1}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}.$$

is a solution of the singular heat equation $\mathcal{H}_\gamma u(x,t)=0$ on $\mathbb{R}_+\times(0,\infty)$. The constant $A(\gamma)$ is chosen such that

$$\int_0^\infty E_\gamma(x,t)x^\gamma\,dx=1,\quad t>0.$$

Also we have

$$E_\gamma(x,t)\to\delta_\gamma(x),\quad t\to+0\quad\text{in}\quad S'_{ev},$$

where δ_γ is given by (8).

Let $\varphi\in C_{ev,b}(\mathbb{R}_+)$, then

$$u(x,t)=(G_t^\gamma\varphi)(x)=(E_\gamma*\varphi)_\gamma(x)=\int_0^\infty\left({}^\gamma T_x^y E_\gamma(x,t)\right)\varphi(y)y^\gamma\,dy \tag{13}$$

is a solution in $\mathbb{R}_+$ of the Cauchy problem for the singular heat equation

$$\begin{cases}(B_\gamma)_x u(x,t)=u_t(x,t)\\ u(x,0)=\varphi(x).\end{cases} \tag{14}$$

In (13) we use generalized translation (6). Then, taking into account (10), for $t>0$ we obtain

$$\Gamma_\gamma(x,y,t):={}^\gamma T_x^y E_\gamma(x,t)=A(\gamma)t^{-\frac{1+\gamma}{2}}e^{-\frac{x^2+y^2}{4t}}i_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right), \tag{15}$$

and for $t>0$ the solution (13) of the Cauchy problem for the singular heat equation (14) becomes

$$u(x,t)=A(\gamma)t^{-\frac{1+\gamma}{2}}\int_0^\infty e^{-\frac{x^2+y^2}{4t}}i_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right)\varphi(y)y^\gamma\,dy=\int_0^\infty\Gamma_\gamma(x,y,t)\varphi(y)y^\gamma\,dy. \tag{16}$$

The function $\Gamma_\gamma(x,y,t)$ is the *generalized heat kernel*. Given $y>0$, the function $u(x,t)=\Gamma_\gamma(x,y,t)$ solves the generalized heat equation $\mathcal{H}_\gamma u=0$.

We can write the equality corresponding to (9):

$$\lim_{t\to0}\Gamma_\gamma(x,y,t)={}^\gamma T_x^y\delta_\gamma(x).$$

Statement 3. *The function $\Gamma_\gamma(x,y,t)$ has the following properties for $x,y,t>0$:*

1. $\int_0^\infty\Gamma_\gamma(x,y,t)x^\gamma\,dx=1$.
2. $|\Gamma_\gamma(x,y,t)|\le C_\gamma t^{-\frac{\gamma}{2}}e^{-\frac{(x-y)^2}{4t}}$.
3. $\int_0^\infty E_\gamma(y,s)\Gamma_\gamma(x,y,t)y^\gamma\,dy=E_\gamma(x,t+s)$, $t,s>0$.

Proof.

1. Indeed, using the formula [13]

$$\int\limits_0^\infty {}^\gamma T_x^y f(x) g(y) y^\gamma\, dy = \int\limits_0^\infty f(y)\, {}^\gamma T_x^y g(x) y^\gamma\, dy,$$

we obtain

$$\int\limits_0^\infty \Gamma_\gamma(x,y,t) x^\gamma\, dx = \int\limits_0^\infty {}^\gamma T_x^y E_\gamma(x,t)\cdot x^\gamma\, dx = \int\limits_0^\infty {}^\gamma E_\gamma(x,t)\cdot x^\gamma\, dx = 1.$$

2. Using the property of I_ν, we obtain

$$\big|\Gamma_\gamma(x,y,t)\big| = \left|A(\gamma) t^{-\frac{1+\gamma}{2}} e^{-\frac{x^2+y^2}{4t}} i_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right)\right|$$

$$\leq \left|A(\gamma) t^{-\frac{1+\gamma}{2}} e^{-\frac{x^2+y^2}{4t}} \cdot e^{\frac{xy}{4t}} \frac{1}{\sqrt{2\pi\cdot \dfrac{xy}{2t}}}\right| \leq C_\gamma t^{-\frac{\gamma}{2}} e^{-\frac{(x-y)^2}{4t}}.$$

3. Let us find Hankel transform of $E_\gamma(x,t)$ by x for $t>0$,

$$(F_\gamma)_x\big[E_\gamma(x,t)\big](\xi,t) = \int\limits_0^\infty E_\gamma(x,t)\, j_{\frac{\gamma-1}{2}}(x\xi) x^\gamma\, dx$$

$$= \frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\dfrac{\gamma+1}{2}\right)}{\xi^{\frac{\gamma-1}{2}}} \cdot \frac{1}{2^\gamma \Gamma\left(\dfrac{\gamma+1}{2}\right)} t^{-\frac{1+\gamma}{2}} \int\limits_0^\infty e^{-\frac{x^2}{4t}} J_{\frac{\gamma-1}{2}}(x\xi) x^{\frac{\gamma+1}{2}}\, dx$$

$$= \frac{\xi^{\frac{1-\gamma}{2}}}{(2t)^{\frac{1+\gamma}{2}}} \int\limits_0^\infty e^{-\frac{x^2}{4t}} J_{\frac{\gamma-1}{2}}(x\xi) x^{\frac{\gamma+1}{2}}\, dx.$$

Using formula 2.12.9.3 in [16] we obtain

$$(F_\gamma)_x\big[E_\gamma(x,t)\big](\xi,t) = \frac{\xi^{\frac{1-\gamma}{2}}}{(2t)^{\frac{1+\gamma}{2}}} \cdot 2^{\frac{\gamma+1}{2}} \xi^{\frac{\gamma-1}{2}} \frac{1}{t^{-\frac{\gamma+1}{2}}} e^{-t\xi^2} = e^{-t\xi^2}.$$

Therefore,

$$(F_\gamma)_x\big[E_\gamma(x,t)\big](\xi,t) = e^{-t\xi^2}. \tag{17}$$

Next, for $t>0$ using (17) we can write

$$(F_\gamma)_x \int\limits_0^\infty E_\gamma(y,s)\Gamma_\gamma(x,y,t) y^\gamma\, dy = (F_\gamma)_x \int\limits_0^\infty E_\gamma(y,s)\big({}^\gamma T_x^y E_\gamma(x,t)\big) y^\gamma\, dy$$

$$= (F_\gamma)_x\big[E_\gamma(x,t)\big](\xi,t)(F_\gamma)_x\big[E_\gamma(x,s)\big](\xi,s) = e^{-t\xi^2} e^{-s\xi^2} = e^{-(t+s)\xi^2}.$$

So, taking into account (17) we obtain

$$\int\limits_0^\infty E_\gamma(y,s)\Gamma_\gamma(x,y,t) y^\gamma\, dy = (F_\gamma)_\xi^{-1} e^{-(t+s)\xi^2} = E_\gamma(x,t+s).$$

For $\varphi \in C_{ev,b}(\mathbb{R}_+)$, consider the operator

$$\mathcal{H}_t^\gamma \varphi(x) = \begin{cases} \int\limits_0^\infty \Gamma_\gamma(x,y,t)\varphi(y)y^\gamma\,dy & \text{if } t>0 \\ \varphi(x) & \text{if } t=0. \end{cases}$$

From Statement 3 is follows that $\mathcal{H}_t^\gamma$ is well defined and continuous on $(0,\infty)$.

We have the following theorems to characterize $\mathcal{H}_t^\gamma$.

Theorem 1. *If $\varphi \in S_{ev}$, then $u(x,t)=\mathcal{H}_t^\gamma\varphi(x)$ is a Schwartz function in x having a continuous even continuation in x and sartisfying the Cauchy problem (14).*

Proof. Indeed, $u(x,t)$ satisfies $\mathcal{H}_\gamma u(x,t)$=0 since it is a generalized convolution with $E_\gamma(x,t)$. Let us show that for fixed $x_0 \in \mathbb{R}_+$

$$\lim_{x\to x_0, t\to +0} \int\limits_0^\infty \left({}^\gamma T_x^y E_\gamma(x,t)\right)\varphi(y)y^\gamma\,dy = \varphi(x_0).$$

For any $\varepsilon>0$ we can choose $\delta>0$ such that $|\varphi(x)-\varphi(x_0)|<\varepsilon$ for all $x\in\mathbb{R}_+$ such that $|x-x_0|<\delta$. We have

$$\big|u(x,t)-\varphi(x_0)\big| \le \int\limits_0^\delta \left({}^\gamma T_x^y E_\gamma(x,t)\right)\big|\varphi(y)-\varphi(x_0)\big|y^\gamma\,dy + \int\limits_\delta^\infty \left({}^\gamma T_x^y E_\gamma(x,t)\right)\big|\varphi(y)-\varphi(x_0)\big|y^\gamma\,dy.$$

Taking into account the fact that for $z\to\infty$ we have $I_\nu(z)\sim\frac{e^z}{\sqrt{2\pi z}}$, we see that the second integral tends to zero:

$$\int\limits_\delta^\infty \left({}^\gamma T_x^y E_\gamma(x,t)\right)\big|\varphi(y)-\varphi(x_0)\big|y^\gamma\,dy$$

$$\le 2\|\varphi\|_{\infty,\gamma}\int\limits_\delta^\infty {}^\gamma T_x^y E_\gamma(x,t)\cdot y^\gamma\,dy$$

$$= Ct^{-\frac{1+\gamma}{2}}e^{-\frac{x^2}{4t}}\int\limits_\delta^\infty e^{-\frac{y^2}{4t}} i_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right)\cdot y^\gamma\,dy = C\cdot\frac{1}{t}\cdot e^{-\frac{x^2}{4t}}\int\limits_\delta^\infty e^{-\frac{y^2}{4t}} I_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right)\cdot y^{\frac{\gamma+1}{2}}\,dy$$

$$\le C\cdot\frac{1}{\sqrt{t}}\cdot\int\limits_\delta^\infty e^{-\frac{(x-y)^2}{4t}}\cdot y^{\frac{\gamma+1}{2}}\,dy \le C\cdot\frac{1}{\sqrt{t}}\cdot\int\limits_\delta^\infty e^{-\frac{(y-x_0)^2}{16t}}\cdot y^{\frac{\gamma+1}{2}}\,dy$$

$$= C\cdot t^{\frac{\gamma+1}{2}}\cdot\int\limits_{\frac{\delta-x_0}{4\sqrt{t}}}^\infty e^{-z^2}\cdot(z+x_0)^{\frac{\gamma+1}{2}}\,dz\to 0,\quad t\to+0.$$

As for the first integral, we get

$$\int\limits_0^\delta \left({}^\gamma T_x^y E_\gamma(x,t)\right)\big|\varphi(y)-\varphi(x_0)\big|y^\gamma\,dy \le \varepsilon\int\limits_0^\infty E_\gamma(y,t)y^\gamma\,dy = \varepsilon;$$

therefore $|u(x,t)-\varphi(x_0)|<\varepsilon$ and $\lim\limits_{x\to x_0,t\to+0} u(x,t)=\varphi(x_0)$. So $u(x,0)=\varphi(x)$.

From (17)

$$E_\gamma(x,t) = (F_\gamma^{-1})_\xi\left[e^{-t\xi^2}\right](x) = \frac{2^{1-\gamma}}{\Gamma^2\left(\frac{\gamma+1}{2}\right)}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)e^{-t\xi^2}\xi^\gamma\,d\xi.$$

Then for $t>0$, using the self-adjointness of generalized translation (see [13]) and (5), we obtain

$$\begin{aligned}\mathcal{H}_t^\gamma\varphi(x) &= \int\limits_0^\infty \left({}^\gamma T_x^y E_\gamma(x,t)\right)\varphi(y)y^\gamma\,dy = \int\limits_0^\infty E_\gamma(y,t)\,{}^\gamma T_x^y\varphi(x)y^\gamma\,dy \\ &= \frac{2^{1-\gamma}}{\Gamma^2\left(\frac{\gamma+1}{2}\right)}\int\limits_0^\infty\left(\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(y\xi)e^{-t\xi^2}\xi^\gamma\,d\xi\right){}^\gamma T_x^y\varphi(x)y^\gamma\,dy \\ &= \frac{2^{1-\gamma}}{\Gamma^2\left(\frac{\gamma+1}{2}\right)}\int\limits_0^\infty\left(\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(y\xi)\,{}^\gamma T_x^y\varphi(x)y^\gamma\,dy\right)e^{-t\xi^2}\xi^\gamma\,d\xi \\ &= \frac{2^{1-\gamma}}{\Gamma^2\left(\frac{\gamma+1}{2}\right)}\int\limits_0^\infty (F_\gamma)_y\left[{}^\gamma T_x^y\varphi(x)\right](\xi)\cdot e^{-t\xi^2}\xi^\gamma\,d\xi.\end{aligned}$$

Using formula of the Hankel transform of generalized translation (see [15]), we obtain

$$\mathcal{H}_t^\gamma\varphi(x) = \frac{2^{1-\gamma}}{\Gamma^2\left(\frac{\gamma+1}{2}\right)}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\,F_\gamma\left[\varphi(x)\right](\xi)\cdot e^{-t\xi^2}\xi^\gamma\,d\xi = \frac{2^{1-\gamma}}{\Gamma^2\left(\frac{\gamma+1}{2}\right)}F_\gamma\left[F_\gamma[\varphi](\xi)\cdot e^{-t\xi^2}\right](x).$$

Since the Schwartz space is invariant under the Hankel transform, we see that $\mathcal{H}_t^\gamma\varphi$ is a Schwartz function if φ is a Schwartz function. Also if φ is even, then $\mathcal{H}_t^\gamma\varphi$ is even too.

Theorem 2. *If $\varphi\in S_{ev}$, then $u(x,t)=\mathcal{H}_t^\gamma\varphi(x)$ has the following properties*:

1. $\mathcal{H}_{t+s}^\gamma\varphi = \mathcal{H}_t^\gamma\mathcal{H}_s^\gamma\varphi$,
2. $\|\mathcal{H}_t^\gamma\varphi-\varphi\|_{\gamma,\infty}\to 0$, $t\to 0$.

Proof.

1. To prove the semigroup property $\mathcal{H}_{t+s}^\gamma\varphi = \mathcal{H}_t^\gamma\mathcal{H}_s^\gamma\varphi$, note that for $t,s>0$, by Statement 3

$$\begin{aligned}\mathcal{H}_{t+s}^\gamma\varphi(x) &= \int\limits_0^\infty \Gamma_\gamma(x,y,t+s)\varphi(y)y^\gamma\,dy = \int\limits_0^\infty {}^\gamma T_x^y E_\gamma(x,t+s)\varphi(y)y^\gamma\,dy \\ &= \int\limits_0^\infty {}^\gamma T_x^y\left(\int\limits_0^\infty E_\gamma(z,s)\Gamma_\gamma(x,z,t)z^\gamma\,dz\right)\varphi(y)y^\gamma\,dy \\ &= \int\limits_0^\infty\left(\int\limits_0^\infty E_\gamma(z,s)\,{}^\gamma T_x^y\,{}^\gamma T_x^z E_\gamma(x,t)z^\gamma\,dz\right)\varphi(y)y^\gamma\,dy.\end{aligned}$$

Next, using the associativity and self-adjointness of generalized translation (see [13]) we get

$$\mathcal{H}_{t+s}^\gamma\varphi(x) = \int\limits_0^\infty\left(\int\limits_0^\infty E_\gamma(z,s)\,{}^\gamma T_z^y\,{}^\gamma T_x^z E_\gamma(x,t)z^\gamma\,dz\right)\varphi(y)y^\gamma\,dy$$

$$= \int_0^\infty \left(\int_0^\infty {}^\gamma T_z^y E_\gamma(z,s)\, {}^\gamma T_x^z E_\gamma(x,t) z^\gamma\, dz \right) \varphi(y) y^\gamma\, dy$$

$$= \int_0^\infty {}^\gamma T_x^z E_\gamma(x,t) \left(\int_0^\infty {}^\gamma T_z^y E_\gamma(z,s) \varphi(y) y^\gamma\, dy \right) z^\gamma\, dz$$

$$= \int_0^\infty \Gamma_\gamma(x,z,t) \left(\int_0^\infty \Gamma_\gamma(y,z,s) \varphi(y) y^\gamma\, dy \right) z^\gamma\, dz = \mathcal{H}_t^\gamma \mathcal{H}_s^\gamma \varphi(x).$$

2. Now consider the norm

$$\|\mathcal{H}_t^\gamma \varphi - \varphi\|_{\gamma,\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}_+^n}{}_\gamma |\mathcal{H}_t^\gamma \varphi - \varphi| \le A_\gamma \int_0^\infty (1-e^{-ty^2}) y^\gamma dy \to 0, \qquad t \to 0.$$

4. POLYNOMIAL SOLUTION OF THE SINGULAR HEAT EQUATION

When we deal with infinite-dimensional Hilbert space of functions it is desirable to have an orthonormal basis which provides a connection of this Hilbert space with the space of weighted squared-integrable sequences. This section is concerned with orthonormal systems of polynomials which are associated with the Bessel operator. Polynomial solutions are important in the development of numerical techniques.

Let consider the Cauchy problem for the singular heat equation when $\varphi(x) = x^{2n}$, $n \in \mathbb{N} \cup \{0\}$, $x > 0$:

$$\begin{cases} (B_\gamma)_x u(x,t) = u_t(x,t) \\ u(x,0) = x^{2n}. \end{cases}$$

Using (16) and taking into account formula 2.15.5.4 in [16], we see that for $t > 0$

$$u(x,t) = A(\gamma) t^{-\frac{1+\gamma}{2}} \int_0^\infty e^{-\frac{x^2+y^2}{4t}} i_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right) y^{2n+\gamma}\, dy = \frac{1}{2tx^{\frac{\gamma-1}{2}}} e^{-\frac{x^2}{4t}} \int_0^\infty e^{-\frac{y^2}{4t}} I_{\frac{\gamma-1}{2}}\left(\frac{xy}{2t}\right) y^{2n+\frac{\gamma+1}{2}}\, dy$$

$$= \frac{1}{2tx^{\frac{\gamma-1}{2}}} e^{-\frac{x^2}{4t}} \frac{n! \left(\dfrac{x}{2t}\right)^{\frac{\gamma-1}{2}}}{2^{\frac{\gamma+1}{2}} \left(\dfrac{1}{4t}\right)^{n+\frac{\gamma+1}{2}}} e^{\frac{x^2}{4t}} L_n^{\frac{\gamma-1}{2}}\left(-\frac{x^2}{4t}\right) = 2^{2n} n! t^n L_n^{\frac{\gamma-1}{2}}\left(-\frac{x^2}{4t}\right),$$

where $L_n^\alpha(z)$ is the generalized Laguerre polynomial defined by confluent hypergeometric functions and Kummer's transformation

$$L_n^\alpha(z) := \binom{n+\alpha}{n} M(-n, \alpha+1, z);$$

here $\binom{n+\alpha}{n}$ is a generalized binomial coefficient. Kummer's function of the first kind M is a generalized hypergeometric series:

$$M(a,b,z) = \sum_{n=0}^\infty \frac{a^{(n)} z^n}{b^{(n)} n!} = {}_1F_1(a;b;z),$$

where $a^{(0)} = 1$, $a^{(n)} = a(a+1)(a+2)\cdots(a+n-1)$, is the rising factorial. Another form for these generalized Laguerre polynomials of degree n is

$$L_n^\alpha(x) = \sum_{i=0}^n (-1)^i \binom{n+\alpha}{n-i} \frac{x^i}{i!}.$$

Therefore, the polynomial solution of the singular heat equation is

$$u(x,t) = 2^{2n} n! t^n L_n^{\frac{\gamma-1}{2}} \left(-\frac{x^2}{4t}\right). \tag{18}$$

The generalized Laguerre polynomials L_n^α are orthogonal over $[0,\infty)$ with respect to the measure with weighting function $x^\alpha e^{-x}$:

$$\int\limits_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x)\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta.

So using polynomial solution (18) of the singular heat equation we can obtain a solution of the Cauchy problem (14), if $\varphi(x) = \sum_{n=0}^\infty a_n x^{2n}$, $a_n \in \mathbb{R}$.

Example. Let us consider the problem

$$\begin{cases} (B_\gamma)_x u(x,t) = u_t(x,t) \\ u(x,0) = \cos(x). \end{cases}$$

Since $\cos(x) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} x^{2n}$, taking into account (18), we immediately obtain

$$u(x,t) = \sum_{n=0}^\infty \frac{(-1)^n 2^{2n} n!}{(2n)!} t^n L_n^{\frac{\gamma-1}{2}} \left(-\frac{x^2}{4t}\right).$$

5. FRACTIONAL POWER OF THE BESSEL OPERATOR

For some problems, it is convenient to use symbolic calculus with symbol B_γ; for example, we can use $(B_\gamma)^n$, $n \in \mathbb{N}$, for the iterated operator:

$$(B_\gamma)^n = \underbrace{B_\gamma\, B_\gamma \ldots B_\gamma}_{n}.$$

Thus, taking into account (11), the generalized translation formula [13]

$$^\gamma T_x^y f(x) = \sum_{m=0}^\infty \frac{\Gamma\left(\dfrac{\gamma+1}{2}\right)}{m!\Gamma\left(\dfrac{\gamma+1}{2}+m\right)} \left(\frac{y}{2}\right)^{2m} (B_\gamma)^n f(x)$$

can be written in the form

$$^\gamma T_x^y f(x) = \sum_{m=0}^\infty \frac{\Gamma\left(\dfrac{\gamma+1}{2}\right)}{m!\Gamma\left(\dfrac{\gamma+1}{2}+m\right)} \left(\frac{y\sqrt{B_\gamma}}{2}\right)^{2m} f(x) = i_{\frac{\gamma-1}{2}}\left(y\sqrt{B_\gamma}\right) f(x);$$

i.e., the normalized modified Bessel function of the first kind $i_{\frac{\gamma-1}{2}}(y\sqrt{B_\gamma})$ of the operator $\sqrt{B_\gamma}$ is the generalized translation.

Let us explain how to understand $\sqrt{B_\gamma}$. In [17], a negative fractional power of B_γ was given in the factorized form $B_\gamma^{-\alpha} = F_\gamma^{-1}(x^{-2\alpha} F_\gamma)$, where $x > 0$, F_γ is the Hankel transform (2) and F_γ^{-1} is its inverse (3). We can write an explicit formula

$$(B_\gamma^{-\alpha}\varphi)(x) = \frac{2^{1-2\alpha}\Gamma\left(\dfrac{\gamma+1}{2}-\alpha\right)}{\Gamma\left(\dfrac{\gamma+1}{2}\right)\Gamma(\alpha)} \int\limits_0^\infty ({}^\gamma T_x^y \varphi)(x) x^{2\alpha-1}\, dy.$$

The operator $B_\gamma^{-\alpha}$ is called the *Bessel–Riesz fractional integral*.

Positive fractional power of the Bessel operator B_γ for $0 < \alpha < 1$ is given then as the inverse operator of $B_\gamma^{-\alpha}$ by [17]

$$\left(\mathbf{B}_\gamma^\alpha \varphi\right)(x) = \frac{1}{d_\gamma(\alpha)} \int\limits_0^\infty \frac{\varphi(x) - ({}^\gamma T_x^y \varphi)(x)}{y^{1+2\alpha}}\, dy, \tag{19}$$

where

$$d_\gamma(\alpha) = \frac{\pi\,\Gamma\left(\dfrac{\gamma+1}{2}\right)}{2^{2\alpha+1}\,\Gamma\left(\dfrac{1+\gamma}{2}+\alpha\right)\Gamma\left(\dfrac{1}{2}+\alpha\right)} \frac{1}{\sin\alpha\pi}.$$

CONCLUSIONS

In this paper, our aim was to find and study the solution of the Cauchy problem for the singular heat equation. In particular, we obtained a polynomial solution, which makes it possible to get a solution of the singular heat equation with good enough initial values without having to evaluate the generalized convolution.

FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. Kipriyanov, I.A. and Ivanov, L.A., Fundamental solutions of homogeneous B-hyperbolic equations, *Sib. Math. J.*, 1980, vol. 21, no. 4, pp. 552–557.
2. Kipriyanov, I.A., *Singulyarnye ellipticheskie kraevye zadachi* (Singular Elliptic Boundary Value Problems), Moscow: Nauka, 1997.
3. Kipriyanov, I.A. and Kononenko, V.I., Fundamental solutions of B-elliptic equations, *Differ. Uravn.*, 1967, vol. 3, no. 1, pp. 114–129.
4. Shishkina, E.L., The Dirichlet problem for an elliptic singular equation, *Complex Variables Elliptic Equat.*, 2020, vol. 65, no. 7, pp. 1210–1218. https://doi.org/10.1080/17476933.2019.1588259
5. Zaitseva, N.V., First initial–boundary value problem for B-hyperbolic equation, *Lobachevskii J. Math.*, 2019, vol. 40, no. 2, pp. 240–247. https://doi.org/10.1134/S1995080219020161
6. Zaitseva, N.V., Keldysh-type problems with integral conditions for two-dimensional B-hyperbolic equations, *Moscow Univ. Math. Bull.,* 2021, vol. 76, no. 5, pp. 220–228. https://doi.org/10.3103/S0027132221050077
7. Zaitseva, N.V., *Smeshannye zadachi s integral'nymi usloviyami dlya giperbolicheskikh uravnenii s operatorom Besselya* (Mixed Problems with Integral Conditions for Hyperbolic Equations with the Bessel Operator), Moscow: Izd. Mosk. Gos. Univ., 2021.

8. Zhitomirskii, Ya.I., Cauchy's problem for systems of linear partial differential equations with differential operators of Bessel type, *Mat. Sb. (N.S.)*, 1955, vol. 36 (78), no. 2, pp. 299–310.
9. Muravnik, A.B., Functional differential parabolic equations: Integral transformations and qualitative properties of solutions of the Cauchy problem, *J. Math. Sci.,* 2016, vol. 216, no. 3, pp. 345–496. https://doi.org/10.1007/s10958-016-2904-0
10. Khushtova, F.G., Third boundary-value problem in the half-strip for the B-parabolic equation, *Math. Notes*, 2021, vol. 109, no. 2, pp. 292–301. https://doi.org/10.1134/S0001434621010338
11. Shishkina, E.L. and Sitnik, S.M., General form of the Euler–Poisson–Darboux equation and application of the transmutation method, *Electron. J. Differ. Equat.*, 2017, vol. 2017, no. 177, pp. 1–20.
12. Shishkina, E.L., Solution of the singular Cauchy problem for a general inhomogeneous Euler–Poisson–Darboux equation, *Carpathian J. Math.,* 2018, vol. 34, no. 2, pp. 255–267. https://doi.org/10.37193/CJM.2018.02.14
13. Levitan, B.M., Expansion in Fourier series and integrals with Bessel functions, *Usp. Mat. Nauk*, 1951, vol. 6, no. 2 (42), pp. 102–143.
14. Platonov, S.S., Bessel harmonic analysis and approximation of functions on the half-line, *Izv. Ross. Akad. Nauk. Ser. Mat.*, 2007, vol. 71, no. 5, pp. 149–196.
15. Shishkina, E.L. and Sitnik, S.M., *Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics*, Amsterdam: Elsevier, 2020.
16. Prudnikov, A.P., Brychkov, Yu.A., and Marichev, O.I., *Integrals and Series. 1. Elementary Functions*, New York: Gordon & Breach, 1992.
17. Durdiev, D.K., Shishkina, E.L., and Sitnik, S.M., Fractional powers of Bessel operator and its numerical calculation, *Chelyab. Phys. Math. J.*, 2021, vol. 6, no. 2, pp. 172–189.