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The Integral Transforms Composition Method for Generalized Index Shifted Transmutations

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Abstract—This paper deals with transmutation theory and how we can use the integral transforms composition method (ITCM) to develop generalized transmutation operators. With the ITCM, we can create various transmutation operators by combining different integral transforms like Hankel, Y, Mellin, Laplace, and Fourier, sine- and cosine-Fourier, and other generalized transforms. The paper applies the ITCM to derive connection formulas for solutions of both singular and nonsingular differential equations. It concludes that using the ITCM to construct transmutations is a valuable and practical approach for finding connection formulas and explicit solutions for a wide range of differential equations, including those with Bessel operators.

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1. INTRODUCTION AND PRELIMINARIES

Before dealing with the main aim of this paper, let's start by clarifying some important tools needed for the establishment of our results. Initially, we will cover fundamental topics such as the special functions, integral transforms, transmutation theory, and Bessel operator.

1.1. On the Special Functions

Here are some elementary definitions and brief information on the special functions and classes of functions. Let ${}_2\mathcal{F}_1$ be the hypergeometric function defined by the power series

$$ {}_2\mathcal{F}_1(a,b;c;t)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!},\quad \text{where}\quad |t|<1. \tag{1}$$

The Bessel functions of the first and second kind, respectively, J_ν and Y_ν, of order ν, are defined as follow, by there series expansion around $t=0$:

$$J_\nu(t)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\left(\frac{t}{2}\right)^{2m+\nu}, \tag{2}$$

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and

$$Y_\nu(t) = \frac{\cos \pi\nu J_\nu(t) - J_{-\nu}(t)}{\sin \pi\nu}, \tag{3}$$

where ν is non-integer.

Normalized Bessel function of the first kind j_ν is defined by the formula

$$j_\nu(x) = \frac{2^\nu \Gamma(\nu+1)}{x^\nu}\ J_\nu(x), \tag{4}$$

where J_ν is Bessel function of the first kind. The best general reference here is [17].

We denote by $\mathbb{H}_\nu$ the Struve function given by the following power series expansion

$$\mathbb{H}_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+\frac{3}{2})\,\Gamma(m+\nu+\frac{3}{2})} \left(\frac{t}{2}\right)^{2m+\nu+1}. \tag{5}$$

Having disposed of the functions' definitions (2), (3), and (5), we can now introduce integral transforms involving them as kernels.

1.2. Integral Transforms

The one-dimentional *Hankel transform* of a function $f \in L_{1,\nu}(\mathbb{R}^1_+)$ is defined as

$$H_\nu[f](\xi) = H_\nu[f(x)](\xi) = \widehat{f}(\xi) = \int\limits_0^\infty f(x)\, j_{\frac{\nu-1}{2}}(x\xi) x^\nu dx, \tag{6}$$

where $\nu > 0$, and j_ν is the normalized Bessel function of the first kind (4).

From now on, we regard $f \in \mathbb{S}$, where $\mathbb{S}$ is the space of rapidly decreasing functions on $(0,\infty)$

$$\mathbb{S} = \left\{ f \in C^\infty(0,\infty) : \sup_{t\in(0,\infty)} \left| t^a D^b f(t) \right| < \infty \quad \forall a, b \in Z_+ \right\}.$$

Let the Hankel and $\mathcal{Y}$-transforms be the integral transforms of order ν of a function f defined as follow

$$\left(\mathcal{H}_\nu f\right)(x) = \int\limits_0^{+\infty} (xt)^{\frac{1}{2}} J_\nu(xt) f(t) dt, \tag{7}$$

$$\left(\mathcal{Y}_\nu f\right)(x) = \int\limits_0^{+\infty} (xt)^{\frac{1}{2}} Y_\nu(xt) f(t) dt, \tag{8}$$

$$\left(\mathcal{S}_\nu f\right)(x) = \int\limits_0^{+\infty} (xt)^{\frac{1}{2}} \mathbb{H}_\nu(xt) f(t) dt. \tag{9}$$

It is known that (7), (8), and (9) are invertible transforms, where $(\mathcal{H}_\nu)^{(-1)} = \mathcal{H}_\nu$ and $(\mathcal{Y}_\nu)^{(-1)} = \mathcal{S}_\nu$.

1.3. Transmutation Theory

The overall focus of this paper is mainly on the investigation of these fractional powers of the Bessel operators and their applications to the transmutation theory via integral transforms and transmutation methods.

We start this section by giving an overview of one of the most important parts of our research, "Transmutation Theory". Studies of the linear second order differential equations arise in numerous models and problems of mathematics, physics, and many other scientific fields, while the ones of first order are easily solved.

Throughout the centuries, there was no general method of solving linear ordinary second order differential equations with variable coefficients, and this situation creates a gap in our understanding by the antique mathematicians from the epoch of N. Tartaglia, G. Cardano, and L. Ferrari. At that time, the problem of a closed form solution of such differential equations was not even viewed as among the most important mathematical problems. It is possible that this was because there was no expectation that these types of problems could be solved. To reduce the difficult problem to a simpler one, the idea was to construct an operator to relate solutions of the equation with constant coefficients to solutions of the equation with variable coefficients. This operator is called a "transmutation operator".

Let's consider the following second order linear differential operator

$$\mathcal{L} := -\frac{d^2}{dx^2} + q(x), \tag{10}$$

where q is an L^2-function defined on a finite interval.

The next equation is called the one-dimensional Shrödinger equation or the Sturm–Liouville equation

$$\mathcal{L}y(x) = \gamma y(x), \quad \gamma \in \mathbb{C}, \tag{11}$$

taking in consideration that Liouville transformation reduces a large variety of linear ordinary second order equations to this form.

An intertwining operator is sought to relate $\mathcal{L}$ to the simplest linear second order expression $\mathcal{B} = -\frac{d^2}{dx^2}$ by the formula $\mathcal{L}T = T\mathcal{B}$.

Definition 1. *Let (A, B) be a given pair of operators. An operator T is called transmutation (or intertwining) operator if the following property is valid, on elements of some functional spaces*

$$TA = BT. \tag{12}$$

1.4. The Differential Bessel Operator

The differential Bessel operator is given by

$$B_\nu = D^2 + \frac{\nu}{x}D, \quad \nu \geq 0, \quad D := \frac{d}{dx}, \tag{13}$$

and its fractional powers $(B_\nu)^\alpha, \alpha \in \mathbb{R}$, have been studied in many papers. However, in the majority of them, they were defined implicitly as a power function multiplication under Hankel transform. In [10], fractional powers of the Bessel operators are given in the form

$$(B^{-\alpha}_{\nu,b-}f)(x)=\frac{1}{\Gamma(2\alpha)}\int\limits_x^b \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right) f(y)dy, \tag{14}$$

for $f \in C^{[2\alpha]+1}(0, b]$, $b \in (0, +\infty)$, and

$$(B^{-\alpha}_{\nu,a+}f)(x)=\frac{1}{\Gamma(2\alpha)}\int\limits_a^x \left(\frac{y}{x}\right)^\nu \left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right) f(y)dy, \tag{15}$$

for $f \in C^{[2\alpha]+1}[a, +\infty)$, $a\in(0, +\infty)$.

The operators (14) and (15) are called *The right and Left-sided fractional Bessel integrals*. They are given explicitly in the integral form without using integral transforms in their definitions.

In order to ensure the readers that their definitions really define fractional powers of the Bessel differential operator (13), the authors made simple calculations for two special cases, and they found that for $\nu = 0$,

$$(B_{0,b-}^{-\alpha} f)(x) = (I_{b-}^{2\alpha} f)(x), \quad (B_{0,a+}^{-\alpha} f)(x)y = (I_{a+}^{2\alpha} f)(x),$$

where $I_{b-}^{2\alpha}$ and $I_{a+}^{2\alpha}$ are the right-sided and left-sided Riemann–Liouville fractional integrals given, respectively, by (16) and (17)

$$I_{b^-}^{\alpha}[f](\xi) = \frac{1}{\Gamma(\alpha)} \int\limits_{\xi}^{b} (\xi - \lambda)^{\alpha-1} f(\lambda) d\lambda, \quad \alpha > 0 \quad \text{and} \quad \xi \in [a, b), \tag{16}$$

and

$$I_{a^+}^{\alpha}[f](\xi) = \frac{1}{\Gamma(\alpha)} \int\limits_{a}^{\xi} (\xi - \lambda)^{\alpha-1} f(\lambda) d\lambda, \quad \alpha > 0 \quad \text{and} \quad \xi \in (a, b], \tag{17}$$

$I_{a^+}^{\alpha}[f]$ and $I_{b^-}^{\alpha}[f]$ are defined on (a, b) for $f \in L^1(a, b; \mathbb{R})$.

1.5. The Integral Transform Composition Method

What is the integral transform composition method (ITCM), and how does it work? In transmutation theory, explicit operators were derived based on different ideas and methods, often not connecting altogether. Therefore, there is an urgent need in transmutation theory to develop a general method for obtaining known and new classes of transmutations.

In this subsection, we give such a method for constructing transmutation operators. We call this method Integral Transform Composition Method, or ITCM for short. The method is based on the representation of transmutation operators as compositions of basic integral transforms. The Integral Transform Composition Method (ITCM) gives the algorithm not only for constructing new transmutation operators, but also for all now explicitly known classes of transmutations, including Poisson, Sonine, Vekua–Erdelyi–Lowndes, Buschman–Erdelyi, Sonin–Katrakhov, and Poisson–Katrakhov ones, cf. [1–4, 6, 7, 13–16] as well as the classes of elliptic, hyperbolic and parabolic transmutation operators introduced by R. Carroll [1–3]. This method, with many applications and examples, was essentially introduced and developed by S.M. Sitnik.

The formal algorithm of ITCM is next. Let us take as input a pair of arbitrary operators A and B, and also connect with them generalized Fourier transforms F_A and F_B, which are invertible and act by the formulas

$$F_A A = g(t) F_A, \quad F_B B = g(t) F_B, \tag{18}$$

t is a dual variable and g is an arbitrary function with suitable properties. It is often convenient to choose $g(t) = -t^2$ or $g(t) = -t^{\alpha}$, $\alpha \in \mathbb{R}$.

Then, the essence of ITCM is to obtain formally a pair of transmutation operators P and S as the method output by the next formulas

$$S = F_B^{-1} \frac{1}{w(t)} F_A, \quad P = F_A^{-1} w(t) F_B \tag{19}$$

with arbitrary function $w(t)$. When P and S are transmutation operators intertwining A and B:

$$SA = BS, \quad PB = AP. \tag{20}$$

A formal checking of (20) can be obtained by direct substitution. The main difficulty is the calculation of compositions (19) in an explicit integral form, as well as the choice of domains of operators P and S.

The main advantages of ITCM have been listed in [5, 11].

One obstacle to which to apply ITCM is the next one: we know acting of classical integral transforms usually on standard spaces like L_2, L_p, and C^k, variable exponent Lebesgue spaces and so on. However, for application of transmutations to differential equations, we usually need more conditions to hold, say, at zero or at infinity. For these problems, we may first construct a transmutation by ITCM and then expand it to the needed functional classes.

Let us stress that formulas of the type (19) of course are not new for integral transforms and its applications to differential equations. *However, ITCM is new when applied to transmutation theory!* In other fields of integral transforms and connected differential equations, theory compositions (19) for the choice of classical Fourier transform leads to famous pseudo-differential operators with symbol function $w(t)$. For the choice of the classical Fourier transform and the function $w(t) = (\pm it)^{-s}$ we get fractional integrals on the whole real axis, while for $w(t) = |x|^{-s}$ we get M. Riesz potential, also for $w(t) = (1 + t^2)^{-s}$ in formulas (19) we get Bessel potential and for $w(t) = (1 \pm it)^{-s}$—modified Bessel potentials [9].

The next choice for algorithm of ITCM

$$A = B = B_\nu, \quad F_A = F_B = H_\nu, \quad g(t) = -t^2, \quad w(t) = j_\nu(st) \tag{21}$$

leads to generalized translation operators of Delsarte [16], for this case we have to choose in the algorithm defined by (18) and (19) the above values (21) in which B_ν is the Bessel operator (13), H_ν is the Hankel transform (6), and j_ν is the normalized Bessel function (4).

It is possible to apply ITCM instead of classical approaches for getting fractional powers of Bessel operators [15, 16]. Therefore, we may conclude that the method we consider in the paper for obtaining transmutations—ITCM is effective; it is connected to many known methods and problems, it gives all known classes of explicit transmutations and works as a tool to construct new classes of transmutations. Application of ITCM needs the following three steps:

- **First step**: For a given pair of operators A and B and connected generalized Fourier transforms F_A and F_B define and calculate a pair of transmutations P and S by basic formulas (18) and (19).
- **Second step**: Derive exact conditions and find classes of functions for which transmutations obtained by Step 1 satisfy proper intertwining properties.
- **Third step**: Apply now correctly defined transmutations by the first an second steps on proper classes of functions to deriving connection formulas for solutions of differential equations.

1.6. Some Previous Results

In this subsection, we collect some of our main results on the transmutation theory from [5, 11, 12]. We start by investigating composition operators of the form [11]

$$T = \mathcal{H}_\mu \mathcal{Y}_\nu, \quad T = \mathcal{Y}_\mu \mathcal{Y}_\nu, \quad T = \mathcal{Y}_\mu \mathcal{H}_\nu, \quad T = \mathcal{H}_\mu \mathcal{H}_\nu, \tag{22}$$

where $\mathcal{H}_\mu$ and $\mathcal{Y}_\nu$ are the *Hankel* and *$\mathcal{Y}$-transforms* defined by (7) and (8). The operators T defined in (22) commute with the differential operator

$$L_\nu = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{\nu^2}{x^2}, \tag{23}$$

respecting the common transmutation property $T\,L_\nu = L_\mu\,T$. The compositions $\mathcal{H}_\mu \mathcal{Y}_\nu$ and $\mathcal{Y}_\mu \mathcal{H}_\nu$ are considered as generalizations of Hilbert operators. It is clear that

$$\mathcal{Y}_\nu[\mathcal{H}_\nu f](x) = \frac{2}{\pi}\int\limits_0^\infty \frac{tf(t)}{x^2 - t^2}\left(\frac{x}{t}\right)^{\frac{1}{2}-\nu} dt = -\frac{2}{\pi}\int\limits_0^\infty \frac{tf(t)}{t^2 - x^2}\left(\frac{x}{t}\right)^{-\nu+\frac{1}{2}} dt = -\mathcal{H}_{-\nu}[\mathcal{Y}_{-\nu} f](x), \tag{24}$$

so, only one of them can be considered.

For the special case $\nu = \mp\frac{1}{2}$, it is easy to see that the operator $\mathcal{H}_\nu\, \mathcal{Y}_\nu$ is equal to Hilbert transform on semi-axes

$$\mathcal{H}_{\frac{1}{2}}\, [\mathcal{Y}_{\frac{1}{2}} f](x) = \frac{2}{\pi} \int\limits_0^\infty \frac{x f(t)}{t^2 - x^2} dt.$$

The norms of $\mathcal{H}_\nu$, $\mathcal{Y}_\nu$, and their compositions in $L_2(0,\infty)$ are also studied in this section and it is shown that $||\mathcal{H}_\nu \mathcal{Y}_\nu||_{L_2} = ||\mathcal{Y}_\nu \mathcal{H}_\nu||_{L_2}$.

In [12], the next operators constructed by ITCM may be considered as generalized composition operators of 22 for $\nu = \mu$.

Let $f \in L^2(0,\infty)$,

- $x, y > 0$, $Re\, s < 0$, and $Re\,(s + \nu + 1) > 0$. Then, for the transmutation operator T_1 obtained by ITCM [5], such that $(T_1 f)\,(x) = \mathcal{H}_\nu \left[(-t^2)^s \mathcal{H}_\nu f\right](x)$. The next integral representation is true

$$\mathcal{H}_\nu \left[(-t^2)^s \mathcal{H}_\nu f\right](x) = C_{s,\nu}\, x^{-2s-\nu-\frac{3}{2}} \int\limits_0^x {}_2F_1\left(\nu + s + 1, s + 1; \nu + 1; \frac{y^2}{x^2}\right) y^{\nu+\frac{1}{2}} f(y)\, dy$$
$$+ C_{s,\nu} x^{\nu+\frac{1}{2}} \int\limits_x^{+\infty} {}_2F_1\left(\nu + s + 1, s + 1; \nu + 1; \frac{x^2}{y^2}\right) y^{-2s-\nu-\frac{3}{2}}\, f(y)\, dy, \tag{25}$$

where $C_{s,\nu} = (-1)^s 2^{2s+1} \dfrac{\Gamma(\nu + s + 1)}{\Gamma(-s)\Gamma(\nu + 1)}$ and ${}_2F_1$ is the Gauss hypergeometric function.

- $x, y > 0$, $Re\, s < 0$, and $|Re\,\nu| - Re\nu < Re\,[2(s + 1)] < 2$, we have

$$(T_2 f)\,(x) = \mathcal{H}_\nu \left[(-t^2)^s \mathcal{Y}_\nu f\right](x),$$

where the next integral representation is true

$$(T_2 f)(x) = \frac{(-1)^{s+1}}{2\pi} \left(\frac{2}{x}\right)^{2(s+1)} \cos(\nu\pi) \frac{\Gamma(-\nu)\Gamma(s + 1 + \nu)}{\Gamma(-s)}$$
$$\times \int\limits_0^x \left(\frac{y}{x}\right)^{\nu-\frac{1}{2}} {}_2F_1\left(s + 1, s + 1 + \nu; 1 + \nu; \frac{y^2}{x^2}\right) y\, f(y)\, dy$$
$$+ \frac{(-1)^{s+1}}{2\pi} \left(\frac{2}{x}\right)^{2(s+1)} \frac{\Gamma(\nu)\Gamma(s + 1)}{\Gamma(\nu - s)} \int\limits_0^x \left(\frac{x}{y}\right)^{\nu+\frac{1}{2}} {}_2F_1\left(s + 1, s + 1 - \nu; 1 - \nu; \frac{y^2}{x^2}\right) y\, f(y)\, dy$$
$$+ \frac{(-1)^s}{2\pi} 2^{2(s+1)} \cos(s\pi) \frac{\Gamma(s + 1 + \nu)\Gamma(s + 1)}{\Gamma(\nu + 1)}$$
$$\times \int\limits_x^{+\infty} \left(\frac{x}{y}\right)^{\nu+\frac{1}{2}} {}_2F_1\left(s + 1, s + 1 - \nu; 1 - \nu; \frac{x^2}{y^2}\right) y^{-2s-1}\, f(y)\, dy. \tag{26}$$

- $x, y > 0$, $Re\, s < 0$, and $|Re\,\nu| - Re\,\nu < Re\,[2(s + 1)] < 2$,

$$T_3(f) = \mathcal{Y}_\nu (-t^2)^s \mathcal{H}_\nu(f), \tag{27}$$

where

$$(T_3 f)(x) = \frac{(-1)^s}{2\pi} \left(\frac{2}{x}\right)^{2(s+1)} \cos(s\pi) \frac{\Gamma(s + 1 + \nu)\Gamma(s + 1)}{\Gamma(\nu + 1)}$$

$$\times \int_0^x \left(\frac{y}{x}\right)^{\nu-\frac{1}{2}} {}_2F_1\left(s+1, s+1+\nu; 1+\nu; \frac{y^2}{x^2}\right) y\, f(y)\, dy$$

$$+ \frac{(-1)^{s+1}2^{2s+1}}{\pi}\cos(\nu\pi)\frac{\Gamma(-\nu)\Gamma(s+1+\nu)}{\Gamma(-s)}$$

$$\times \int_x^{+\infty} \left(\frac{x}{y}\right)^{\nu+\frac{1}{2}} y^{-2(s+1)}{}_2F_1\left(s+1, s+1+\nu; 1+\nu; \frac{x^2}{y^2}\right) y\, f(y)\, dy$$

$$+ \frac{(-1)^{s+1}2^{2s+1}}{\pi}\frac{\Gamma(\nu)\Gamma(s+1)}{\Gamma(\nu-s)}$$

$$\times \int_x^{+\infty} \left(\frac{y}{x}\right)^{\nu-\frac{1}{2}} y^{-2(s+1)}{}_2F_1\left(s+1, s+1-\nu; 1-\nu; \frac{x^2}{y^2}\right) y\, f(y)\, dy, \tag{28}$$

- $|Re\,\nu|^2 < Re\,[2(s+1)] < 2$ and $Re\, s < 0$. Then, for the transmutation operator T_4 obtained by ITCM, such that $T_4(f) = \mathcal{Y}_\nu(-t^2)^s\mathcal{Y}_\nu(f)$. The next integral representation is true

$$\left[\mathcal{Y}_\nu(-t^2)^s\mathcal{Y}_\nu f\right](x) = (-1)^s\frac{2^{2s+1}}{\pi^2}x^{\nu+\frac{1}{2}}\cos(\nu\pi)\cos\left((s+1)\pi\right)\Gamma(-\nu)\Gamma(s+1+\nu)\Gamma(s+1)$$

$$\times \int_x^{+\infty} y^{-2(s+1)-\nu+\frac{1}{2}}\, f(y)\, {}_2F_1\left(s+1+\nu, s+1; 1+\nu; \frac{x^2}{y^2}\right) dy$$

$$+ (-1)^s\frac{2^{2s+1}}{\pi^2}x^{-\nu+\frac{1}{2}}\cos\left((s+1-\nu)\pi\right)\Gamma(\nu)\Gamma(s+1-\nu)\Gamma(s+1)$$

$$\times \int_x^{+\infty} y^{-2(s+1)+\nu+\frac{1}{2}}\, f(y)\, {}_2F_1\left(s+1, s+1-\nu; 1-\nu; \frac{x^2}{y^2}\right) dy. \tag{29}$$

In [5], we study the composition (19), of integral transforms with Bessel functions kernels and different indices $\mathcal{H}_\mu$ and $\mathcal{Y}_\nu$ defined by (7) and (8). By applying ITCM of the form

$$T^{(\varphi)}_{\nu,\mu} = F^{-1}_\mu\Big(\varphi(t)F_\nu\Big),$$

we obtain an interesting and important family of transmutations including index shift B-hyperbolic transmutations, "descent" operators, classical Sonine and Poisson-type transmutations, explicit integral representations for fractional powers of the Bessel operator, generalized translations of Delsarte and others, such that $T^{(\varphi)}_{\nu,\mu}B_\nu = B_\mu T^{(\varphi)}_{\nu,\mu}$.

A class of transmutation operators introduced in this paper, for $\varphi(t) = (-t^2)^s$, $s < 0$. They all be have as transmutations, namely as shift parameter operator for the Bessel operator.

For $\varphi(t) = Ct^\alpha$, $C \in \mathbb{R}$ does not depend on t and $T^{(\varphi)}_{\nu,\mu} = T^{(\alpha)}_{\nu,\mu}$ we proved the following integral representation

$$\left(T^{(\alpha)}_{\nu,\mu}f\right)(x) = C\,\frac{2^{\alpha+1}\Gamma\left(\frac{\alpha+\mu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)}\left[\frac{x^{-1-\mu-\alpha}}{\Gamma\left(-\frac{\alpha}{2}\right)}\int_0^x f(y){}_2F_1\left(\frac{\alpha+\mu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{y^2}{x^2}\right) y^\nu dy\right.$$

$$\left. + \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu-\alpha}{2}\right)}\int_x^\infty f(y){}_2F_1\left(\frac{\alpha+\mu+1}{2}, \frac{\alpha+\mu-\nu}{2}+1; \frac{\mu+1}{2}; \frac{x^2}{y^2}\right) y^{\nu-\mu-\alpha-1}dy\right],$$

where ${}_2F_1$ is the Gauss hypergeometric function.

2. GENERALIZATIONS OF FRACTIONAL BESSEL OPERATORS WITH TWO INDICES μ AND ν

The next operators constructed by ITCM [5] may be considered as generalized fractional powers of the Bessel operator introduced in [12], with two indices μ and ν as they annihilate integer powers of classical Bessel operators.

Theorem 1. *Let f be a proper function, x, $y>0$, $Re\, s<0$, $Re\,[2(s+1)+\mu+\nu]>0$, and $Re\,[2(s+1)]<2$. Then, for the transmutation operator T_5 obtained by ITCM, such that $T_5(f)=\mathcal{H}_\nu(-t^2)^s\mathcal{H}_\mu(f)$. The next integral representation is true*

$$\left[\mathcal{H}_\nu(-t^2)^s\mathcal{H}_\mu f\right](x)=(-1)^s 2^{2s+1}x^{-\mu-2s-\frac{3}{2}}\frac{\Gamma(\frac{\mu+\nu}{2}+s+1)}{\Gamma(\frac{\nu-\mu}{2}-s)\Gamma(\mu+1)}$$
$$\times\int_0^x f(y)\,{}_2F_1\left(\frac{\mu+\nu}{2}+s+1,\frac{\mu-\nu}{2}+s+1;\mu+1;\frac{y^2}{x^2}\right)y^{\mu+\frac{1}{2}}dy$$
$$+(-1)^s 2^{2s+1}x^{\nu+\frac{1}{2}}\frac{\Gamma(\frac{\nu+\mu}{2}+s+1)}{\Gamma(\frac{\mu-\nu}{2}-s)\Gamma(\nu+1)}$$
$$\times\int_x^{+\infty} f(y)\,{}_2F_1\left(\frac{\nu+\mu}{2}+s+1,\frac{\nu-\mu}{2}+s+1;\nu+1;\frac{x^2}{y^2}\right)y^{-\nu-2s-\frac{3}{2}}dy, \qquad (30)$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Proof. We have

$$\left[\mathcal{H}_\nu(-t^2)^s\mathcal{H}_\mu f\right](x)=(-1)^s\int_0^{+\infty}(xt)^{\frac{1}{2}}J_\nu(xt)t^{2s}dt\int_0^{+\infty}(yt)^{\frac{1}{2}}J_\mu(yt)\,f(y)\,dy$$
$$=(-1)^s\int_0^x\left(\frac{x}{y}\right)^{\frac{1}{2}}yf(y)dy\int_0^{+\infty}t^{2(s+1)-1}\,J_\mu(yt)\,J_\nu(xt)\,dt$$
$$+(-1)^s\int_x^{+\infty}\left(\frac{x}{y}\right)^{\frac{1}{2}}yf(y)dy\int_0^{+\infty}t^{2(s+1)-1}\,J_\mu(yt)\,J_\nu(xt)\,dt.$$

Using [8] (formula 2.12.31, p. 209)

$$\int_0^{+\infty}x^{\alpha-1}\,J_\mu(bx)\,J_\nu(cx)\,dx=A^\alpha_{\mu,\nu},\quad b,\quad c,\quad Re\,(\alpha+\mu+\nu)>0,\quad Re\,\alpha<2;$$

$$A^\alpha_{\mu,\nu}=2^{\alpha-1}b^{-\nu-\alpha}c^\nu\frac{\Gamma(\frac{\nu+\mu+\alpha}{2})}{\Gamma(\frac{\mu-\nu-\alpha}{2}+1)\Gamma(\nu+1)}\,{}_2F_1\left(\frac{\nu+\mu+\alpha}{2},\frac{\nu-\mu+\alpha}{2};\nu+1;\frac{c^2}{b^2}\right),\quad 0<c<b,$$
$$A^\alpha_{\mu,\nu}=2^{\alpha-1}b^\mu c^{-\mu-\alpha}\frac{\Gamma(\frac{\alpha+\mu+\nu}{2})}{\Gamma(\frac{\nu-\mu-\alpha}{2}+1)\Gamma(\mu+1)}\,{}_2F_1\left(\frac{\alpha+\mu+\nu}{2},\frac{\alpha+\mu-\nu}{2};\mu+1;\frac{b^2}{c^2}\right),$$
$$0<b<c, \qquad (31)$$

for $\alpha=2(s+1)$, $x=t$, $b=y$, and $c=x$, we get

$$\int_0^{+\infty}t^{2s+1}\,J_\mu(yt)\,J_\nu(xt)\,dt=A^{2(s+1)}_{\mu,\nu},$$
$$x,y>0,\quad Re\,s<0,\quad Re\,[2(s+1)+\mu+\nu]>0,\quad Re\,[2(s+1)]<2,$$

where

$$A^{2(s+1)}_{\mu,\nu} = 2^{2s+1} y^{-\nu-2(s+1)} x^{\nu} \frac{\Gamma(\frac{\nu+\mu}{2}+s+1)}{\Gamma(\frac{\mu-\nu}{2}-s)\Gamma(\nu+1)} {}_2F_1\left(\frac{\nu+\mu}{2}+s+1, \frac{\nu-\mu}{2}+s+1; \nu+1; \frac{x^2}{y^2}\right),$$

for $0 < x < y$, and

$$A^{2(s+1)}_{\mu,\nu} = 2^{2s+1} y^{\mu} x^{-\mu-2(s+1)} \frac{\Gamma(\frac{\mu+\nu}{2}+s+1)}{\Gamma(\frac{\nu-\mu}{2}-s)\Gamma(\mu+1)} {}_2F_1\left(\frac{\mu+\nu}{2}+s+1, \frac{\mu-\nu}{2}+s+1; \mu+1; \frac{y^2}{x^2}\right),$$

for $0 < y < x$. Hence,

$$\begin{aligned}\left[\mathcal{H}_\nu(-t^2)^s\mathcal{H}_\mu f\right](x) &= (-1)^s \int\limits_0^x \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1} J_\mu(yt)\, J_\nu(xt)\, dt \\ &+ (-1)^s \int\limits_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1} J_\mu(yt)\, J_\nu(xt)\, dt \\ &= (-1)^s 2^{2s+1} x^{-\mu-2s-\frac{3}{2}} \frac{\Gamma(\frac{\mu+\nu}{2}+s+1)}{\Gamma(\frac{\nu-\mu}{2}-s)\Gamma(\mu+1)} \\ &\times \int\limits_0^x f(y)\, {}_2F_1\left(\frac{\mu+\nu}{2}+s+1, \frac{\mu-\nu}{2}+s+1; \mu+1; \frac{y^2}{x^2}\right) y^{\mu+\frac{1}{2}} dy \\ &+ (-1)^s 2^{2s+1} x^{\nu+\frac{1}{2}} \frac{\Gamma(\frac{\nu+\mu}{2}+s+1)}{\Gamma(\frac{\mu-\nu}{2}-s)\Gamma(\nu+1)} \\ &\times \int\limits_x^{+\infty} f(y)\, {}_2F_1\left(\frac{\nu+\mu}{2}+s+1, \frac{\nu-\mu}{2}+s+1; \nu+1; \frac{x^2}{y^2}\right) y^{-\nu-2s-\frac{3}{2}} dy.\end{aligned}$$

□

Theorem 2. *Let f be a proper function, $x, y > 0$, $Re\, s < 0$, and $|Re\,\mu| + |Re\,\nu| < Re\,[2(s+1)] < 2$. Then, for the transmutation operator T_6 obtained by ITCM, such that $T_6(f) = \mathcal{Y}_\nu(-t^2)^s\mathcal{Y}_\mu(f)$. The next integral representation is true*

$$\begin{aligned}&\left[\mathcal{Y}_\nu(-t^2)^s\mathcal{Y}_\mu f\right](x) \\ &= (-1)^s \frac{2^{2s+1}}{\pi^2 x^{2s+\mu+\frac{3}{2}}} \cos(\mu\pi) \cos\left((s+1+\frac{\mu-\nu}{2})\pi\right) \Gamma(-\mu)\Gamma(s+1+\frac{\mu+\nu}{2})\Gamma\left(s+1+\frac{\mu-\nu}{2}\right) \\ &\times \int\limits_x^{+\infty} f(y) {}_2F_1\left(s+1+\frac{\mu+\nu}{2}, s+1+\frac{\mu-\nu}{2}; 1+\mu; \frac{y^2}{x^2}\right) y^{\mu+\frac{1}{2}} dy \\ &+ (-1)^s \frac{2^{2s+1}}{\pi^2 x^{2s-\mu+\frac{3}{2}}} \cos\left((s+1-\frac{\mu+\nu}{2})\pi\right) \Gamma(\mu)\Gamma\left(s+1-\frac{\mu-\nu}{2}\right)\Gamma\left(s+1-\frac{\mu+\nu}{2}\right) \\ &\times \int\limits_x^{+\infty} f(y) {}_2F_1\left(s+1-\frac{\mu-\nu}{2}, s+1-\frac{\mu+\nu}{2}; 1-\mu; \frac{y^2}{x^2}\right) y^{-\mu+\frac{1}{2}} dy, \end{aligned} \tag{32}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Proof. We have

$$\left[\mathcal{Y}_\nu(-t^2)^s\mathcal{Y}_\mu f\right](x) = (-1)^s \int\limits_0^{+\infty} (xt)^{\frac{1}{2}} Y_\nu(xt) t^{2s} dt \int\limits_0^{+\infty} (yt)^{\frac{1}{2}} Y_\mu(yt)\, f(y)\, dy$$

$$= (-1)^s \int\limits_0^{+\infty} (xy)^{\frac{1}{2}} f(y) dy \int\limits_0^{+\infty} t^{2s+1} Y_\nu(xt)\, Y_\mu(yt)\, dt$$

$$= (-1)^s \int\limits_0^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, Y_\mu(yt)\, Y_\nu(xt)\, dt.$$

Using [8] (formula 2.13.20, p. 278) and (2), we have

$$\int\limits_0^{+\infty} x^{\alpha-1}\, Y_\mu(bx)\, Y_\nu(cx)\, dx = \frac{2^{\alpha-1} b^\mu}{\pi^2 c^{\alpha+\mu}} \cos(\mu\pi) \cos\left(\frac{\alpha+\mu-\nu}{2}\pi\right)$$
$$\Gamma(-\mu)\Gamma\left(\frac{\alpha+\mu+\nu}{2}\right)\Gamma\left(\frac{\alpha+\mu-\nu}{2}\right) {}_2F_1\left(\frac{\alpha+\mu+\nu}{2}, \frac{\alpha+\mu-\nu}{2}; 1+\mu; \frac{b^2}{c^2}\right)$$
$$+ \frac{2^{\alpha-1} c^{\mu-\alpha}}{\pi^2 b^\mu} \cos\left(\frac{\alpha-\mu-\nu}{2}\pi\right)$$
$$\Gamma(\mu)\Gamma\left(\frac{\alpha-\mu+\nu}{2}\right)\Gamma\left(\frac{\alpha-\mu-\nu}{2}\right) {}_2F_1\left(\frac{\alpha-\mu+\nu}{2}, \frac{\alpha-\mu-\nu}{2}; 1-\mu; \frac{b^2}{c^2}\right),$$
$$0 < c < b, \quad |Re\,\mu| + Re\,\nu| < Re\,\alpha < 2. \tag{33}$$

For $\alpha = 2(s+1)$, $x = t$, $b = y$, $c = x$, and $Re\, s < 0$, we obtain

$$\int\limits_0^{+\infty} t^{2s+1}\, Y_\mu(yt)\, Y_\nu(xt)\, dt = \frac{2^{2s+1} y^\mu}{\pi^2 x^{2(s+1)+\mu}} \cos(\mu\pi) \cos\left((s+1+\frac{\mu-\nu}{2})\pi\right)$$
$$\Gamma(-\mu)\Gamma\left(s+1+\frac{\mu+\nu}{2}\right)\Gamma\left(s+1+\frac{\mu-\nu}{2}\right) {}_2F_1\left(s+1+\frac{\mu+\nu}{2}, s+1+\frac{\mu-\nu}{2}; 1+\mu; \frac{y^2}{x^2}\right)$$
$$+ \frac{2^{2s+1} x^{\mu-2(s+1)}}{\pi^2 y^\mu} \cos\left((s+1-\frac{\mu+\nu}{2})\pi\right)$$
$$\Gamma(\mu)\Gamma\left(s+1-\frac{\mu-\nu}{2}\right)\Gamma\left(s+1-\frac{\mu+\nu}{2}\right) {}_2F_1\left(s+1-\frac{\mu-\nu}{2}, s+1-\frac{\mu+\nu}{2}; 1-\mu; \frac{y^2}{x^2}\right),$$
$$0 < x < y, \quad |Re\,\mu + Re\,\nu| < Re\,[2(s+1)] < 2.$$

Thus,

$$\left[\mathcal{Y}_\nu(-t^2)^s \mathcal{Y}_\mu f\right](x) = (-1)^s \int\limits_0^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, Y_\mu(yt)\, Y_\nu(xt)\, dt$$
$$= (-1)^s \frac{2^{2s+1}}{\pi^2 x^{2s+\mu+\frac{3}{2}}} \cos(\mu\pi) \cos\left((s+1+\frac{\mu-\nu}{2})\pi\right) \Gamma(-\mu)\Gamma\left(s+1+\frac{\mu+\nu}{2}\right)\Gamma\left(s+1+\frac{\mu-\nu}{2}\right)$$
$$\times \int\limits_x^{+\infty} f(y) {}_2F_1\left(s+1+\frac{\mu+\nu}{2}, s+1+\frac{\mu-\nu}{2}; 1+\mu; \frac{y^2}{x^2}\right) y^{\mu+\frac{1}{2}} dy$$
$$+ (-1)^s \frac{2^{2s+1}}{\pi^2 x^{2s-\mu+\frac{3}{2}}} \cos\left((s+1-\frac{\mu+\nu}{2})\pi\right) \Gamma(\mu)\Gamma\left(s+1-\frac{\mu-\nu}{2}\right)\Gamma\left(s+1-\frac{\mu+\nu}{2}\right)$$
$$\times \int\limits_x^{+\infty} f(y) {}_2F_1\left(s+1-\frac{\mu-\nu}{2}, s+1-\frac{\mu+\nu}{2}; 1-\mu; \frac{y^2}{x^2}\right) y^{-\mu+\frac{1}{2}} dy.$$

□

Theorem 3. *Let f be a proper function, $x,\, y>0$; $Re\, s<0$, $|Re\,\nu|-Re\,\mu<Re\,[2(s+1)]<2$; $T_7(f)=\mathcal{Y}_\nu(-t^2)^s\mathcal{H}_\mu(f)$. The next integral representation is true*

$$\left[\mathcal{Y}_\nu(-t^2)^s\mathcal{H}_\mu f\right](x)=(-1)^{s+1}\frac{2^{2s+1}}{\pi x^{2s+\mu+\frac{3}{2}}}\cos\left((s+1+\frac{\mu-\nu}{2})\pi\right)\frac{\Gamma(s+1+\frac{\mu+\nu}{2})\Gamma(s+1+\frac{\mu-\nu}{2})}{\Gamma(\mu+1)}$$
$$\times\int\limits_0^x {}_2F_1\left(s+1+\frac{\mu+\nu}{2},s+1+\frac{\mu-\nu}{2};\mu+1;\frac{y^2}{x^2}\right)y^{\mu+\frac{1}{2}}f(y)dy$$
$$+(-1)^{s+1}\frac{2^{2s+1}\,x^{\nu+\frac{1}{2}}}{\pi}\cos(\nu\pi)\frac{\Gamma(-\nu)\Gamma(s+1+\frac{\mu+\nu}{2})}{\Gamma(-s-\frac{\mu-\nu}{2})}$$
$$\times\int\limits_x^{+\infty} {}_2F_1\left(s+1-\frac{\mu-\nu}{2},s+1+\frac{\mu+\nu}{2};1+\nu;\frac{x^2}{y^2}\right)y^{-\nu-2s-\frac{3}{2}}f(y)dy$$
$$+(-1)^{s+1}\frac{2^{2s+1}\,x^{-\nu+\frac{1}{2}}}{\pi}\frac{\Gamma(\nu)\Gamma(s+1+\frac{\mu-\nu}{2})}{\Gamma(-s+\frac{\mu+\nu}{2})}$$
$$\times\int\limits_x^{+\infty} {}_2F_1\left(s+1-\frac{\mu+\nu}{2},s+1+\frac{\mu-\nu}{2};1-\nu;\frac{x^2}{y^2}\right)y^{\nu-2s-\frac{3}{2}}f(y)dy, \quad (34)$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Proof. We have

$$\left[\mathcal{Y}_\nu(-t^2)^s\mathcal{H}_\mu f\right](x)=\int\limits_0^{+\infty}(xt)^{\frac{1}{2}}Y_\nu(xt)\left[(-t^2)^s\int\limits_0^{+\infty}(yt)^{\frac{1}{2}}J_\mu(yt)f(y)dy\right]dt$$
$$=(-1)^s\int\limits_0^{+\infty}\left(\frac{x}{y}\right)^{\frac{1}{2}}yf(y)dy\int\limits_0^{+\infty}t^{2(s+1)-1}J_\mu(yt)\,Y_\nu(xt)\,dt$$
$$=(-1)^s\int\limits_0^{x}\left(\frac{x}{y}\right)^{\frac{1}{2}}yf(y)dy\int\limits_0^{+\infty}t^{2(s+1)-1}\,J_\mu(yt)\,Y_\nu(xt)\,dt$$
$$+(-1)^s\int\limits_x^{+\infty}\left(\frac{x}{y}\right)^{\frac{1}{2}}yf(y)dy\int\limits_0^{+\infty}t^{2(s+1)-1}\,J_\mu(yt)\,Y_\nu(xt)\,dt.$$

Using formula 4. 2.13.15 from [8] (p. 272) of the form

$$\int\limits_0^{+\infty}x^{\alpha-1}\,J_\mu(bx)\,Y_\nu(cx)\,dx$$
$$=-\frac{2^{\alpha-1}\,b^\mu}{\pi c^{\alpha+\mu}}\cos\left(\frac{\alpha+\mu-\nu}{2}\pi\right)\frac{\Gamma(\frac{\alpha+\mu+\nu}{2})\Gamma(\frac{\alpha+\mu-\nu}{2})}{\Gamma(\mu+1)}\,{}_2F_1\left(\frac{\alpha+\mu+\nu}{2},\frac{\alpha+\mu-\nu}{2};\mu+1;\frac{b^2}{c^2}\right),$$
$$\text{for}\quad 0<b<c; \quad (35)$$

$$\int\limits_0^{+\infty}x^{\alpha-1}\,J_\mu(bx)\,Y_\nu(cx)\,dx$$
$$=-\frac{2^{\alpha-1}\,c^\nu}{\pi b^{\alpha+\nu}}\cos(\nu\pi)\frac{\Gamma(-\nu)\Gamma(\frac{\alpha+\mu+\nu}{2})}{\Gamma(1-\frac{\alpha+\mu-\nu}{2})}\,{}_2F_1\left(\frac{\alpha-\mu+\nu}{2},\frac{\alpha+\mu+\nu}{2};1+\nu;\frac{c^2}{b^2}\right)$$

$$- \frac{2^{\alpha-1}\, b^{\nu-\alpha}}{\pi c^{\nu}} \frac{\Gamma(\nu)\Gamma(\frac{\alpha+\mu-\nu}{2})}{\Gamma(1+\frac{\nu+\mu-\alpha}{2})} \, {}_2F_1\left(\frac{\alpha-\mu-\nu}{2}, \frac{\alpha+\mu-\nu}{2}; 1-\nu; \frac{c^2}{b^2}\right), \quad \text{for} \quad 0<c<b, \qquad (36)$$

for $\alpha = 2(s+1)$, $x = t$, $b = y$, and $c = x$ we obtain

$$\int\limits_0^{+\infty} t^{2s+1}\, J_\mu(yt)\, Y_\nu(xt)\, dt$$

$$= -\frac{2^{2s+1}\, y^{\mu}}{\pi x^{2(s+1)+\mu}} \cos\left((s+1+\frac{\mu-\nu}{2})\pi\right) \frac{\Gamma(s+1+\frac{\mu+\nu}{2})\Gamma(s+1+\frac{\mu-\nu}{2})}{\Gamma(\mu+1)}$$

$${}_2F_1\left(s+1+\frac{\mu+\nu}{2}, s+1+\frac{\mu-\nu}{2}; \mu+1; \frac{y^2}{x^2}\right), \quad \text{for} \quad 0<y<x;$$

$$\int\limits_0^{+\infty} t^{2s+1}\, J_\mu(yt)\, Y_\nu(xt)\, dt = -\frac{2^{2s+1}\, x^{\nu}}{\pi y^{2(s+1)+\nu}} \cos(\nu\pi) \frac{\Gamma(-\nu)\Gamma(s+1+\frac{\mu+\nu}{2})}{\Gamma(-s-\frac{\mu-\nu}{2})}$$

$$\times {}_2F_1\left(s+1-\frac{\mu-\nu}{2}, s+1+\frac{\mu+\nu}{2}; 1+\nu; \frac{x^2}{y^2}\right) - \frac{2^{2s+1}\, y^{\nu-2(s+1)}}{\pi x^{\nu}} \frac{\Gamma(\nu)\Gamma(s+1+\frac{\mu-\nu}{2})}{\Gamma(-s+\frac{\mu+\nu}{2})}$$

$$\times {}_2F_1\left(s+1-\frac{\mu+\nu}{2}, s+1+\frac{\mu-\nu}{2}; 1-\nu; \frac{x^2}{y^2}\right), \quad \text{for} \quad 0<x<y;$$

$$x, y > 0 \quad Re\, s < 0; \quad |Re\,\nu| - Re\,\mu < Re\,[2(s+1)] < 2.$$

Thus,

$$\left[\mathcal{Y}_\nu(-t^2)^s \mathcal{H}_\mu f\right](x) = (-1)^s \int\limits_0^x \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\mu(yt)\, Y_\nu(xt)\, dt$$

$$+ (-1)^s \int\limits_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\mu(yt)\, Y_\nu(xt)\, dt$$

$$= (-1)^{s+1} \frac{2^{2s+1}}{\pi x^{2s+\mu+\frac{3}{2}}} \cos\left((s+1+\frac{\mu-\nu}{2})\pi\right) \frac{\Gamma(s+1+\frac{\mu+\nu}{2})\Gamma(s+1+\frac{\mu-\nu}{2})}{\Gamma(\mu+1)}$$

$$\times \int\limits_0^x {}_2F_1\left(s+1+\frac{\mu+\nu}{2}, s+1+\frac{\mu-\nu}{2}; \mu+1; \frac{y^2}{x^2}\right) y^{\mu+\frac{1}{2}} f(y) dy$$

$$+ (-1)^{s+1} \frac{2^{2s+1}\, x^{\nu+\frac{1}{2}}}{\pi} \cos(\nu\pi) \frac{\Gamma(-\nu)\Gamma(s+1+\frac{\mu+\nu}{2})}{\Gamma(-s-\frac{\mu-\nu}{2})}$$

$$\times \int\limits_x^{+\infty} {}_2F_1\left(s+1-\frac{\mu-\nu}{2}, s+1+\frac{\mu+\nu}{2}; 1+\nu; \frac{x^2}{y^2}\right) y^{-\nu-2s-\frac{3}{2}} f(y) dy$$

$$+ (-1)^{s+1} \frac{2^{2s+1}\, x^{-\nu+\frac{1}{2}}}{\pi} \frac{\Gamma(\nu)\Gamma(s+1+\frac{\mu-\nu}{2})}{\Gamma(-s+\frac{\mu+\nu}{2})}$$

$$\times \int\limits_x^{+\infty} {}_2F_1\left(s+1-\frac{\mu+\nu}{2}, s+1+\frac{\mu-\nu}{2}; 1-\nu; \frac{x^2}{y^2}\right) y^{\nu-2s-\frac{3}{2}} f(y) dy.$$

□

Theorem 4. *Let f be a proper function, $x, y > 0$, $Re\, s < 0$, and $|Re\,\mu| - Re\,\nu < Re\,[2(s+1)] < 2$. Then, for the transmutation operator T_8 obtained by ITCM, such that $T_8(f) = \mathcal{H}_\nu(-t^2)^s\mathcal{Y}_\mu(f)$ the next integral representation is true*

$$\left[\mathcal{H}_\nu(-t^2)^s\mathcal{Y}_\mu f\right](x) = (-1)^{s+1}\frac{2^{2s+1}}{\pi x^{2s+\mu+\frac{3}{2}}}\cos(\mu\pi)\frac{\Gamma(-\mu)\Gamma(s+1+\frac{\nu+\mu}{2})}{\Gamma(-s-\frac{\nu-\mu}{2})}$$
$$\times\int_0^x f(y)\,{}_2F_1\left(s+1-\frac{\nu-\mu}{2}, s+1+\frac{\nu+\mu}{2}; 1+\mu; \frac{y^2}{x^2}\right)y^{\mu+\frac{1}{2}}dy$$
$$+(-1)^{s+1}\frac{2^{2s+1}}{\pi x^{2s-\mu+\frac{3}{2}}}\frac{\Gamma(\mu)\Gamma(s+1+\frac{\nu-\mu}{2})}{\Gamma(\frac{\mu+\nu}{2}-s)}$$
$$\times\int_0^x f(y)\,{}_2F_1\left(s+1-\frac{\nu+\mu}{2}, s+1+\frac{\nu-\mu}{2}; 1-\mu; \frac{y^2}{x^2}\right)y^{-\mu+\frac{1}{2}}dy$$
$$+(-1)^{s+1}\frac{2^{2s+1}\,x^{\nu+\frac{1}{2}}}{\pi}\cos\left((s+1+\frac{\nu-\mu}{2})\pi\right)\frac{\Gamma(s+1+\frac{\nu+\mu}{2})\Gamma(s+1+\frac{\nu-\mu}{2})}{\Gamma(\nu+1)}$$
$$\times\int_x^{+\infty} f(y)\,{}_2F_1\left(s+1+\frac{\nu+\mu}{2}, s+1+\frac{\nu-\mu}{2}; \nu+1; \frac{x^2}{y^2}\right)y^{-2s-\nu-\frac{3}{2}}dy, \quad (37)$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Proof. We have

$$\left[\mathcal{H}_\nu(-t^2)^s\mathcal{Y}_\mu f\right](x) = \int_0^{+\infty}(xt)^{\frac{1}{2}}J_\nu(xt)\left[(-t^2)^s\int_0^{+\infty}(yt)^{\frac{1}{2}}Y_\mu(yt)f(y)dy\right]dt$$
$$= (-1)^s\int_0^x\left(\frac{x}{y}\right)^{\frac{1}{2}}yf(y)dy\int_0^{+\infty}t^{2(s+1)-1}\,J_\nu(xt)\,Y_\mu(yt)\,dt$$
$$+(-1)^s\int_x^{+\infty}\left(\frac{x}{y}\right)^{\frac{1}{2}}yf(y)dy\int_0^{+\infty}t^{2(s+1)-1}\,J_\nu(xt)\,Y_\mu(yt)\,dt.$$

Using (35) and (36) for $\alpha = 2(s+1)$, $\mu = \nu$, $\nu = \mu$, $x = t$, $b = x$, and $c = y$, we obtain

$$\int_0^{+\infty}x^{2(s+1)-1}\,J_\nu(xt)\,Y_\mu(yt)\,dt$$
$$= -\frac{2^{2s+1}\,x^\nu}{\pi y^{2(s+1)+\nu}}\cos\left((s+1+\frac{\nu-\mu}{2})\pi\right)\frac{\Gamma(s+1+\frac{\nu+\mu}{2})\Gamma(s+1+\frac{\nu-\mu}{2})}{\Gamma(\nu+1)}$$
$$\times\,{}_2F_1\left(s+1+\frac{\nu+\mu}{2}, s+1+\frac{\nu-\mu}{2}; \nu+1; \frac{x^2}{y^2}\right), \quad \text{for} \quad 0 < x < y.$$

$$\int_0^{+\infty}t^{2(s+1)-1}\,J_\nu(xt)\,Y_\nu(yt)\,dt = -\frac{2^{2s+1}\,y^\mu}{\pi x^{2(s+1)+\mu}}\cos(\mu\pi)\frac{\Gamma(-\mu)\Gamma(s+1+\frac{\nu+\mu}{2})}{\Gamma(-s-\frac{\nu-\mu}{2})}$$
$$\times\,{}_2F_1\left(s+1-\frac{\nu-\mu}{2}, s+1+\frac{\nu+\mu}{2}; 1+\mu; \frac{y^2}{x^2}\right)$$
$$-\frac{2^{2s+1}\,x^{\mu-2(s+1)}}{\pi y^\mu}\frac{\Gamma(\mu)\Gamma(s+1+\frac{\nu-\mu}{2})}{\Gamma(\frac{\mu+\nu}{2}-s)}$$

$$\times\, {}_2F_1\left(s+1-\frac{\nu+\mu}{2}, s+1+\frac{\nu-\mu}{2}; 1-\mu; \frac{y^2}{x^2}\right), \quad \text{for} \quad 0<y<x;$$

$x, y > 0$, $Re\, s < 0$, and $|Re\, \mu| - Re\, \nu < Re\,[2(s+1)] < 2$. Thus,

$$\left[\mathcal{H}_\nu(-t^2)^s \mathcal{Y}_\mu f\right](x) = (-1)^s \int_0^x \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int_0^{+\infty} t^{2(s+1)-1}\, J_\nu(xt)\, Y_\mu(yt)\, dt$$

$$+ (-1)^s \int_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int_0^{+\infty} t^{2(s+1)-1}\, J_\nu(xt)\, Y_\mu(yt)\, dt$$

$$= (-1)^{s+1} \frac{2^{2s+1}}{\pi x^{2s+\mu+\frac{3}{2}}} \cos(\mu\pi) \frac{\Gamma(-\mu)\Gamma(s+1+\frac{\nu+\mu}{2})}{\Gamma(-s-\frac{\nu-\mu}{2})}$$

$$\times \int_0^x f(y)_2F_1\left(s+1-\frac{\nu-\mu}{2}, s+1+\frac{\nu+\mu}{2}; 1+\mu; \frac{y^2}{x^2}\right) y^{\mu+\frac{1}{2}} dy$$

$$+ (-1)^{s+1} \frac{2^{2s+1}}{\pi x^{2s-\mu+\frac{3}{2}}} \frac{\Gamma(\mu)\Gamma(s+1+\frac{\nu-\mu}{2})}{\Gamma(\frac{\mu+\nu}{2}-s)}$$

$$\times \int_0^x f(y)_2F_1\left(s+1-\frac{\nu+\mu}{2}, s+1+\frac{\nu-\mu}{2}; 1-\mu; \frac{y^2}{x^2}\right) y^{-\mu+\frac{1}{2}} dy$$

$$+ (-1)^{s+1} \frac{2^{2s+1}\, x^{\nu+\frac{1}{2}}}{\pi} \cos\left((s+1+\frac{\nu-\mu}{2})\pi\right) \frac{\Gamma(s+1+\frac{\nu+\mu}{2})\Gamma(s+1+\frac{\nu-\mu}{2})}{\Gamma(\nu+1)}$$

$$\times \int_x^{+\infty} f(y)_2F_1\left(s+1+\frac{\nu+\mu}{2}, s+1+\frac{\nu-\mu}{2}; \nu+1; \frac{x^2}{y^2}\right) y^{-2s-\nu-\frac{3}{2}} dy.$$

□

They all act as transmutations, namely as shift parameter operator for the Bessel operator $TB_\mu = B_\nu T$.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. R. W. Carroll, *Transmutation and Operator Differential Equations,* Vol. 37 of *Mathematics Studies* (North-Holland, Amsterdam, 1979).
2. R. W. Carroll, *Transmutation, Scattering Theory and Special Functions,* Vol. 69 of *Mathematics Studies* (North-Holland, Amsterdam, 1982).
3. R. W. Carroll, *Transmutation Theory and Applications,* Vol. 117 of *Mathematics Studies* (North Holland, Amsterdam, 1985).
4. R. W. Carroll and R. E. Showalter, *Singular and Degenerate Cauchy Problems* (Academic Press, New York, 1976).
5. A. Fitouhi, I. Jebabli, E. L. Shishkina, and S. M. Sitnik, "Applications of the integral transforms composition method to wave-type singular differential equations and index shift transmutations," Electron. J. Differ. Equat. **2018** (130), 1–27 (2018).

6. V. V. Katrakhov and S. M. Sitnik, "Composition method of construction of B-elliptic, B-parabolic and B-hyperbolic transmutation operators," Dokl. Akad. Nauk **337**, 307–311 (1994).
7. V. V. Katrakhov and S. M. Sitnik, "The transmutation method and boundary-value problems for singular elliptic equations," Contemp. Math. Fundam. Direct. **4**, 211–426 (2018).
8. A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series,* Vol. 2 *Special Functions* (Gordon and Breach, New York, 1990).
9. S. G. Samko, A. A. Kilbas, and O. L. Marichev, *Fractional Integrals and Derivatives, Theory and Applications* (Gordon and Breach Science, Amsterdam, 1993).
10. E. L. Shishkina and S. M. Sitnik, "On fractional powers of Bessel operators," J. Inequal. Spec. Funct. **8** (1), 49–67 (2017).
11. E. L. Shishkina, S. M. Sitnik, and I. Jebabli, "Composition of Hankel type transforms via integral transforms composition method," Sib. Electron. Math. Rep. **8**, 884–900 (2021) https://doi.org/10.33048/semi.2021.18.067
12. S. M. Sitnik and I. Jebabli, "On applications of integral transforms composition method," Lobachevskii J. Math. **44**, 1029–1042 (2023). https://doi.org/10.1134/S1995080223030319
13. S. M. Sitnik, "Factorization and norm estimation in weighted Lebesgue spaces of Buschman–Erdelyi operators," Dokl. Akad. Nauk **320**, 1326–1330 (1991).
14. S. M. Sitnik, "Transmutations and applications: A survey," arXiv: 1012.3741 (2010).
15. S. M. Sitnik and E. L. Shishkina, *Transmutation Operators Method for Differential Equations with Bessel Operator* (Fizmatlit, Moscow, 2019) [in Russian].
16. S. M. Sitnik and E. L. Shishkina, *Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics,* Part of *Mathematics in Science and Engineering* (Academic, Elsevier, 2020).
17. G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge Univ. Press, Cambridge, 1966).