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A Special Type of Multi-Dimensional Integral Transform with Fox H-Function in Lebesgue-type Weighted Spaces

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Abstract—This paper is devoted to the study of multi-dimensional integral transform with Fox H-function in kernels in weighted spaces of Lebesgue measurable functions in the domain $\mathbb{R}^n_+$ with positive coordinates. By using the technique of the multidimensional Mellin transformation, mapping properties such as the boundedness, the range, the representations of the considered transform are established. Research results generalize those obtained earlier for the corresponding one-dimensional transformation.

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Dedicated to memory of the prominent mathematician Valeriy Vyacheslavovich Katrakhov on the occasion of his 75th jubilee

1. INTRODUCTION

We consider the multi-dimensional H-integral transform ([1], formula (1))

$$(\mathrm{H}f)(\mathbf{x}) = \int\limits_{\mathbf{0}}^{\infty} \mathrm{H}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}} \left[\mathbf{xt} \,\middle|\, \begin{matrix} (\mathbf{a}_i, \overline{\alpha}_i)_{1,p} \\ (\mathbf{b}_j, \overline{\beta}_j)_{1,q} \end{matrix} \right] f(\mathbf{t})\mathbf{dt}, \quad \mathbf{x} > 0; \tag{1}$$

where (see [1–3]; [4], Ch. 28; [5], Ch. 1; [6, 7]) $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$; $\mathbf{t} = (t_1, t_2, ..., t_n) \in \mathbb{R}^n$, $\mathbb{R}^n$ is the n-dimensional Euclidean space; $\mathbf{x} \cdot \mathbf{t} = \sum_{n=1}^{n} x_n t_n$ denotes their scalar product; in particular, $\mathbf{x} \cdot \mathbf{1} = \sum_{n=1}^{n} x_n$ for $\mathbf{1} = (1, 1, ..., 1)$. The inequality $\mathbf{x} > \mathbf{t}$ means that $x_1 > t_1, ..., x_n > t_n$, and inequalities $\geq$, $<$, $\leq$ have similar meanings; $\int_{\mathbf{0}}^{\infty} = \int_{\mathbf{0}}^{\infty} \int_{\mathbf{0}}^{\infty} \cdots \int_{\mathbf{0}}^{\infty}$; by $\mathbb{N} = \{1, 2, ...\}$, we denote the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\mathbb{N}_0^n = \mathbb{N}_0 \times ... \times \mathbb{N}_0$; $k = (k_1, k_2, ..., k_n) \in \mathbb{N}_0^n$ ($k_i \in \mathbb{N}_0$, $i = 1, 2, ..., n$) is a multi-index with $k! = k_1! \cdots k_n!$ and $|k| = k_1 + ... + k_n$; $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} > 0\}$; for $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n) \in \mathbb{R}^n_+$ $\mathbf{D}^\kappa = \frac{\partial^{|\kappa|}}{(\partial x_1)^{\kappa_1} \cdots (\partial x_n)^{\kappa_n}}$; $\mathbf{dt} = dt_1 \cdots dt_n$; $\mathbf{t}^\kappa = t^{\kappa_1} t^{\kappa_2} \cdots t^{\kappa_n}$; $f(\mathbf{t}) = f(t_1, t_2, ..., t_n)$; $\mathbb{C}^n$ ($n \in \mathbb{N}$) be the n-dimensional space of n complex numbers $z = (z_1, z_2, \cdots, z_n)$ ($z_j \in \mathbb{C}$, $j = 1, 2, \cdots, n$); $\overline{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{C}^n$; $\overline{h} = (h_1, h_2, ..., h_n) \in \mathbb{R}^n_+$; $\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} = \frac{d}{dx_1 \cdot dx_2 \cdots dx_n}$; $\mathbf{m} = (m_1, m_2, ..., m_n) \in \mathbb{N}_0^n$ and $m_1 = m_2 = ... = m_n$; $\mathbf{n} = (\overline{n}_1, \overline{n}_2, ..., \overline{n}_n) \in \mathbb{N}_0^n$ and $\overline{n}_1 = \overline{n}_2 = ... = \overline{n}_n$; $\mathbf{p} = (p_1, p_2, ..., p_n) \in \mathbb{N}_0^n$ and $p_1 = p_2 = ... = p_n$; $\mathbf{q} = (q_1, q_2, ..., q_n) \in \mathbb{N}_0^n$ and $q_1 = q_2 = ... = q_n$ ($0 \leq \mathbf{m} \leq \mathbf{q}$, $0 \leq \mathbf{n} \leq \mathbf{p}$);

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$\mathbf{a}_i = (a_{i_1}, a_{i_2}, ..., a_{i_n})$, $1 \le i \le \mathbf{p}$, $a_{i_1}, a_{i_2}, ..., a_{i_n} \in \mathbb{C}$ $(i_1 = 1, 2, ..., p_1; ...; i_n = 1, 2, ..., p_n)$;
$\mathbf{b}_j = (b_{j_1}, b_{j_2}, ..., b_{j_n})$, $1 \le j \le \mathbf{q}$, $b_{j_1}, b_{j_2}, ..., b_{j_n} \in \mathbb{C}$ $(j_1 = 1, 2, ..., q_1; ...; j_n = 1, 2, ..., q_n)$;
$\overline{\alpha}_i = (\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_n})$, $1 \le i \le \mathbf{p}$, $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_n} \in \mathbb{R}_1^+$ $(i_1 = 1, 2, ..., p_1; ...; i_n = 1, 2, ..., p_n)$;
$\overline{\beta}_j = (\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_n})$, $1 \le j \le \mathbf{q}$, $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_n} \in \mathbb{R}_1^+$ $(j_1 = 1, 2, ..., q_1; ...; j_n = 1, 2, ..., q_n)$.

The function in the kernel of (1)

$$\mathrm{H}^{\mathbf{m},\,\mathbf{n}}_{\mathbf{p},\,\mathbf{q}}\left[\mathbf{x}\mathbf{t}\,\middle|\,\begin{matrix}(\mathbf{a}_i, \overline{\alpha}_i)_{1,p}\\(\mathbf{b}_j, \overline{\beta}_j)_{1,q}\end{matrix}\right] = \prod_{k=1}^{n} H^{m_k,\,\overline{n}_k}_{p_k,\,q_k}\left[x_k t_k\,\middle|\,\begin{matrix}(a_{i_k}, \alpha_{i_k})_{1,p_k}\\(b_{j_k}, \beta_{j_k})_{1,q_k}\end{matrix}\right] \tag{2}$$

is the product of H-functions $H^{m,n}_{p,q}[z]$:

$$H^{m,n}_{p,q}[z] \equiv H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i, \alpha_i)_{1,p}\\(b_j, \beta_j)_{1,q}\end{matrix}\right] = \frac{1}{2\pi i}\int_L \mathcal{H}^{m,n}_{p,q}(s) z^{-s} ds, \quad z \ne 0, \tag{3}$$

where

$$\mathcal{H}^{m,n}_{p,q}(s) \equiv \mathcal{H}^{m,n}_{p,q}\left[\begin{matrix}(a_i, \alpha_i)_{1,p}\\(b_j, \beta_j)_{1,q}\end{matrix}|s\right] = \frac{\prod_{j=1}^{m}\Gamma(b_j + \beta_j s)\prod_{i=1}^{n}\Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i + \alpha_i s)\prod_{j=m+1}^{q}\Gamma(1 - b_j - \beta_j s)}. \tag{4}$$

In the representation (3), L is a specially chosen infinite contour, and the empty products, if any, are taken to be one.

The H-function (3) is the most general of the known special functions and includes, as special cases, elementary functions, special functions of hypergeometric and Bessel type, as well as the Meyer G-function. One may find its properties, for example, in the books ([8], Ch. 2; [9], Ch. 1; [10], Section 8.3; [11]; [12], Ch. 1–Ch. 4; [21, 23]).

In the paper [1], we have already considered the integral transformation (1), where we characterize the existence, the boundedness and representation properties of the H-transform on Lebesgue-type weighted spaces $\mathfrak{L}_{\overline{\nu},\overline{2}}$ of functions $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ on $\mathbb{R}^{\mathbf{n}}_+$, such that

$$||f||_{\overline{\nu},\overline{2}} = \left\{\int_{\mathbb{R}^1_+} x_n^{2\nu_n - 1}\left\{\cdots\left\{\int_{\mathbb{R}^1_+} x_2^{2\nu_2-1}\int_{\mathbb{R}^1_+} x_1^{2\nu_1-1}|f(x_1, ..., x_n)|^2 dx_1]dx_2\right\}\cdots\right\}dx_n\right\}^{1/2} < \infty,$$

$\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{R}^n$, $\nu_1 = \nu_2 = ... = \nu_n$, and $\overline{2} = (2, 2, ..., 2)$.

The present work is devoted to extending the above results from $\overline{r} = 2$ to any $\overline{r} \ge 1$. Moreover, we will deal with the study of properties of the transform (1) on Lebesgue-type weighted spaces $\mathfrak{L}_{\overline{\nu},\overline{r}}$ of functions $f(\mathbf{x})$, such that

$$||f||_{\overline{\nu},\overline{r}} = \left\{\int_{\mathbb{R}^1_+} x_n^{\nu_n r_n - 1}\left\{\cdots\left\{\int_{\mathbb{R}^1_+} x_2^{\nu_2 r_2 - 1}\right.\right.\right.$$

$$\left.\left.\left.\times\int_{\mathbb{R}^1_+} x_1^{\nu_1 r_1 - 1}|f(x_1, ..., x_n)|^{r_1} dx_1]^{r_2/r_1} dx_2\right\}^{r_3/r_2}\cdots\right\}^{r_n/r_{n-1}} dx_n\right\}^{1/r_n} < \infty,$$

$\overline{r} = (r_1, r_2, ..., r_n) \in \mathbb{R}^n$, $1 \le \overline{r} < \infty$, $r_1 = r_2 = ... = r_n$; $\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{R}^n$, $\nu_1 = \nu_2 = ... = \nu_n$.

Research results for the transformation (1) generalize those obtained earlier for the corresponding one-dimensional transformation (see [12], Ch. 4.1)

$$(\mathrm{H}f)(x) = \int_{\mathbf{0}}^{\infty} \mathrm{H}^{m,\,n}_{p,\,q}\left[xt\,\middle|\,\begin{matrix}(a_i, \alpha_i)_{1,p}\\(b_j, \beta_j)_{1,q}\end{matrix}\right] f(t)dt, \quad x > 0; \tag{5}$$

in the space $\mathfrak{L}_{\nu,r}$ of Lebesgue measurable functions f on $\mathbb{R}^1_+ = (0,\infty)$ for which

$$||f||_{\nu,r} = \left\{ \int_0^\infty |t^\nu f(t)|^r \frac{dt}{t} \right\}^{1/r} < \infty \quad (1 \le r < \infty, \nu \in \mathbb{R}).$$

The H-transform (5) generalizing many integral transforms: transforms with the Meijer G-function, Laplace and Hankel transforms, transforms with Gauss hypergeometric function, transforms with other hypergeometric and Bessel functions in the kernels. One may find a survey of results and bibliography in this field for one-dimensional case in the monograph ([12], Sections 6–8). Besides it the type of integral transforms considered in this paper generalizes well-known transforms with Legendre function kernels [17] and Buschman–Erdélyi operators [18–19]. The class of operators with Fox function kernels are also important in transmutation theory [20–22], fractional integrodifferentiation operators and applications [23–24]. Especially note the reference [20] which contains a detailed survey of V.V. Katrakhov's main results.

2. PRELIMINARIES

The properties of the H-function $H^{m,n}_{p,q}[z]$ (3) depend on the numbers ([12], formulas 1.1.7–1.1.15):

$$a^* = \sum_{i=1}^n \alpha_i - \sum_{i=n+1}^p \alpha_i + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j; \quad \Delta = \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i; \tag{6}$$

$$\delta = \prod_{i=1}^p \alpha_i^{-\alpha_i} \prod_{j=1}^q \beta_j^{\beta_j}; \tag{7}$$

$$\mu = \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2}; \tag{8}$$

$$a_1^* = \sum_{j=1}^m \beta_j - \sum_{i=n+1}^p \alpha_i; \quad a_2^* = \sum_{i=1}^n \alpha_i - \sum_{j=m+1}^q \beta_j; \quad a_1^* + a_2^* = a^*, \quad a_1^* - a_2^* = \Delta; \tag{9}$$

$$\xi = \sum_{j=1}^m b_j - \sum_{j=m+1}^q b_j + \sum_{i=1}^n a_i - \sum_{i=n+1}^p a_i; \tag{10}$$

$$c^* = m + n - \frac{p+q}{2}. \tag{11}$$

An empty sum in (6), (8)–(10) and an empty product in (7), if they occur, are taken to be zero and one, respectively.

There hold the following assertions.

Lemma 1 ([12], Lemma 1.2). *For σ, $t \in \mathbb{R}$, there holds the estimate*

$$|\mathcal{H}^{m,n}_{p,q}(\sigma + it)| \sim C|t|^{\Delta\sigma + \mathrm{Re}(\mu)} \exp^{-\pi[|t|a^* + \mathrm{Im}(\xi)\mathrm{sgn}(t)]/2} \quad (|t| \to \infty) \tag{12}$$

uniformly in σ on any bounded interval in $\mathbb{R}$, where

$$C = (2\pi)^{c^*} \exp^{-c^* - \Delta\sigma - \mathrm{Re}(\mu)} \delta^\sigma \prod_{i=1}^p \alpha_i^{1/2 - \mathrm{Re}(a_i)} \prod_{j=1}^q \beta_j^{\mathrm{Re}(b_j) - 1/2}$$

and ξ and c^* are defined in (10) and (11).

Lemma 2 ([12], Lemma 3.3). *There holds the estimate as $|t| \to \infty$,*

$$\mathcal{H}'(\sigma + it) = \mathcal{H}(\sigma + it) \left[\log\delta + a_1^* \log(it) - a_2^* \log(-it) + \frac{\mu + \Delta\sigma}{it} + O(1/t^2) \right]. \tag{13}$$

Definition 1 ([12], Definition 3.2). W*e say that a function m belongs to the class $\mathcal{A}$, if there are extended real number $\varphi(m)$ and $\psi(m)$ with $\varphi(m) < \psi(m)$ such that*

(a) $m(s)$ is analytic in the strip $\varphi(m) < \mathrm{Re}(s) < \psi(m)$;

(b) $m(s)$ is bounded in every closed substrip $\sigma_1 \leq Re(s) \leq \sigma_2$, where $\varphi(m) \leq \sigma_1 \leq \sigma_2 \leq \psi(m)$;

(c) $|m'(\sigma+it)| = O(|t|^{-1})$ as $|t| \to \infty$ for $\varphi(m) < \sigma < \psi(m)$.

For two Banach space X and Y we use the notation $[X, Y]$ to denote the collection of bounded linear operators from X to Y, and $[X, Y]$ is abbreviated to $[X]$.

Theorem 1 ([12], Theorem 3.1). *Suppose $m \in \mathcal{A}$. Then, there is the transform $T_m \in \mathfrak{L}_{\nu,r}$ with $\varphi(m) < \nu < \psi(m)$ and $1 < r < \infty$ so that, if $f \in \mathfrak{L}_{\nu,r}$ with $\varphi(m) < \nu < \psi(m)$ and $1 < r \leq 2$, there holds the relation*

$$(\mathfrak{M}T_m f)(s) = m(s)(\mathfrak{M}f)(s) \quad (\mathrm{Re}(s) = \nu).$$

For $\varphi(m) < \nu < \psi(m)$ and $1 < r \leq 2$, the transform T_m is one-to-one on $\mathfrak{L}_{\nu,r}$, except when $m = 0$. If $1/m \in \mathcal{A}$, then for $max[\varphi(m), \varphi(1/m)] < \nu < min[\psi(m), \psi(1/m)]$ and for $1 < r < \infty$, T_m maps $\mathfrak{L}_{\nu,r}$ one-to-one onto itself, and for the inverse operator T_m^{-1} there holds the formula $T_m^{-1} = T_{1/m}$.

Multidimensional Mellin integral transform $(\mathfrak{M}f)(\mathbf{x})$ of function $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$, $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+$, is determined by the formula

$$(\mathfrak{M}f)(\mathbf{s}) = \int\limits_0^\infty f(\mathbf{t})\mathbf{t}^{\mathbf{s}-1}\mathbf{dt}, \quad \mathrm{Re}(\mathbf{s}) = \overline{\nu}, \tag{14}$$

$\mathbf{s} = (s_1, s_2, ..., s_n) \in \mathbb{C}^n$. The inverse multidimensional Mellin transform has the form

$$(\mathfrak{M}^{-1}g)(\mathbf{x}) = \frac{1}{(2\pi i)^n} \int\limits_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \cdots \int\limits_{\gamma_n - i\infty}^{\gamma_n + i\infty} \mathbf{x}^{-\mathbf{s}} g(\mathbf{s}) \mathrm{d}\mathbf{s}, \tag{15}$$

$\mathbf{x} \in \mathbb{R}^n_+$, $\gamma_j = \mathrm{Re}(s_j)$ $(j = 1, \cdots, n)$. The theory of multidimensional integral transformations (14) and (15) can be recognized, for example, in books ([5], Ch. 1; [13]; [14]). Let $\mathbf{W}_\delta$, $\mathbf{R}$ be elementary operators (see [5], Chapter 1)

$$(\mathbf{W}_\delta f)(\mathbf{x}) = f\left(\frac{\mathbf{x}}{\delta}\right), \quad \mathbf{x} \in \mathbb{R}^n, \quad \delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}^n_+; \tag{16}$$

$$(\mathbf{R}f)(\mathbf{x}) = \frac{1}{\mathbf{x}} f\left(\frac{1}{\mathbf{x}}\right). \tag{17}$$

Operators (16) and (17) have the the following properties.

Lemma 3 ([3], Lemma 2; [6], Lemma 2.1). *Let $\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{R}^n$ $(\nu_1 = \nu_2 = ... = \nu_n)$ and $1 \leq \overline{r} < \infty$.*

(a) $\mathbf{W}_\delta$ is a bounded isomorphism of $\mathfrak{L}_{\overline{\nu},\overline{r}}$ onto itself, and if $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ $(1 \leq \overline{r} \leq 2)$, then

$$(\mathfrak{M}\mathbf{W}_\delta f)(\mathbf{s}) = \delta^{\mathbf{s}}(\mathfrak{M}f)(\mathbf{s}) \quad (\mathrm{Re}(\mathbf{s}) = \overline{\nu}). \tag{18}$$

(b) $\mathbf{R}$ is an isometric isomorphism of $\mathfrak{L}_{\overline{\nu},\overline{r}}$ onto $\mathfrak{L}_{1-\overline{\nu},\overline{r}}$, and if $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ $(1 \leq \overline{r} \leq 2)$, then

$$(\mathfrak{M}\mathbf{R}f)(\mathbf{s}) = (\mathfrak{M}f)(1 - \mathbf{s}) \quad (\mathrm{Re}\,\mathbf{s} = \overline{\nu}). \tag{19}$$

To formulate the results for the H-transform (1) we need the following constants, analogical for one-dimensional case defined via the parameters of the H-function (3) ([12], (3.4.1), (3.4.2), (1.1.7)–(1.1.13)):

let $\widetilde{\alpha} = (\widetilde{\alpha}_1, \widetilde{\alpha}_2, ..., \widetilde{\alpha}_n)$ and $\widetilde{\beta} = (\widetilde{\beta}_1, \widetilde{\beta}_2, ..., \widetilde{\beta}_n)$, where

$$\widetilde{\alpha}_1 = \begin{cases} -\min\limits_{1 \leq j_1 \leq m_1} \left[\frac{\mathrm{Re}(b_{j_1})}{\beta_{j_1}}\right], & m_1 > 0, \\ -\infty, & m_1 = 0, \end{cases} \qquad \widetilde{\beta}_1 = \begin{cases} \min\limits_{1 \leq i_1 \leq \overline{n}_1} \left[\frac{1 - \mathrm{Re}(a_{i_1})}{\alpha_{i_1}}\right], & \overline{n}_1 > 0, \\ \infty, & \overline{n}_1 = 0, \end{cases}$$

$$\widetilde{\alpha}_2=\begin{cases}-\min\limits_{1\le j_2\le m_2}\left[\frac{\mathrm{Re}(b_{j_2})}{\beta_{j_2}}\right], & m_2>0,\\ -\infty, & m_2=0,\end{cases}\qquad \widetilde{\beta}_2=\begin{cases}\min\limits_{1\le i_2\le \overline{n}_2}\left[\frac{1-\mathrm{Re}(a_{i_2})}{\alpha_{i_2}}\right], & \overline{n}_2>0,\\ \infty, & \overline{n}_2=0,\end{cases}$$

and so on

$$\widetilde{\alpha}_n=\begin{cases}-\min\limits_{1\le j_n\le m_n}\left[\frac{\mathrm{Re}(b_{j_n})}{\beta_{j_n}}\right], & m_n>0,\\ -\infty, & m_2=0,\end{cases}\qquad \widetilde{\beta}_n=\begin{cases}\min\limits_{1\le i_n\le \overline{n}_n}\left[\frac{1-\mathrm{Re}(a_{i_n})}{\alpha_{i_n}}\right], & \overline{n}_n>0,\\ \infty, & \overline{n}_n=0;\end{cases}\tag{20}$$

let $a^*=(a_1^*,a_2^*,...,a_n^*)$, $\Delta=(\Delta_1,\Delta_2,...,\Delta_n)$, and

$$a_1^*=\sum_{i=1}^{\overline{n}_1}\alpha_{i_1}-\sum_{i=\overline{n}_1+1}^{p_1}\alpha_{i_1}+\sum_{j=1}^{m_1}\beta_{j_1}-\sum_{j=m_1+1}^{q_1}\beta_{j_1},\quad \Delta_1=\sum_{j=1}^{q_1}\beta_{j_1}-\sum_{i=1}^{p_1}\alpha_{i_1},$$

$$a_2^*=\sum_{i=1}^{\overline{n}_2}\alpha_{i_2}-\sum_{i=\overline{n}_2+1}^{p_2}\alpha_{i_2}+\sum_{j=1}^{m_2}\beta_{j_2}-\sum_{j=m_1+1}^{q_2}\beta_{j_2},\quad \Delta_2=\sum_{j=1}^{q_2}\beta_{j_2}-\sum_{i=1}^{p_2}\alpha_{i_2},$$

and so on

$$a_n^*=\sum_{i=1}^{\overline{n}_n}\alpha_{i_n}-\sum_{i=\overline{n}_n+1}^{p_n}\alpha_{i_n}+\sum_{j=1}^{m_n}\beta_{j_n}-\sum_{j=m_n+1}^{q_n}\beta_{j_n};\quad \Delta_n=\sum_{j=1}^{q_n}\beta_{j_n}-\sum_{i=1}^{p_n}\alpha_{i_n};\tag{21}$$

also let $\overline{\delta}=(\delta_1,\delta_2,...,\delta_n)$ and

$$\delta_1=\prod_{i=1}^{p_1}\alpha_{i_1}^{-\alpha_{i_1}}\prod_{j=1}^{q_1}\beta_{j_1}^{\beta_{j_1}},\quad \delta_2=\prod_{i=1}^{p_2}\alpha_{i_2}^{-\alpha_{i_2}}\prod_{j=1}^{q_2}\beta_{j_2}^{\beta_{j_2}},...,\delta_n=\prod_{i=1}^{p_n}\alpha_{i_n}^{-\alpha_{i_n}}\prod_{j=1}^{q_n}\beta_{j_n}^{\beta_{j_n}};\tag{22}$$

let $\overline{\mu}=(\mu_1,\mu_2,...,\mu_n)$ and

$$\mu_1=\sum_{j=1}^{q_1}b_{j_1}-\sum_{i=1}^{p_1}a_{i_1}+\frac{p_1-q_1}{2},\mu_2=\sum_{j=1}^{q_2}b_{j_2}-\sum_{i=1}^{p_2}a_{i_2}+\frac{p_2-q_2}{2},...,$$

$$\mu_n=\sum_{j=1}^{q_n}b_{j_n}-\sum_{i=1}^{p_n}a_{i_n}+\frac{p_n-q_n}{2};\tag{23}$$

let $\widetilde{a}^*=(\widetilde{a}_1^*,\widetilde{a}_2^*,...,\widetilde{a}_n^*)$, $\hat{a}^*=(\hat{a}_1^*,\hat{a}_2^*,...,\hat{a}_n^*)$

$$\widetilde{a}_1^*=\sum_{j=1}^{m_1}\beta_{j_1}-\sum_{i=\overline{n}_1+1}^{p_1}\alpha_{i_1},\quad \hat{a}_2^*=\sum_{i=1}^{\overline{n}_1}\alpha_{i_1}-\sum_{i=m_1+1}^{q_1}\beta_{j_1},$$

$$\widetilde{a}_2^*=\sum_{j=1}^{m_2}\beta_{j_2}-\sum_{i=\overline{n}_2+1}^{p_2}\alpha_{i_2},\quad \hat{a}_2^*=\sum_{i=1}^{\overline{n}_2}\alpha_{i_2}-\sum_{i=m_2+1}^{q_2}\beta_{j_2},$$

and so on

$$\widetilde{a}_n^*=\sum_{j=1}^{m_n}\beta_{j_n}-\sum_{i=\overline{n}_n+1}^{p_2}\alpha_{i_n},\quad \hat{a}_n^*=\sum_{i=1}^{\overline{n}_n}\alpha_{i_n}-\sum_{i=m_n+1}^{q_n}\beta_{j_n};\tag{24}$$

and

$$\widetilde{a}_k^*+\hat{a}_k^*=a_k^*,\quad \widetilde{a}_k^*-\hat{a}_k^*=\Delta_k\quad (k=1,2,...,n).\tag{25}$$

The exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of a function $\overline{\mathcal{H}}_{\mathbf{p},\mathbf{q}}^{\mathbf{m},\mathbf{n}}(\mathbf{s})$:

$$\overline{\mathcal{H}}_{\mathbf{p},\mathbf{q}}^{\mathbf{m},\mathbf{n}}(\mathbf{s})\equiv\overline{\mathcal{H}}_{\mathbf{p},\mathbf{q}}^{\mathbf{m},\mathbf{n}}\left[\begin{array}{c}(\mathbf{a}_i,\overline{\alpha}_i)_{1,\mathbf{p}}\\(\mathbf{b}_j,\overline{\beta}_j)_{1,\mathbf{q}}\end{array}\middle|\,\mathbf{s}\right]=\prod_{k=1}^{n}\mathcal{H}_{p_k,q_k}^{m_k,\overline{n}_k}\left[\begin{array}{c}(a_{i_k},\alpha_{i_k})_{1,p_k}\\(b_{j_k},\beta_{j_k})_{1,q_k}\end{array}\middle|\,s_k\right],\tag{26}$$

is called a set of vectors $\overline{\nu}=(\nu_1,\nu_2,...,\nu_n)\in\mathbb{R}^n$ $(\nu_1=\nu_2=...=\nu_n)$, such that $\widetilde{\alpha}_k<1-\nu_k<\widetilde{\beta}_k$ $(k=1,2,...n)$, where the parameters $\widetilde{\alpha}_k$, $\widetilde{\beta}_k(k=1,2,...,n)$ are defined by formulas (20), and functions $\mathcal{H}^{m_k,\overline{n}_k}_{p_k,q_k}(s_k)$ $(k=1,2,...,n)$ of the view (4) have zeros on lines $\mathrm{Re}(s_k)<1-\nu_k$ $(k=1,2,...,n)$, respectively (see [2], (61)).

Applying multidimensional Mellin transform (14) to (1), formally we obtain

$$(\mathfrak{M}\mathrm{H}f)(\mathbf{s})=\overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\begin{matrix}(\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}}\\(\mathbf{b}_j,\beta_j)_{1,\mathbf{q}}\end{matrix}|\mathbf{s}\right](\mathfrak{M}f)(1-\mathbf{s}). \tag{27}$$

Theorem 2 ([1], Theorem 3). *Suppose that*

$$\widetilde{\alpha}_k<1-\nu_k<\widetilde{\beta}_k;\quad \nu_k=\nu_l,\quad k\neq l\quad (k,l=1,2,...,n); \tag{28}$$

and that either of the conditions

$$a_k^*>0\quad (k=1,2,...,n); \tag{29}$$

or

$$a_k^*=0,\quad \Delta_k[1-\nu_k]+\mathrm{Re}(\mu_k)\leq 0\quad (k=1,2,...,n) \tag{30}$$

holds. Then, we have the following results:

(a) There exists a one-to-one transform $\mathrm{H}\in[\mathfrak{L}_{\overline{\nu},\overline{2}},\ \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$ *so that the relation (27) holds for* $\mathrm{Re}(\mathbf{s})=1-\overline{\nu}$ *and* $f\in\mathfrak{L}_{\overline{\nu},\overline{2}}$. *If* $a_k^*=0$, $\Delta_k[1-\nu_k]+Re(\mu_k)=0$ $(k=1,2,...,n)$, *and* $\overline{\nu}$ *does not belong to an exceptional set* $\mathcal{E}_{\overline{\mathcal{H}}}$, *then the operator H maps* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *onto* $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$.

(b) If $f\in\mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g\in\mathfrak{L}_{\overline{\nu},\overline{2}}$, *then for H there holds the relation*

$$\int\limits_0^\infty f(\mathbf{x})\big(\mathrm{H}g\big)(\mathbf{x})\mathbf{dx}=\int\limits_0^\infty\big(\mathrm{H}f\big)(\mathbf{x})g(\mathbf{x})d\mathbf{x}. \tag{31}$$

(c) Let $f\in\mathfrak{L}_{\overline{\nu},\overline{2}}$, $\overline{\lambda}=(\lambda_1,\lambda_2,...,\lambda_n)\in\mathbb{C}^n$, $\overline{h}=(h_1,h_2,...,h_n)\in\mathbb{R}^n_+$. *If* $Re(\overline{\lambda})>(1-\overline{\nu})\overline{h}-1$, *then Hf is given by formula*

$$\big(\mathrm{H}f\big)(\mathbf{x})=\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}}$$
$$\times\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}}\int\limits_0^\infty \mathrm{H}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\mathbf{xt}\left|\begin{matrix}(-\overline{\lambda},\overline{h}),(\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}}\\(\mathbf{b}_j,\beta_j)_{1,\mathbf{q}},\ (-\overline{\lambda}-1,\overline{h})\end{matrix}\right.\right]f(\mathbf{t})\mathbf{dt}. \tag{32}$$

When $\mathrm{Re}(\overline{\lambda})<(1-\overline{\nu})\overline{h}-1$, $\mathrm{H}f$ is given by $\big(\mathrm{H}f\big)(\mathbf{x})=-\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}}$

$$\times\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}}\int\limits_0^\infty \mathrm{H}^{\mathbf{m}+1,\mathbf{n}}_{\mathbf{p}+1,\mathbf{q}+1}\left[\mathbf{xt}\left|\begin{matrix}(\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}},(-\overline{\lambda},\overline{h})\\(-\overline{\lambda}-1,\overline{h}),(\mathbf{b}_j,\beta_j)_{1,\mathbf{q}}\end{matrix}\right.\right]f(\mathbf{t})\mathbf{dt}. \tag{33}$$

(d) The transform H is independent of $\overline{\nu}$ *in the sense that, for* $\overline{\nu}$ *and* $\widetilde{\overline{\nu}}$ *satisfying the assumptions (28), and either (29) or (30), and for the respective transforms H on* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\widetilde{H}$ *on* $\mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}}$ *given in (27), then* $Hf=\widetilde{H}f$ *for* $f\in\mathfrak{L}_{\overline{\nu},\overline{2}}\bigcap\mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}}$.

3. $\mathfrak{L}_{\overline{\nu},\overline{r}}$-THEORY OF THE H-TRANSFORM WHEN $a^*=\Delta=0$ AND $\mathrm{Re}(\overline{\mu})=0$

In this section, based on the existence of the H-transform on the space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ which is garanteed in Theorem 2 for some $\overline{\nu}=(\nu_1,\nu_2,...,\nu_n)\in\mathbb{R}^n$ $(\nu_1=\nu_2=...=\nu_n)$, and $a_k^*=\Delta_k=0$, $\mathrm{Re}(\mu_k)\leq 0$ $(k=1,2,...n)$, we prove that such a transform can be extended to $\mathfrak{L}_{\overline{\nu},\overline{r}}$ for $\overline{r}=(r_1,r_2,...,r_n)\in\mathbb{R}^n$, $1<\overline{r}<\infty$, $r_1=r_2=...=r_n$; $\overline{\nu}=(\nu_1,\nu_2,...,\nu_n)\in\mathbb{R}^n$, $\nu_1=\nu_2=...=\nu_n$, such that $\mathrm{H}\in[\mathfrak{L}_{\overline{\nu},\overline{r}},\mathfrak{L}_{1-\overline{\nu},\overline{s}}]$ for a certain range of the value $\overline{s}=(s_1,s_2,...,s_n)$. The results will be different in cases $\mathrm{Re}(\mu_k)=0$ and $\mathrm{Re}(\mu_k)\neq 0$ $(k=1,2,...n)$. We consider the former case.

Theorem 3. *Let* $a_k^* = 0$, $\Delta_k = 0$ $(k = 1, 2, ...n)$; $\mathrm{Re}(\mu_k) = 0$ $(k = 1, 2, ...n)$, *and*

$$\widetilde{\alpha}_k < 1 - \nu_k < \widetilde{\beta}_k; \quad \nu_k = \nu_l, \quad k \neq l \quad (k, l = 1, 2, ..., n).$$

Let $1 < r_k < \infty, r_k = r_l, k \neq l$ $(k, l = 1, 2, ..., n)$.

(a) The transform H defined on $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *can be extended to* $\mathfrak{L}_{\overline{\nu},\overline{r}}$ *as an element of* $[\mathfrak{L}_{\overline{\nu},\overline{r}}, \mathfrak{L}_{1-\overline{\nu},\overline{r}}]$.

(b) If $1 < r_k \leq 2$, $r_k = r_l$, $k \neq l$ $(k, l = 1, 2, ..., n)$, *then the transform H is one-to-one on* $\mathfrak{L}_{\overline{\nu},\overline{r}}$ *and there holds the equality (27), namely,*

$$\big(\mathfrak{M}\mathrm{H}f\big)(\mathbf{s}) = \overline{\mathcal{H}}(\mathbf{s})\big(\mathfrak{M}f\big)(1 - \mathbf{s}) \quad (\mathrm{Re}(\mathbf{s}) = 1 - \overline{\nu}). \tag{34}$$

(c) If $\overline{\nu} \notin \mathcal{E}_{\overline{\mathcal{H}}}$, *then H is a one-to-one transform on* $\mathfrak{L}_{\overline{\nu},\overline{r}}$ *onto* $\mathfrak{L}_{1-\overline{\nu},\overline{r}}$, *i.e.*,

$$\mathrm{H}(\mathfrak{L}_{\overline{\nu},\overline{r}}) = \mathfrak{L}_{1-\overline{\nu},\overline{r}}. \tag{35}$$

(d) If $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ *and* $g \in \mathfrak{L}_{\overline{\nu},\overline{r}'}$ *and* $\overline{r}' = \overline{r}/(\overline{r} - 1)$, *then the relation (31) holds:*

$$\int_0^\infty f(\mathbf{x})\big(\mathrm{H}g\big)(\mathbf{x})\mathbf{dx} = \int_0^\infty \big(\mathrm{H}f\big)(\mathbf{x})g(\mathbf{x})d\mathbf{x}. \tag{36}$$

(e) If $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$, $\overline{\lambda} \in \mathbb{C}^n$ *and* $\overline{h} > 0$, *then Hf is given by*

$$\big(\mathrm{H}f\big)(\mathbf{x}) = \overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}}$$
$$\times \frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int_0^\infty \mathrm{H}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\mathbf{xt}\,\middle|\, \begin{matrix} (-\overline{\lambda}, \overline{h}), (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}},\ (-\overline{\lambda} - 1, \overline{h}) \end{matrix}\right] f(\mathbf{t})\mathbf{dt} \tag{37}$$

for $\mathrm{Re}(\overline{\lambda}) > (1 - \overline{\nu})\overline{h} - 1$, *while*

$$\big(\mathrm{H}f\big)(\mathbf{x}) = -\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}}$$
$$\times \frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int_0^\infty \mathrm{H}^{\mathbf{m}+1,\mathbf{n}}_{\mathbf{p}+1,\mathbf{q}+1}\left[\mathbf{xt}\,\middle|\, \begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}}, (-\overline{\lambda}, \overline{h}) \\ (-\overline{\lambda} - 1, \overline{h}), (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}} \end{matrix}\right] f(\mathbf{t})\mathbf{dt} \tag{38}$$

for $\mathrm{Re}(\overline{\lambda}) < (1 - \overline{\nu})\overline{h} - 1$.

Proof. Since $\widetilde{\alpha}_k < 1 - \nu_k < \widetilde{\beta}_k$, $\nu_k = \nu_l$, $k \neq l\,(k, l = 1, 2, ..., n)$, and $\Delta_k[1 - \overline{\nu}_k] + \mathrm{Re}(\overline{\mu}_k) \leq 0$ $(k = 1, 2, ..., n)$, then, according to Theorem 2, the transform H is defined on $\mathfrak{L}_{\overline{\nu},\overline{2}}$. We denote by $\overline{\mathcal{H}}_0(\mathbf{s})$ the function

$$\overline{\mathcal{H}}_0(\mathbf{s}) = \delta^{-\mathbf{s}}\overline{\mathcal{H}}(\mathbf{s}) = \prod_{k=1}^n \mathcal{H}_0(s_k) = \prod_{k=1}^n \delta_k^{-s_k}\mathcal{H}(s_k), \tag{39}$$

where δ_k $(k = 1, 2, ..., n)$ are defined in (22) and functions $\mathcal{H}(s_k)$ $(k = 1, 2, ..., n)$ are of the view (4). It follows from (12) that

$$\mathcal{H}_0(\sigma_k + it_k) \sim \prod_{j=1}^{q_k} \beta_{j_k}^{\mathrm{Re}(b_{j_k})-1/2} \prod_{i=1}^{p_k} \alpha_{i_k}^{1/2-\mathrm{Re}(a_{i_k})} (2\pi)^{c^*} e^{-c^*} e^{-\pi \mathrm{Im}(\xi_k)\mathrm{sgn}(t_k)/2} \quad (|t_k| \to \infty) \tag{40}$$

are uniformly in σ_k $(k = 1, 2, ..., n)$ in any bounded interval of $\mathbb{R}$. Therefore, $\mathcal{H}_0(s_k)$ $(k = 1, 2, ..., n)$ are analytic in the strips $\widetilde{\alpha}_k < \mathrm{Re}(s_k) < \widetilde{\beta}_k$, and if $\widetilde{\alpha}_k < \sigma_k^1 \leq \sigma_k^2 < \widetilde{\beta}_k$, then $\mathcal{H}_0(s_k)$ are bounded in the strips $\sigma_k^1 \leq \mathrm{Re}(s_k) \leq \sigma_k^2$ $(k = 1, 2, ..., n)$. Since parameters (21) $a_k^* = \Delta_k = 0$ $(k = 1, 2, ...n)$, then in accordance with (24), (25) $\widetilde{a}_k^* = -\hat{a}_k^* = \Delta_k/2 = 0$ $(k = 1, 2, ..., n)$. Then, from (39) and (13), we have

$$\mathcal{H}_0'(\sigma_k + it_k) = \mathcal{H}_0(\sigma_k + it_k)\left[-\log \delta_k + \frac{\mathcal{H}_0'(\sigma_k + it_k)}{\mathcal{H}_0(\sigma_k + it_k)}\right]$$

$$= \mathcal{H}_0(\sigma_k + it_k)\left[\frac{\mathrm{Im}(\mu_k)}{t_k} + O\left(\frac{1}{t_k^2}\right)\right] = O\left(\frac{1}{t_k}\right) \quad (|t_k| \to \infty) \tag{41}$$

for $\widetilde{\alpha}_k < \sigma_k < \widetilde{\beta}_k$ $(k = 1, 2, ..., n)$. Thus, $\mathcal{H}_0(s_k)$ $(k = 1, 2, ...n)$ belong to the class $\mathcal{A}$ (see Definition 1) with $\widetilde{\alpha}_k(\mathcal{H}_0) = \widetilde{\alpha}_k$ and $\widetilde{\beta}_k(\mathcal{H}_0) = \widetilde{\beta}_k$ $(k = 1, 2, ...n)$.

Therefore, by virtue of the Theorem 1, there is a transform $T_{\mathcal{H}_0(s_k)} \in [\mathfrak{L}_{1-\nu_k, r_k}]$ for $1 < r_k < \infty$ and $\widetilde{\alpha}_k < 1 - \nu_k < \widetilde{\beta}_k$ $(k = 1, 2, ...n)$. Let $\mathrm{T} = T_{\overline{\mathcal{H}}_0(\mathbf{s})} = \prod_{k=1}^{n} T_{\mathcal{H}_0(s_k)}$. When $1 < \overline{r} \leq 2$, then T is one-to-one transform on $\mathfrak{L}_{1-\overline{\nu},\overline{r}}$ and the relation

$$\big(\mathfrak{M}\mathrm{T}f\big)(\mathbf{s}) = \overline{\mathcal{H}}_0(\mathbf{s})\big(\mathfrak{M}f\big)(\mathbf{s}) \quad (\mathrm{Re}(\mathbf{s}) = 1 - \overline{\nu}) \tag{42}$$

holds for $f \in \mathfrak{L}_{1-\overline{\nu},\overline{r}}$. Let

$$\mathrm{H}_0 = \mathbf{W}_\delta \mathrm{T}\mathbf{R}, \tag{43}$$

where $\mathbf{W}_\delta$ and $\mathbf{R}$ are given in (16) and (17). According to Lemmas 3(b) and 3(a), $\mathbf{R} \in [\mathfrak{L}_{\overline{\nu},\overline{r}}, \mathfrak{L}_{1-\overline{\nu},\overline{r}}]$, $\mathbf{W}_\delta \in [\mathfrak{L}_{1-\overline{\nu},\overline{r}}]$ and, hence, $\mathrm{H}_0 \in [\mathfrak{L}_{\overline{\nu},\overline{r}}, \mathfrak{L}_{1-\overline{\nu},\overline{r}}]$ for $\widetilde{\alpha} < 1 - \overline{\nu} < \widetilde{\beta}$ and $1 < \overline{r} < \infty$, too. When $\widetilde{\alpha} < 1 - \overline{\nu} < \widetilde{\beta}, 1 < \overline{r} \leq 2$ and $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$, it follows from (43), (18), (42), (19), and (39) that

$$\begin{aligned}\big(\mathfrak{M}\mathrm{H}_0 f\big)(\mathbf{s}) &= \big(\mathfrak{M}\mathbf{W}_\delta \mathrm{T}\mathbf{R}f\big)(\mathbf{s}) = \delta^{\mathbf{s}}\big(\mathfrak{M}\mathrm{T}\mathbf{R}f\big)(\mathbf{s}) = \delta^{\mathbf{s}}\overline{\mathcal{H}}_0(\mathbf{s})\big(\mathfrak{M}\mathbf{R}f\big)(\mathbf{s}) \\ &= \delta^{\mathbf{s}}\overline{\mathcal{H}}_0(\mathbf{s})\big(\mathfrak{M}f\big)(1-\mathbf{s}) = \overline{\mathcal{H}}(\mathbf{s})\big(\mathfrak{M}f\big)(1-\mathbf{s})\end{aligned} \tag{44}$$

for $\mathrm{Re}(\mathbf{s}) = 1 - \overline{\nu}$. In particular, for $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ Theorem 2 (a), (27) and (44) imply the equality

$$\big(\mathfrak{M}\mathrm{H}_0 f\big)(\mathbf{s}) = \big(\mathfrak{M}\mathrm{H}f\big)(\mathbf{s}) \quad (\mathrm{Re}(\mathbf{s}) = 1 - \overline{\nu}). \tag{45}$$

Thus, $\mathrm{H}_0 f = \mathrm{H}f$ for $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and, therefore, if $\widetilde{\alpha} < 1 - \overline{\nu} < \widetilde{\beta}, \mathrm{H} = \mathrm{H}_0$ on $\mathfrak{L}_{\overline{\nu},\overline{2}}$ by Theorem 2(d). Since $\mathfrak{L}_{\overline{\nu},\overline{2}} \cap \mathfrak{L}_{\overline{\nu},\overline{r}}$ is dense in $\mathfrak{L}_{\overline{\nu},\overline{r}}$ (see [16], Lemma 2.2), H can be extended to $\mathfrak{L}_{\overline{\nu},\overline{r}}$ and, if we denote it there by H again, $\mathrm{H} \in [\mathfrak{L}_{\overline{\nu},\overline{r}}, \mathfrak{L}_{1-\overline{\nu},\overline{r}}]$. This completes the proof of assertion (a) of the theorem.

The property (b) is clear from the fact that the operator T above and the operators $\mathbf{W}_\delta$ and $\mathbf{R}$ are one-to-one and (34) follows from (44).

Let us prove (c). Since $\mathbf{R}(\mathfrak{L}_{\overline{\nu},\overline{r}}) = \mathfrak{L}_{1-\overline{\nu},\overline{r}}$ and $\mathbf{W}_\delta(\mathfrak{L}_{1-\overline{\nu},\overline{r}}) = \mathfrak{L}_{1-\overline{\nu},\overline{r}}$, then the onto map property $\mathrm{H}(\mathfrak{L}_{\overline{\nu},\overline{r}}) = \mathfrak{L}_{1-\overline{\nu},\overline{r}}$ holds if and only if $\mathrm{T}(\mathfrak{L}_{1-\overline{\nu},\overline{r}}) = \mathfrak{L}_{1-\overline{\nu},\overline{r}}$. To prove this, it should be noted that the abscissas of the zeros of $\mathcal{H}(s_k)$ divide the interval $(\widetilde{\alpha}_k, \widetilde{\beta}_k)$ $(k = 1, 2, ..., n)$ into disjoint open intervals, where and thereafter $\mathcal{H}_0(s_k)$ in (39) is renamed $\mathcal{H}(s_k)$ $(k = 1, 2, ..., n)$. Let $(\widetilde{\alpha}_k^1, \widetilde{\beta}_k^1)$ be one such interval $(k = 1, 2, ..., n)$. Then, each function $\frac{1}{\mathcal{H}(s_k)}$ is analytic in $\widetilde{\alpha}_k^1 < \mathrm{Re}(s_k) < \widetilde{\beta}_k^1$ $(k = 1, 2, ..., n)$. In view of (40) we have

$$\frac{1}{\mathcal{H}_0(\sigma_k + it_k)} \sim \prod_{j=1}^{q_k} \beta_{j_k}^{1/2 - \mathrm{Re}(b_{j_k})} \prod_{i=1}^{p_k} \alpha_{i_k}^{\mathrm{Re}(a_{i_k}) - 1/2} (2\pi)^{-c^*} e^{c^*} e^{\pi \mathrm{Im}(\xi_k)\mathrm{sgn}(t_k)/2} \quad (|t_k| \to \infty).$$

So, if we take $\widetilde{\alpha}_k^1 < \sigma_k^1 \leq \sigma_k^2 < \widetilde{\beta}_k^1$, then $\frac{1}{\mathcal{H}(s_k)}$ is bounded in the strip $\sigma_k^1 \leq \mathrm{Re}(s_k) \leq \sigma_k^2$ $(k = 1, 2, ..., n)$. The qualities

$$\left\{\frac{1}{\mathcal{H}_k}\right\}'(\sigma_k + it_k) = -\frac{\mathcal{H}_k'(\sigma_k + it_k)}{\mathcal{H}_k^2(\sigma_k + it_k)}$$

$(k = 1, 2, ..., n)$ imply by (40) and (41) that

$$\left\{\frac{1}{\mathcal{H}_k}\right\}'(\sigma_k + it_k) = O\left(\frac{1}{t_k}\right) \quad (|t_k| \to \infty)$$

for $\widetilde{\alpha}_k^1 < \sigma_k < \widetilde{\beta}_k^1$. Thus, $\frac{1}{\mathcal{H}(s_k)}$ $(k = 1, 2, ...n)$ belong to the class $\mathcal{A}$ with $\widetilde{\alpha}_k(1/\mathcal{H}(s_k)) = \widetilde{\alpha}_k^1$ and $\widetilde{\beta}_k(1/\mathcal{H}(s_k)) = \widetilde{\beta}_k^1$ $(k = 1, 2, ...n)$. Then, for $\widetilde{\alpha}_k^1 < \nu_k < \widetilde{\beta}_k^1$ and $1 < r_k < \infty$ it follows from Theorem

1 that the transforms $T_{1/\mathcal{H}(s_k)}$ are one-to-one on $\mathfrak{L}_{\nu_k,r_k}$ and $T(\mathfrak{L}_{\nu_k,r_k}) = \mathfrak{L}_{\nu_k,r_k}$ $(k = 1, 2, ...n)$. Let $\mathrm{T} = T_{1/\overline{\mathcal{H}}(\mathbf{s})} = \prod_{k=1}^{n} T_{1/\mathcal{H}(s_k)}$. Then, we have that $\mathrm{T}(\mathfrak{L}_{\overline{\nu},\overline{r}}) = \mathfrak{L}_{\overline{\nu},\overline{r}}$. But if $\nu_k \notin \mathcal{E}_{\mathcal{H}}$, then the value $1-\nu_k$ does not coincide with the abscissa of any zero of $\mathcal{H}(s_k)$, and, hence, $1-\nu_k$ lies in such $(\widetilde{\alpha}_k^1, \widetilde{\beta}_k^1)$ $(k = 1, 2, ...n)$. Therefore, H is one-to-one transform on $\mathfrak{L}_{\overline{\nu},\overline{r}}$ and $\mathrm{H}(\mathfrak{L}_{\overline{\nu},\overline{r}}) = \mathfrak{L}_{1-\overline{\nu},\overline{r}}$. The assertion (c) of the theorem is thus proved.

Now we prove (36). If $\widetilde{\alpha} < 1-\overline{\nu} < \widetilde{\beta}$, then by using the Holder inequality ([12], 1.3.4.(1))

$$\left|\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})g(\mathbf{x})\mathbf{dx}\right| \leq \left(\int_{\mathbf{a}}^{\mathbf{b}} |f(\mathbf{x})|^{\mathbf{p}}\mathbf{dx}\right)^{1/\mathbf{p}} \left(\int_{\mathbf{a}}^{\mathbf{b}} |g(\mathbf{x})|^{\mathbf{p}'}\mathbf{dx}\right)^{1/\mathbf{p}'} \times \left(\frac{1}{\mathbf{p}} + \frac{1}{\mathbf{p}'} = 1, -\infty \leq \mathbf{a} < \mathbf{b} \leq \infty\right), \tag{46}$$

we have

$$\left|\int_0^\infty f(\mathbf{x})(\mathrm{H}g)(\mathbf{x})\mathbf{dx}\right| = \left|\int_0^\infty [\mathbf{x}^{\overline{\nu}-1/\overline{r}} f(\mathbf{x})][\mathbf{x}^{1/\overline{r}-\overline{\nu}}(\mathrm{H}g)(\mathbf{x})]\mathbf{dx}\right|$$
$$\leq ||f||_{\overline{\nu},\overline{r}} ||\mathrm{H}g||_{1-\overline{\nu},\overline{r}'} \leq \mathrm{K}||f||_{\overline{\nu},\overline{r}}||g||_{\overline{\nu},\overline{r}'} \quad \left(\frac{1}{\overline{r}} + \frac{1}{\overline{r}'} = 1\right),$$

where K is a bound for $\mathrm{H} \in [\mathfrak{L}_{\overline{\nu},\overline{r}'}, \mathfrak{L}_{1-\overline{\nu},\overline{r}'}]$. Hence, the left-hand side of (36) represents a bounded functional on $\mathfrak{L}_{\overline{\nu},\overline{r}} \times \mathfrak{L}_{\overline{\nu},\overline{r}'}$. Similarly it is proved that the right-hand side of (36) represents such a functional on $\mathfrak{L}_{\overline{\nu},\overline{r}} \times \mathfrak{L}_{\overline{\nu},\overline{r}'}$. By virtue of Theorem 2 (b), if $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and $g \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, (31) is also true. By ([16], Lemma 2.2), $\mathfrak{L}_{\overline{\nu},\overline{r}} \cap \mathfrak{L}_{\overline{\nu},\overline{2}}$ is dense in $\mathfrak{L}_{\overline{\nu},\overline{r}}$ and hence (31) is true for $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ and $g \in \mathfrak{L}_{\overline{\nu},\overline{r}'}$ with $1 < \overline{r} < \infty$ and $\widetilde{\alpha} < 1-\overline{\nu} < \widetilde{\beta}$. This completes the proof of the assertion (d) of the theorem.

Finally we prove (e). If $\mathrm{Re}(\overline{\lambda}) > (1-\overline{\nu})\overline{h} - 1$, then the function

$$g_{\mathbf{x}}(\mathbf{t}) = \begin{cases} \mathbf{t}^{(\overline{\lambda}+1)/\overline{h}-1}, & 0 < \mathbf{t} < \mathbf{x}; \\ 0, & \mathbf{t} > \mathbf{x}; \end{cases} = \begin{cases} t_1^{(\lambda_1+1)/h_1-1} \cdots t_n^{(\lambda_n+1)/h_n-1}, & 0 < t_k < x_k; \\ 0, & t_k > x_k \quad (k = 1, 2, ...n); \end{cases} \tag{47}$$

belongs to $\mathfrak{L}_{\overline{\nu},\mathbf{s}}$ for $1 \leq \mathbf{s} < \infty$. When $\mathbf{s} = \overline{2}$, we may apply Theorem 2 (c) for $g_{\mathbf{x}} \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and we have

$$(\mathrm{H}g_{\mathbf{x}})(\mathbf{y}) = \overline{h}\mathbf{y}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{d}}{\mathbf{dy}} \mathbf{y}^{(\overline{\lambda}+1)/\overline{h}} \int_0^{\mathbf{x}} \mathrm{H}_{\mathbf{p}+1,\mathbf{q}+1}^{\mathbf{m},\mathbf{n}+1}\left[\mathbf{yt}\middle| \begin{matrix} (-\overline{\lambda}, \overline{h}), & (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}}, & (-\overline{\lambda}-1, \overline{h}) \end{matrix}\right] \mathbf{t}^{(\overline{\lambda}+1)/\overline{h}-1}\mathbf{dt}$$
$$= \overline{h}\mathbf{y}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{d}}{\mathbf{dy}} \int_0^{\mathbf{xy}} \mathrm{H}_{\mathbf{p}+1,\mathbf{q}+1}^{\mathbf{m},\mathbf{n}+1}\left[\mathbf{t}\middle| \begin{matrix} (-\overline{\lambda}, \overline{h}), & (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}}, & (-\overline{\lambda}-1, \overline{h}) \end{matrix}\right] \mathbf{t}^{(\overline{\lambda}+1)/\overline{h}-1}\mathbf{dt}$$
$$= \overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \mathrm{H}_{\mathbf{p}+1,\mathbf{q}+1}^{\mathbf{m},\mathbf{n}+1}\left[\mathbf{xy}\middle| \begin{matrix} (-\overline{\lambda}, \overline{h}), & (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}}, & (-\overline{\lambda}-1, \overline{h}) \end{matrix}\right]$$

almost everywhere. For $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ with $\widetilde{\alpha} < 1-\overline{\nu} < \widetilde{\beta}$ and $1 < \overline{r} < \infty$ and for the above $g_{\mathbf{x}} \in \mathfrak{L}_{\overline{\nu},\overline{r}'}$, we have from the previous result (d) that

$$\int_0^{\mathbf{x}} \mathbf{t}^{(\overline{\lambda}+1)/\overline{h}-1}(\mathrm{H}f)(\mathbf{t})\mathbf{dt} = \int_0^\infty (\mathrm{H}f)(\mathbf{t})g_{\mathbf{x}}(\mathbf{t})\mathbf{dt} = \int_0^\infty f(\mathbf{t})(\mathrm{H}g_{\mathbf{x}})(\mathbf{t})\mathbf{dt}$$
$$= \overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int_0^\infty \mathrm{H}_{\mathbf{p}+1,\mathbf{q}+1}^{\mathbf{m},\mathbf{n}+1}\left[\mathbf{xt}\middle| \begin{matrix} (-\overline{\lambda}, \overline{h}), & (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}}, & (-\overline{\lambda}-1, \overline{h}) \end{matrix}\right] f(\mathbf{t})\mathbf{dt}.$$

From here, after differentiation with respect to $\mathbf{x}$, we arrive at (37). In the case $\mathrm{Re}(\overline{\lambda}) < (1-\overline{\nu})\overline{h} - 1$, the relation (38) is proved similarly if we use the function

$$h_{\mathbf{x}}(\mathbf{t}) = \begin{cases} 0, & 0 < \mathbf{t} < \mathbf{x}; \\ \mathbf{t}^{(\overline{\lambda}+1)/\overline{h}-1}, & \mathbf{t} > \mathbf{x}; \end{cases}$$

instead of the function $g_{\mathbf{x}}(\mathbf{t})$. Thus, the theorem is proved.

4. CONCLUSIONS

The multi-dimensional integral transform with Fox H-function is studied. Conditions are obtained for the boundedness and one-to-one correspondence property of the operator of such transform from one Lebesgue-type weighted spaces of functions to others, and analogue of the formula for integration by parts are proved. For the transform under consideration, various integral representations are established. The results generalize those obtained earlier for the corresponding one-dimensional integral transform and also for some special forms of the considered transforms.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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