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A Cauchy Problem for a Third-Order Pseudo-Parabolic Equation with the Bessel Operator

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Abstract—This article investigates the Cauchy problem for a third-order pseudo-parabolic equation with a Bessel operator. Using the Erdélyi–Kober operator transformation, the Riemann's function for this equation is constructed, which is expressed through the hypergeometric Kampé de Fériet function. In particular, we obtain the Riemann's function for a one-dimensional pseudo-parabolic equation.

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1. INTRODUCTION

Boundary value problems for partial differential equations with singular coefficients have been studied extensively by many mathematicians. The investigation of more complex equations with singular coefficients represents a natural progression towards theoretical generalizations. The value of the theoretical results obtained in such studies increases significantly due to the presence of these equations or their special cases in applications.

A particular class of equations with partial derivatives with coefficients exhibiting singular behavior includes equations with Bessel operators of the form

$$B_\eta^x = x^{-2\eta-1}\frac{d}{dx}\left(x^{2\eta+1}\frac{d}{dx}\right) = \frac{d^2}{dx^2} + \frac{2\eta+1}{x}\frac{d}{dx}.$$

For elliptic, hyperbolic, and parabolic type equations with Bessel operators in one or more variables, I.A. Kipriyanov [1], introduced the corresponding terminology of B-elliptic, B-hyperbolic, and B-parabolic equations. The importance of equations from these classes is also determined by their applications in problems related to axisymmetric potential theory [2, 3], Euler–Poisson–Darboux (EPD) equations [4, 5], Radon transform and tomography [6–8], gas dynamics and acoustics [9], jet theory in hydrodynamics [10], linearized Maxwell–Einstein equations [11, 12], mechanics, theory of elasticity and plasticity [13], and many others.

The most thorough exploration of problems related to equations involving Bessel operators has been conducted by the Voronezh mathematician I.A. Kipriyanov and his students. More detailed information on this subject can be found in the monographs by Katrakhov and Sitnik [14], Sitnik and Shishkina [15].

The foundation of the modern theory of hyperbolic partial differential equations was significantly influenced by B. Riemann's, who obtained an integral representation of the Cauchy problem analogous

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to the representations of boundary value problem solutions for second-order elliptic equations using Green's functions. This representation assumes the existence of an auxiliary function, known as the Riemann's function, which possesses several well-known properties [16, 17].

Subsequently, Riemann's method for hyperbolic equations with two independent variables was developed in the works of Colton [18], Soldatov, Shkhanukov [19], Jokhadze [20], and others. In these studies, the Riemann's function was introduced as a solution to a specific Goursat problem. Zhegalov, Mironov, and Utkina [21, 22], proposed a method for finding the Riemann's function as a solution to an integral equation. Fro more information on this direction we note the works [23–25] and the references therein.

A.A. Andreev and Yu.O. Yakovleva examined and solved Cauchy and Goursat problems for third and fourth-order hyperbolic equations using Riemann's method [26].

It is known that degenerate and singular second-order equations have the peculiarity that classical problem formulations are not always well-posed. The lower-order coefficients significantly influence problem formulation. Such questions for high-order equations with singular coefficients have been scarcely investigated.

In the work of Barenblatt, Zheltov, Kochina [27], a linear pseudoparabolic equation of the form

$$\frac{\partial}{\partial t}(\Delta_x u(x,t) + \lambda u(x,t)) + \Delta_x u(x,t) = 0 \tag{1}$$

was obtained for the first time describing a nonstationary filtration process in a fractured porous medium, where Δ_x is the multidimensional Laplace operator and $\lambda = const \in R$.

A large number of papers are devoted to the study of equations of pseudoparabolic type, a survey of which can be found in the monographs [28–31] and in works [22, 32].

This work is devoted to the study of questions of solvability in the classical sense of an analogue of the Goursat boundary value problem for equation

$$L_\alpha(u) \equiv \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2} + \frac{2\alpha}{x}\frac{\partial u}{\partial x} + \lambda u\right) + \frac{\partial^2 u}{\partial x^2} + \frac{2\alpha}{x}\frac{\partial u}{\partial x} = f(x,t), \tag{2}$$

when $\alpha, \lambda \in R$, and $0 < 2\alpha < 1$, $\lambda > 0$.

Parameter α, participating in the equation (2), determines the order of the singularity of the equation and the problems associated with it. In case $\alpha = 0$, the equation (2) transforms into the one-dimensional equation of Barenblatt, Zheltov, and Kochina (1), and in case $\alpha = (n-1)/2$, we obtain the spherically symmetric case of the equation (1) and in the last case, the variable x plays the role of the variable $r = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ in the spherical coordinate system.

The distinction of our problem from those discussed above lies in augmenting a third-order pseudo-parabolic equation with a Bessel operator, and formulating the corresponding Cauchy problem in the domain $\Omega = \{(x,t);\ 0 < x < t\}$. The exact solution to the problem was found using Riemann's method. To determine the Riemann's function for this equation, the solution to the Goursat problem satisfying homogeneous boundary conditions was employed, using the Erdélyi–Kober operator.

2. FORMULATION OF PROBLEM

In contrast to the cited sources, in this work in the domain $\Omega = \{(x,t) : 0 < x < t\}$, we study the Cauchy problem for the equation with initial data on a non-characteristic line.

Cauchy problem. It is required to find a solution to equation (2) in the domain Ω satisfying the conditions

$$u(x,t)|_{t=x} = \psi_1(x), \quad \left.\frac{\partial u(x,t)}{\partial n}\right|_{t=x} = \psi_2(x), \quad \left.\frac{\partial^2 u(x,t)}{\partial n^2}\right|_{t=x} = \psi_3(x), \quad x > 0, \tag{3}$$

where $\psi_k(x)$ for $(k = 1, 2, 3)$ are given smooth functions, n is the unit normal vector, $\alpha, \lambda \in \mathbb{R}$, such that $0 < \alpha < 1/2$.

To construct the solution to the problem (2)–(3) we apply the Riemann's method. Consider the formulated problem on the plane $\xi O \eta$. By applying Green's identities

$$\xi^{2\alpha}[vL_\alpha(u) - uM_\alpha(v)] = \frac{\partial P}{\partial \xi} - \frac{\partial Q}{\partial \eta},$$

where

$$P=\xi^{2\alpha}\left(v\left(u_{\xi\eta}+u_{\xi}\right)+u\left(v_{\xi\eta}-v_{\xi}\right)\right),\quad Q=\xi^{2\alpha}\left(u_{\xi}v_{\xi}-\lambda vu\right),$$

in the domain $\Omega_1=\{(\xi,\eta):\ x<\xi<t,\ x<\eta<t\}$, we arrive at the following relation:

$$\iint\limits_{\Omega_1}\xi^{2\alpha}\left[vL(u)-uM(v)\right]d\xi\,d\eta=\int\limits_{\Gamma}Q\,d\xi+P\,d\eta,$$

where $\Gamma=\bar{\Omega}_1\backslash\Omega_1$,

$$M_{\alpha}(v)\equiv-\frac{\partial}{\partial\eta}\left(\frac{\partial^2 v}{\partial\xi^2}+\frac{2\alpha}{\xi}\frac{\partial v}{\partial\xi}+\lambda v\right)+\frac{\partial^2 v}{\partial\xi^2}+\frac{2\alpha}{\xi}\frac{\partial v}{\partial\xi}=0 \tag{4}$$

is the adjoint operator to $L_{\alpha}(u)$ is defined as follows.

The Riemann's function of the operator L_{α} is a function $v(x,t;\xi,\eta)$ that satisfies the following conditions:

1) the function $v(x,t;\xi,\eta)\in W$, where $W=\left\{v:\ v\in C^2\left(\overline{\Omega}_1\right),\ v_{\xi\eta},\ v_{\xi\xi\eta},\ v_{\xi}\in C\left(\Omega_1\right)\right\}$;

2) for each $(x,t)\in\Omega_1$, the function $v(x,t;\xi,\eta)$ satisfies the equation $M_{\alpha}(v(x,t;\xi,\eta))=0$,

3) it satisfies the following conditions on the characteristics $\xi=x$ and $\eta=t$:

$$v(x,t;x,\eta)=0, v_{\xi}(x,t;x,\eta)=\omega_1(\eta;x,t),\quad x\le\eta\le t, \tag{5}$$

$$v(x,t;\xi,t)=\omega_2(\xi;x,t),\quad x\le\xi\le t, \tag{6}$$

where $\omega_1(\eta,x,t)$ and $\omega_2(\xi,x,t)$ are solutions to the following Cauchy problems, respectively,

$$\omega_{1\eta}(\eta;x,t)-\omega_1(\eta;x,t)=0, \tag{7}$$

$$\left.\omega_1(\eta;x,t)\right|_{\eta=t}=1,\quad x\le\eta\le t, \tag{8}$$

$$\omega_{2\xi\xi}(\xi;x,t)-\frac{2\alpha}{\xi}\omega_{2\xi}(\xi;x,t)+\lambda\omega_2(\xi;x,t)=0, \tag{9}$$

$$\left.\omega_2(\xi;x,t)\right|_{\xi=x}=0,\quad \left.\omega_{2\xi}(\xi;x,t)\right|_{\xi=x}=1,\quad x\le\xi\le t. \tag{10}$$

The problems (7)–(10) are uniquely solvable. It is easy to demonstrate that the solutions to problem (9) and (10) has the form

$$\omega_1(\eta,x,t)=e^{\eta-t}. \tag{11}$$

Now, let's solve the problem (9) and (10). After the substitution

$$\omega_2(\xi,x,t)=\left(\frac{s}{\sqrt{\lambda}}\right)^{1/2-\alpha}z(s,x,t),\quad s=\sqrt{\lambda}\xi, \tag{12}$$

equation (9) transforms into the equation

$$s^2z''+sz'+\left(s^2-(\alpha-1/2)^2\right)z=0.$$

The given equation is a Bessel-type equation, and its general solution has the form [33]

$$z(s,x,t)=c_1(x,t)\,J_{1/2\,-\alpha}(s)+c_2(x,t)\,J_{\alpha-1/2}\,(s),$$

where $J_{\nu}(z)=\sum\limits_{n=0}^{+\infty}\frac{(-1)^n(z/2\,)^{2n+\nu}}{n!\Gamma(\nu+n+1)}$ is the Bessel function [34], and $c_1(x,t)$ and $c_2(x,t)$ are arbitrary functions depending on (x,t). Considering (12), we have

$$\omega_2(\xi,x,t)=c_1(x,t)\,\xi^{1/2\,-\alpha}J_{1/2\,-\alpha}\left(\sqrt{\lambda}\xi\right)+c_2(x,t)\,\xi^{1/2\,-\alpha}J_{\alpha-1/2}\,\left(\sqrt{\lambda}\xi\right). \tag{13}$$

Substituting (13) into conditions (10), we obtain the following system with respect to $c_1(x,t)$ and $c_2(x,t)$:

$$\begin{cases} c_1(x,t)J_{1/2-\alpha}(\sqrt{\lambda}x)+c_2(x,t)J_{\alpha-1/2}(\sqrt{\lambda}x)=0, \\ c_1(x,t)J_{-1/2-\alpha}(\sqrt{\lambda}x)-c_2(x,t)J_{\alpha+1/2}(\sqrt{\lambda}x)=\frac{x^{\alpha-1/2}}{\sqrt{\lambda}}. \end{cases}$$

Solving this system, we find

$$c_1(x,t)=\frac{x^{\alpha-1/2}\,J_{\alpha-1/2}\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}\left[J_{\alpha-1/2}\left(\sqrt{\lambda}x\right)J_{-1/2-\alpha}\left(\sqrt{\lambda}x\right)+J_{\alpha+1/2}\left(\sqrt{\lambda}x\right)J_{1/2-\alpha}\left(\sqrt{\lambda}x\right)\right]},$$

$$c_2(x,t)=-\frac{x^{\alpha-1/2}\,J_{1/2-\alpha}\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}\left[J_{\alpha-1/2}\left(\sqrt{\lambda}x\right)J_{-1/2-\alpha}\left(\sqrt{\lambda}x\right)+J_{\alpha+1/2}\left(\sqrt{\lambda}x\right)J_{1/2-\alpha}\left(\sqrt{\lambda}x\right)\right]}.$$

Considering formulas [34]

$$J_{\nu}(x)\,J_{1-\nu}(x)+J_{-\nu}(x)\,J_{\nu-1}(x)=\frac{2\sin\nu\pi}{\pi x},$$

we have

$$c_1(x,t)=\frac{\pi x^{\alpha+1/2}\,J_{\alpha-1/2}\left(\sqrt{\lambda}x\right)}{2\cos\alpha\pi},\quad c_2(x,t)=-\frac{\pi x^{\alpha+1/2}\,J_{1/2-\alpha}\left(\sqrt{\lambda}x\right)}{2\cos\alpha\pi}.$$

Substituting the found values of $c_1(x,t)$ and $c_2(x,t)$ into the equation (13), we obtain the representation of the solution to problem (9) and (10) in the from

$$\omega_2(\xi,x,t)=\frac{\pi x}{2\cos(\alpha\pi)}\left(\frac{\xi}{x}\right)^{\alpha+1/2}$$
$$\times\left[J_{\alpha-1/2}\left(\sqrt{\lambda}x\right)J_{1/2-\alpha}\left(\sqrt{\lambda}\xi\right)-J_{1/2-\alpha}\left(\sqrt{\lambda}x\right)J_{\alpha-1/2}\left(\sqrt{\lambda}\xi\right)\right]. \tag{14}$$

Using the Riemann's function $v(x,t;\xi,\eta)$, we can easily obtain the representation of the general solution to equation (2) in triangular domain Ω_1. Indeed, by integrating (4) over the domain Ω_1, where (x,t) is an arbitrary point in Ω, we have

$$u(x,t)=x^{-2\alpha}t^{2\alpha}u(t,t)\,v_\xi(x,t;t,t)-x^{-2\alpha}\int\limits_x^t\xi^{2\alpha}\left[u_\xi(\xi,\xi)\,v_\xi(x,t;\xi,\xi)-\lambda u(\xi,\xi)\,v(x,t;\xi,\xi)\right.$$
$$+\,v(x,t;\xi,\xi)\left(u_{\xi\eta}(\xi,\xi)+u_\xi(\xi,\xi)\right)+u(\xi,\xi)\left(v_{\xi\eta}(x,t;\xi,\xi)-v_\xi(x,t;\xi,\xi)\right)\left.\right]d\xi$$
$$-\,x^{-2\alpha}\iint\limits_{\Omega_1}\xi^{2\alpha}v(x,t;\xi,\eta)\,f(\xi,\eta)\,d\xi d\eta. \tag{15}$$

This represents the solution in the triangular domain Ω_1 using the Riemann function $v(x,t;\xi,\eta)$.

3. CONSTRUCTION OF THE RIEMANN'S FUNCTION

To construct the Riemann's function, we will use the methodology from [23].

Let's consider the following auxiliary problem.

Problem G_0. In the domain $\Omega_0=\{(\xi,\eta):\ 0<\xi<l,\ 0<\eta<h\}$, find a function $u(x,t)$ satisfying equation (2), and homogeneous boundary conditions

$$u(\xi,0)=0,\quad 0\le\xi\le l, \tag{16}$$

$$u(0,\eta)=0,\quad u_\xi(0,\eta)=0,\quad 0\le\eta\le h. \tag{17}$$

If the Riemann's function $v\left(x,t;\xi,\eta\right)$ of problem (2), (16), and (17) is known, then the solution to this problem is represented as [35]

$$u\left(x,t\right)=\iint\limits_{\Omega_0} v\left(x,t;\xi,\eta\right)f\left(\xi,\eta\right)d\xi d\eta \tag{18}$$

Here $\partial\Omega_0$ denotes the boundary of the domain Ω_0, and $f\left(\xi,\eta\right)$ is a given function.

On the other hand, the Riemann function can be defined in another way. Suppose that, constructed by some other method, formula (18) provides the solution to the problem (2), (16), (17) for any sufficiently smooth right-hand side $f(x,t)$. Then, due to the uniqueness of the solution to this problem, the kernel $v\left(x,t;\xi,\eta\right)$ will serve as the Riemann function for the problem (2), (16), and (17).

According to the work [23], to solve the problem (2), (16), and (17) we apply the Erdélyi–Kober fractional order operator. Therefore, let's consider some properties of this operator.

4. FRACTIONAL ORDER ERDÉLYI–KOBER OPERATORS

Various modifications and generalizations of classical fractional integration and differentiation operators are widely used in both theory and applications. Among these modifications are the Erdélyi–Kober operators [36]

$$I_{\eta,\alpha}f(x)=\frac{2x^{-2(\eta+\alpha)}}{\Gamma(\alpha)}\int\limits_0^x\left(x^2-\xi^2\right)^{\alpha-1}\xi^{2\eta+1}f(\xi)d\xi, \tag{19}$$

where $\alpha,\eta\in\mathbb{R}$, $\alpha>0$, $\eta\geq-(1/2)$, $f(x)\in L_1(0,b)$, $b>0$, and $\Gamma(\alpha)$ denotes the gamma function [37]. The main properties of operator (19) can be found in reference [36]. The inverse operator to (19), when $0<\alpha<1$, is given by

$$I^{-1}_{\eta,\alpha}g(x)=\frac{x^{-2\eta-1}}{\Gamma(1-\alpha)}\frac{d}{dx}\int\limits_0^x\left(x^2-s^2\right)^{-\alpha}s^{2(\eta+\alpha)+1}g(s)ds. \tag{20}$$

For operator (19), the following theorem holds [36].

Theorem 1. *Let $\alpha>0$, $\eta\geq-1/2$, $f(x)\in C^2(0,b)$, where $b>0$, and $x^{2\eta+1}f(x)$ is integrable at zero with $\lim\limits_{x\to0}x^{2\eta+1}f'(x)=0$. Then, $B^x_{\eta+\alpha}I_{\eta,\alpha}f(x)=I_{\eta,\alpha}B^x_\eta f(x)$, where*

$$B^x_\eta=x^{-2\eta-1}\frac{d}{dx}x^{2\eta+1}\frac{d}{dx}=\frac{d^2}{dx^2}+\frac{2\eta+1}{x}\frac{d}{dx}$$

is the singular Bessel differential operator.

It should be noted that in works [38–42], the Erdélyi–Kober transformation operator has been applied to solve initial boundary value problems for hyperbolic equations, while in works [23, 43] it has been used to solve initial boundary value problems for parabolic equations with coefficient singularities.

5. APPLICATION OF THE ERDÉLYI–KOBER OPERATOR FOR THE CONSTRUCTION OF THE RIEMANN FUNCTION

Let's assume that the solution to the problem (2), (16), and (17) exists. We seek this solution in the form

$$u\left(x,t\right)=I^x_{-(1/2),\alpha}U\left(x,t\right)=\frac{2x^{1-2\alpha}}{\Gamma\left(\alpha\right)}\int\limits_0^x\left(x^2-\xi^2\right)^{\alpha-1}U\left(\xi,\eta\right)d\xi, \tag{21}$$

where $u\left(x,t\right)$ is the unknown function, and the upper index in the operator indicates the variable on which this operator acts.

Substituting (21) into boundary conditions (16), (17), and then into equation (2), using Theorem 1 and the formula for the inverse operator (20), we obtain the following problem to find the solution $u\,(x,t)$ of the equation

$$U_{xxt} + U_{xx} + \lambda U_t = F\,(x,t)\,, \tag{22}$$

subject to the homogeneous boundary conditions

$$U\,(x,0) = 0, \quad 0 \le x \le l, \tag{23}$$

$$U\,(0,t) = 0, \quad U_x\,(0,t) = 0, \quad 0 \le t \le h, \tag{24}$$

where

$$F\,(x,t) = \left(I^{(x)}_{-1/2\ ,\alpha}\right)^{-1} f\,(x,t) = \frac{1}{\Gamma\,(1-\alpha)} \frac{d}{dx} \int\limits_0^x \left(x^2 - s^2\right)^{-\alpha} s^{2\alpha} f\,(s,t)\,ds. \tag{25}$$

In this case, the solution to the problem (22)–(24) is given by

$$U\,(x,t) = \int\limits_0^t d\eta \int\limits_0^x R\,(x,t;\xi,\eta)\,F\,(\xi,\eta)\,d\xi, \tag{26}$$

where $R\,(x,t;\xi,\eta)$ is the Riemann kernel associated with the Erdélyi–Kober fractional operator.

Here $R\,(x,t;\xi,\eta)$ is the Riemann function associated with problem (22)–(24), constructed in [42] and defined by the formula

$$R\,(x,t;\xi,\eta) = (\xi - x)\,K_0\left(1;\frac{3}{2},1,1;\sigma_1,\sigma_2\right), \tag{27}$$

where

$$K_0\left(a;b,c,b';\sigma_1,\sigma_2\right) = \sum_{m,n=0}^{+\infty} \frac{(a)_{m+n}}{(b)_m (c)_m (b')_n} \frac{{\sigma_1}^m}{m!} \frac{{\sigma_2}^n}{n!}, \tag{28}$$

with $\sigma_1 = -\frac{\lambda}{4}(\xi - x)^2, \quad \sigma_2 = \eta - t.$

This series converges when $|\sigma_1|\,, |\sigma_2| < +\infty$ and $b, c, b', c' \neq 0, -1, -2, \ldots$. It can be expressed as

$$K_0\left(a;b,c,b';\sigma_1,\sigma_2\right) = \sum_{m=0}^{+\infty} \frac{(a)_m}{(b)_m (c)_m} \frac{\sigma_1^m}{m!} {}_1F_1\left(a+m;b';\sigma_2\right) = \sum_{n=0}^{+\infty} \frac{(a)_n}{(b')_n} \frac{\sigma_2^n}{n!} {}_1F_2\left(a+n;b,c;\sigma_1\right),$$

or it can be expressed through the Kampe de Feriet function [44]

$$K_0\left(a;b,c,b';\sigma_1,\sigma_2\right) = F^{1;0;0}_{0;2;1}\left[\begin{matrix} a; & -; & -; \\ -; & b,c, & b'; \end{matrix}\ \sigma,\ \omega\right],$$

where ${}_1F_2\,(a;b,c;z)$ denotes the generalized hypergeometric function, and ${}_1F_1\,(a;b;z)$ denotes the Kummer function [45].

Considering (25), the function $F(\xi,\eta)$ is rewritten as $F(\xi,\eta) = \frac{\partial}{\partial\xi}\bar{F}(\xi,\eta)$, where

$$\bar{F}(\xi,\eta) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^\xi (\xi^2 - s^2)^{-\alpha} s^{2\alpha} f(s,\eta)\,ds.$$

It is obvious that if $f(s,\eta) \in C(\Omega_0)$, then $\bar{F}(0,\eta) = 0$.

Substituting the expressions for $F(\xi,\eta)$ from (25) into equation (26), integrating by parts with respect to the inner integral, and taking into account the equalities $R(x,t;\xi,\eta)|_{\xi=x}=0$ and $\bar{F}(0,\eta)=0$, and then changing the order of integration using the Dirichlet formula, we obtain

$$U(x,t)=-\frac{1}{\Gamma(1-\alpha)}\int\limits_0^t d\eta\int\limits_0^x s^{2\alpha}g(x,t;s,\eta)f(s,\eta)\,ds, \tag{29}$$

where

$$g(x,t;s,\eta)=\int\limits_s^x(\xi^2-s^2)^{-\alpha}R_\xi(x,t;\xi,\eta)\,d\xi. \tag{30}$$

Proceeding further, substituting (29) into (21), we obtain

$$u(x,t)=-\frac{2x^{1-2\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}\int\limits_0^x s^{2\alpha}\,ds\int\limits_0^t f(s,\eta)\,d\eta\int\limits_s^x(x^2-y^2)^{\alpha-1}g(y,t;s,\eta)\,dy. \tag{31}$$

Successively changing the order of integration three times in (31), we find

$$u(x,t)=\int\limits_0^x\int\limits_0^t R_\alpha(x,t;s,\eta)f(s,\eta)\,ds\,d\eta,$$

where

$$R_\alpha(x,t;s,\eta)=-\frac{2x^{1-2\alpha}s^{2\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}\int\limits_s^x(x^2-y^2)^{\alpha-1}g(y,t;s,\eta)\,dy. \tag{32}$$

According to (18), the function $R_\alpha(x,t;s,\eta)$ represents the Riemann's function of problem G_0.
Substituting (30) into (32), we obtain

$$R_\alpha(x,t;s,\eta)=-\frac{2x^{1-2\alpha}s^{2\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}\int\limits_s^x(x^2-y^2)^{\alpha-1}\,dy\int\limits_s^y(\xi^2-s^2)^{-\alpha}R_\xi(y,t;\xi,\eta)\,d\xi. \tag{33}$$

Let's compute the inner integral:

$$h(y,t;\xi,\eta)=\int\limits_s^y(\xi^2-s^2)^{-\alpha}R_\xi(y,t;\xi,\eta)\,d\xi.$$

Due to the equality

$$\frac{\partial}{\partial\xi}R\left(y,t;\xi,\eta\right)=K_0\left(1;\frac{1}{2},1,1;\bar{\sigma}_1,\sigma_2\right),$$

where $\bar{\sigma}_1=-\frac{\lambda}{4}(\xi-y)^2$, the function $h\left(y,t;\xi,\eta\right)$ can be represented as

$$h\left(y,t;\xi,\eta\right)=\int\limits_s^y\left(\xi^2-s^2\right)^{-\alpha}K_0\left(1;\frac{1}{2},1,1;\bar{\sigma}_1,\sigma_2\right)d\xi.$$

Making the change of variables $\xi=y-(y-s)\,\mu$, we obtain

$$h\left(y,t;\xi,\eta\right)=(y+s)^{-\alpha}(y-s)^{1-\alpha}$$

$$\times\int\limits_0^1\mu^{2\alpha}(1-\mu)^{-\alpha}\left(1-\frac{y-s}{y+s}\mu\right)^{-\alpha}K_0\left(1;\frac{1}{2},1,1;\tilde{\sigma}_1,\sigma_2\right)d\mu,$$

where $\tilde{\sigma}_1 = -\frac{\lambda}{4}(y-s)^2$, Utilizing the expansion of the function $K_0\left(a;b,c;b',c';\sigma_1,\sigma_2\right)$ in series (28) and considering the uniform convergence of this series for any argument values, we change the order of integration and summation

$$h\left(y,t;\xi,\eta\right) = (y+s)^{-\alpha}(y-s)^{1-\alpha}\sum_{m,n=0}^{+\infty}\frac{(1)_{m+n}}{(1/2)_m(1)_m(1)_n}\frac{\tilde{\sigma}_1^m}{m!}\frac{\sigma_2{}^n}{n!}$$
$$\times\int_0^1 \mu^{2m}(1-\mu)^{-\alpha}\left(1-\frac{y-s}{y+s}\mu\right)^{-\alpha}d\mu.$$

From here, applying formula [36]

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt = \frac{\Gamma\left(b\right)\Gamma\left(c-b\right)}{\Gamma\left(c\right)}F\left(a,b,c;x\right), \quad c>b>0,$$

we obtain

$$h\left(y,t;\xi,\eta\right) = (y+s)^{-\alpha}(y-s)^{1-\alpha}\sum_{m,n=0}^{+\infty}\frac{(1)_{m+n}}{(1/2)_m(1)_m(1)_n}\frac{\tilde{\sigma}_1^m}{m!}\frac{\sigma_2{}^n}{n!}$$
$$\times\frac{\Gamma\left(2m+1\right)\Gamma\left(1-\alpha\right)}{\Gamma\left(2m-\alpha+2\right)}{}_2F_1\left(\alpha,2m+1;2m-\alpha+1;\sigma_3\right), \tag{34}$$

where $F\left(a,b,c;x\right)$ is the Gaussian hypergeometric function [36], $\sigma_3 = \frac{y-s}{y+s}$.

We apply the following formula to the Gaussian hypergeometric function in expression (34)

$${}_2F_1\left(a,b;c;x\right) = (1-x)^{-a}{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right).$$

Then, (34) takes the following form

$$h\left(y,t;s,\tau\right) = (2s)^{-\alpha}(y-s)^{1-\alpha}\sum_{m,n=0}^{+\infty}\frac{(1)_{m+n}}{(1/2)_m(1)_m(1)_n}\frac{\tilde{\sigma}_1^m}{m!}\frac{\sigma_2{}^n}{n!}$$
$$\times\frac{\Gamma\left(2m+1\right)\Gamma\left(1-\alpha\right)}{\Gamma\left(2m-\alpha+2\right)}{}_2F_1\left(\alpha,1-\alpha;2m-\alpha+2;\sigma_4\right),$$

where $\sigma_4 = \frac{s-y}{2s}$. Substituting the found expression for $h\left(y,t;\xi,\eta\right)$ into (33), we obtain

$$R_\alpha\left(x,t;s,\tau\right) = -\frac{2^{1-\alpha}x^{1-2\alpha}s^\alpha}{\Gamma\left(\alpha\right)\Gamma\left(1-\alpha\right)}\sum_{m,n=0}^{+\infty}\frac{(1)_{m+n}}{(1/2)_m(1)_m(1)_n}\frac{(-\lambda/4)^m}{m!}\frac{\sigma_2{}^n}{n!}$$
$$\times\frac{\Gamma\left(2m+1\right)\Gamma\left(1-\alpha\right)}{\Gamma\left(2m-\alpha+2\right)}\int_s^x\left(x^2-y^2\right)^{\alpha-1}(y-s)^{2m-\alpha+1}{}_2F_1\left(\alpha,1-\alpha;2m-\alpha+2;\sigma_4\right)dy. \tag{35}$$

Let's change the variables of the inner integral in expression (35) to $y = s+(x-s)\rho$ and apply the following formula [36]

$$\int_0^1 \xi^{c-1}(1-\xi)^{\beta-1}(1-zy\xi)^{-\rho}{}_2F_1\left(a,b;c;\omega y\xi\right)d\xi$$
$$= \frac{B\left(c,\beta\right)}{(1-yz)^\rho}F_3\left(\rho,a,\beta,b,c+\beta;\frac{yz}{yz-1};\omega y\right).$$

Then, expression (35) takes the following form

$$R_\alpha(x,t;s,\tau) = x^{-\alpha}s^{\alpha}(x-s)\sum_{m,n=0}^{+\infty}\frac{(1)_{m+n}}{(3/2)_m(1)_m(1)_n}\frac{\hat{\sigma}_1^m}{m!}\frac{\sigma_2{}^n}{n!}F_3(\alpha,\alpha,1-\alpha,1-\alpha,2m+2;\sigma_5,\sigma_6),$$

where $F_3(\alpha,\alpha',\beta,\beta',\gamma,x,y) = \sum\limits_{k,l=0}^{+\infty}\frac{(\alpha)_k(\alpha')_l(\beta)_k(\beta')_l}{(\gamma)_{k+l}}\frac{x^k}{k!}\frac{y^l}{l!}$ is the Horn function [36], $B(c,\beta) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(c+\beta)}$ is the beta function, $\hat{\sigma}_1 = -\frac{\lambda}{4}(x-s)^2$, $\sigma_5 = \frac{x-s}{2x}$, and $\sigma_6 = \frac{s-x}{2s}$. Using the formula [46]

$$F_3\left(\alpha,\alpha',1-\alpha,1-\alpha';\gamma;z,\frac{z}{2z-1}\right)$$
$$= (1-z)^{\gamma-1}(1-2z)^{1-\alpha'}{}_2F_1\left(\frac{1}{2}(\gamma-\alpha-\alpha'+1),\quad \frac{1}{2}(\gamma+\alpha-\alpha');\gamma;4z(1-z)\right),$$

and then sequentially applying formulas [36]

$$F(a,b,2b;x) = \left(\frac{2}{1+\sqrt{1-x}}\right)^{2a}{}_2F_1\left(a,a-b+\frac{1}{2},b+\frac{1}{2};\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2\right),$$
$$F(a,b,c;x) = (1-x)^{-b}F\left(c-a,b,c;\frac{x}{x-1}\right),$$

we obtain

$$R_\alpha(x,t;s,\eta) = (s-x)\left(\frac{s}{x}\right)^{\alpha}\sum_{m,n=0}^{+\infty}\frac{(1)_{m+n}}{(3/2\ )_m(1)_m(1)_n}\frac{\hat{\sigma}_1^m}{m!}\frac{\sigma_2{}^n}{n!}F\left(\alpha,1-\alpha,m+\frac{3}{2};\bar{\sigma}_5\right), \tag{36}$$

where $\bar{\sigma}_5 = -\frac{(x-s)^2}{4xs}$.

The function $R_\alpha(x,t;s,\eta)$, defined by equation (36), is the Riemann's function for problem G_0.

It is straightforward to demonstrate that when $\alpha = 0$, function (36) coincides with the Riemann's function for problems (22)–(24), as defined by equation (27).

Therefore, the Riemann's function for problems (2) and (3) is determined by the formula

$$v(x,t;\xi,\eta) = (\xi-x)\left(\frac{\xi}{x}\right)^{\alpha}\sum_{m,n=0}^{+\infty}\frac{(1)_{m+n}}{(3/2\ )_m(1)_m(1)_n}\frac{\hat{\sigma}_1^m}{m!}\frac{\sigma_2{}^n}{n!}F\left(\alpha,1-\alpha,m+\frac{3}{2};\bar{\sigma}_5\right), \tag{37}$$

where $\sigma_1 = -\frac{\lambda}{4}(\xi-x)^2$, $\sigma_2 = \eta - t$, and $\bar{\sigma}_5 = -\frac{(x-\xi)^2}{4x\xi}$.

Let's demonstrate that function (35) satisfies the conditions (5) and (6).

The first condition (6) is immediately satisfied when $\xi = x$.

The value of the derivative of the function $v(x,y;\xi,\eta)$ with respect to ξ at $\xi = x$ is

$$v_\xi(x,y;\xi,\eta)|_{\xi=x} = \sum_{n=0}^{+\infty}\frac{(\eta-t)^n}{n!}.$$

Since the right-hand side of this equation represents the series expansion of $e^{\eta-t}$, equality (11) follows.

The function $v(x,y;\xi,\eta)$ when $\eta = t$, equals

$$v(x,t;\xi,\eta)|_{\eta=t} = (\xi-x)\left(\frac{\xi}{x}\right)^{\alpha}\sum_{m=0}^{+\infty}\frac{\sigma_1^m}{(3/2\ )_m m!}F(\alpha,1-\alpha,m+3/2\ ;\bar{\sigma}_5). \tag{38}$$

By formula [35]

$$\sum_{m=0}^{+\infty}\frac{y^m}{(c)_m m!}F(a,b,c+m;x) = \Xi_2(a,b,c;x,y),\quad |x|<1$$

the function (38) can be represented as

$$v\left(x,t;\xi,\eta\right)\big|_{\eta=t}=\left(\xi-x\right)\left(\frac{\xi}{x}\right)^{\alpha}\Xi_2\left(\alpha,1-\alpha,3/2\ ;\bar{\sigma}_5,\sigma_1\right), \tag{39}$$

where $\Xi_2\left(a,b,c;x,y\right)=\sum\limits_{m,n=0}^{+\infty}\frac{(a)_m(b)_m}{(c)_{m+n}m!n!}x^m y^n$ is the Humbert function.

Using the notation $\alpha-1/2\ =\nu$ in (14), we obtain

$$\omega_2\left(\xi,x,y\right)=-\frac{\pi x}{2\sin\left(\nu\pi\right)}\left(\frac{\xi}{x}\right)^{\nu+1}\left[J_\nu\left(\sqrt{\lambda}x\right)J_{-\nu}\left(\sqrt{\lambda}\xi\right)-J_{-\nu}\left(\sqrt{\lambda}x\right)J_\nu\left(\sqrt{\lambda}\xi\right)\right.$$

Taking into account formulas [37]

$$J_\nu\left(z\right)=\frac{1}{\Gamma\left(\nu+1\right)}\left(\frac{z}{2}\right)^{\nu}{}_0F_1\left(\nu+1,-\frac{z^2}{4}\right),\quad \Gamma\left(1+\nu\right)\Gamma\left(1-\nu\right)=\frac{\pi\nu}{\sin\left(\pi\nu\right)},$$

we have

$$\omega_2\left(\xi,x,t\right)=\frac{\xi}{2\nu}\left(\frac{\xi}{x}\right)^{\nu}\left[\left(\frac{\xi}{x}\right)^{\nu}{}_0F_1\left(1+\nu;-\frac{\lambda\xi^2}{4}\right){}_0F_1\left(1-\nu;-\frac{\lambda x^2}{4}\right)\right.$$
$$\left.-\left(\frac{x}{\xi}\right)^{\nu}{}_0F_1\left(1-\nu;-\frac{\lambda\xi^2}{4}\right){}_0F_1\left(1+\nu;-\frac{\lambda x^2}{4}\right)\right].$$

From here, applying formula [36]

$${}_0F_1\left(a;pz\right){}_0F_1\left(b;qz\right)=\sum_{n=0}^{\infty}\frac{(pz)^n}{n!(a)_n}F\left(1-a-n,-n;b;\frac{q}{p}\right),$$

we obtain

$$\omega_2\left(\xi,x,t\right)=\frac{\xi}{2\nu}\left(\frac{\xi}{x}\right)^{\nu}\left[\left(\frac{\xi}{x}\right)^{\nu}\sum_{n=0}^{+\infty}\frac{\left(-\frac{\lambda\xi^2}{4}\right)^n}{n!(1+\nu)_n}F\left(-\nu-n,-n,1-\nu;\frac{x^2}{\xi^2}\right)\right.$$
$$\left.-\left(\frac{x}{\xi}\right)^{\nu}\sum_{n=0}^{\infty}\frac{\left(\lambda\xi^2/4\ \right)^n}{n!(1-\nu)_n}F\left(\nu-n,-n,1+\nu;\frac{x^2}{\xi^2}\right)\right].$$

Using formula [46]

$$F\left(a,b,a-b+1;z\right)=\left(1-\sqrt{z}\right)^{-2a}F\left(a,a-b+\frac{1}{2},2a-2b+1;-\frac{4\sqrt{z}}{(1-\sqrt{z})^2}\right),$$

we find

$$\omega_2\left(\xi,x,t\right)=\frac{\xi}{2\nu}\left(\frac{\xi}{x}\right)^{\nu}\sum_{n=0}^{+\infty}\frac{\sigma_1^n}{n!}\left[\frac{(-4\bar{\sigma}_5)^{\nu}}{(1+\nu)_n}F\left(-\nu-n,\frac{1}{2}-\nu,1-2\nu;-\frac{1}{\bar{\sigma}_5}\right)\right.$$
$$\left.-\frac{(-4\bar{\sigma}_5)^{-\nu}}{(1-\nu)_n}F\left(\nu-n,\frac{1}{2}+\nu,1+2\nu;-\frac{1}{\bar{\sigma}_5}\right)\right].$$

Considering formula [36]

$$F\left(a,b,c;z\right)=\frac{\Gamma\left(c\right)\Gamma\left(b-a\right)}{\Gamma\left(b\right)\Gamma\left(c-a\right)}(-z)^{-a}F\left(a,a-c+1,a-b+1;\frac{1}{z}\right)$$
$$+\frac{\Gamma\left(c\right)\Gamma\left(a-b\right)}{\Gamma\left(a\right)\Gamma\left(c-b\right)}(-z)^{-b}F\left(b,b-c+1,b-a+1;\frac{1}{z}\right),$$

we have

$$\omega_2(\xi,x,t)=\frac{\xi}{2\nu}\left(\frac{\xi}{x}\right)^{\nu}\sum_{n=0}^{\infty}\frac{\sigma_1^n}{n!}\left[A_1(-\bar{\sigma}_5)^{-n}\Gamma(1/2\ +n)\,F\left(\nu-n,-\nu-n,\frac{1}{2}-n;\bar{\sigma}_5\right)\right.$$
$$\left.+\ A_2(-\bar{\sigma}_5)^{1/2}\,\Gamma(-1/2\ -n)\,F\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu,n+\frac{3}{2};\bar{\sigma}_5\right)\right],\tag{40}$$

where

$$A_1=\frac{4^{\nu}\Gamma(1-2\nu)}{(1+\nu)_n\Gamma(1+n-\nu)\,\Gamma(1/2\ -\nu)}-\frac{4^{-\nu}\Gamma(1+2\nu)}{(1-\nu)_n\Gamma(1+\nu+n)\,\Gamma(1/2\ +\nu)},$$
$$A_2=\frac{4^{\nu}\Gamma(1-2\nu)}{(1+\nu)_n\Gamma(1/2\ -\nu)\,\Gamma(-\nu-n)}-\frac{4^{-\nu}\Gamma(1+2\nu)}{(1-\nu)_n\Gamma(1/2\ +\nu)\,\Gamma(\nu-n)}.$$

Hence, using the formulas $\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)$ and $\Gamma(z)\,\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$, we find $A_1=0$ and $A_2=(-1)^n\frac{2\nu}{\sqrt{\pi}}$. Substituting the found values of A_1 and A_2 into (40) and considering $\Gamma\left(-\frac{1}{2}-n\right)=\frac{(-1)^n\left(-2\sqrt{\pi}\right)}{(3/2\ )_n}$, after some calculations we obtain

$$\omega_2(\xi,x,y)=-\left|\xi-x\right|\left(\frac{\xi}{x}\right)^{\nu+\frac{1}{2}}\Xi_2\left(1/2\ +\nu,1/2\ -\nu,3/2\ ,\bar{\sigma}_5,\sigma_1\right).\tag{41}$$

If we denote $\alpha-1/2\ =\nu$, then the function (41) coincides with (39). Thus, the formula

$$\Xi_2\left(\alpha,1-\alpha,3/2\ ;-\frac{\lambda}{4}(\xi-x)^2,-\frac{(\xi-x)^2}{4x\xi}\right)$$
$$=\frac{\pi\sqrt{x\xi}}{2(x-\xi)\cos(\alpha\pi)}\left[J_{\alpha-1/2}\ \left(\sqrt{\lambda}x\right)J_{1/2\ -\alpha}\left(\sqrt{\lambda}\xi\right)-J_{1/2\ -\alpha}\left(\sqrt{\lambda}x\right)J_{\alpha-1/2}\ \left(\sqrt{\lambda}\xi\right)\right],$$

which is not found in references, has been proved.

Let's return to the investigation of formula (15). Applying integration by parts to this equation, we obtain

$$u(x,t)=x^{-2\alpha}t^{2\alpha}u(t,t)\,v_{\xi}(x,t;t,t)-x^{-2\alpha}t^{2\alpha}u_{\xi}(t,t)\,v(x,t;t,t)+u_{\xi}(x,x)\,v(x,t;x,x)$$
$$-x^{-2\alpha}t^{2\alpha}u(t,t)\left(v_{\eta}(x,t;t,t)-v(x,t;t,t)\right)+u(x,x)\left(v_{\eta}(x,t;x,x)-v(x,t;x,x)\right)$$
$$+x^{-2\alpha}\int_x^t\left[\left(2\alpha\xi^{2\alpha-1}u_{\xi}(\xi,\xi)+\xi^{2\alpha}u_{\xi\xi}(\xi,\xi)-\xi^{2\alpha}u_{\xi\eta}(\xi,\xi)-2\xi^{2\alpha}u_{\xi}(\xi,\xi)+\lambda\xi^{2\alpha}u(\xi,\xi)\right.\right.$$
$$\left.\left.-\ 2\alpha\xi^{2\alpha-1}u(\xi,\xi)\right)v(x,t;\xi,\xi)-\left(2\alpha\xi^{2\alpha-1}u(\xi,\xi)+\xi^{2\alpha}u_{\xi}(\xi,\xi)\right)v_{\eta}(x,t;\xi,\xi)\right]d\xi$$
$$-\ x^{-2\alpha}\iint_{\Omega_1}\xi^{2\alpha}v(x,t;\xi,\eta)\,f(\xi,\eta)\,d\xi d\eta.\tag{42}$$

The functions u_{ξ}, u_{η}, $u_{\xi\xi}$, and $u_{\xi\eta}$ defined at $\eta=\xi$ are found from the initial conditions of the Cauchy problem

$$u(\xi,\xi)=\psi_1(\xi)\,,\quad u_{\xi}(\xi,\xi)-u_{\eta}(\xi,\xi)=\sqrt{2}\psi_2(\xi)\,,$$
$$u_{\xi}(\xi,\xi)+u_{\eta}(\xi,\xi)={\psi'}_1(\xi)\,,$$
$$u_{\xi\xi}(\xi,\xi)-2u_{\xi\eta}(\xi,\xi)+u_{\eta\eta}(\xi,\xi)=2\psi_3(\xi)\,,$$
$$u_{\xi\xi}(\xi,\xi)+2u_{\xi\eta}(\xi,\xi)+u_{\eta\eta}(\xi,\xi)={\psi''}_1(\xi)\,.$$

Thus, we obtain

$$u_{\xi}(\xi,\xi)=\frac{\sqrt{2}\psi_2(\xi)+{\psi'}_1(\xi)}{2},\quad u_{\eta}(\xi,\xi)=\frac{{\psi'}_1(\xi)-\sqrt{2}\psi_2(\xi)}{2},$$

$$u_{\xi\xi}(\xi,\xi)=\frac{\psi''_1(\xi)+2\sqrt{2}\psi'_2(\xi)+2\psi_3(\xi)}{4},\quad u_{\xi\eta}(\xi,\xi)=\frac{\psi''_1(\xi)-2\psi_3(\xi)}{4}. \tag{43}$$

Considering series (37), the function $v(x,t;\xi,\eta)$ can be represented as

$$v(x,t;\xi,\eta)=(\xi-x)\left(\frac{\xi}{x}\right)^{\alpha}\sum_{k=0}^{+\infty}\frac{(\alpha)_k(1-\alpha)_k}{(3/2\,)_k}\frac{\bar{\sigma}_5^k}{k!}K_0\left(1;\frac{3}{2}+k,1,1;\sigma_1,\sigma_2\right).$$

The following differentiation formulas for series (28) are valid:

$$\frac{\partial}{\partial\sigma_2}K_0\left(a;b,c;b';\sigma_1,\sigma_2\right)=\frac{a}{b'}K_0\left(a+1;b,c;b'+1;\sigma_1,\sigma_2\right),$$

$$\left(\sigma_2\frac{\partial}{\partial\sigma_2}+\left(b'-1\right)\right)K_0\left(a;b,c;b';\sigma_1,\sigma_2\right)=\left(b'-1\right)K_0\left(a;b,c;b'-1;\sigma_1,\sigma_2\right).$$

Using these formulas, we compute the corresponding derivatives of the function $v(x,t;\xi,\eta)$:

$$v(x,t;x,x)=0,\quad v_\eta(x,t;x,x)-v(x,t;x,x)=0,$$

$$v_\eta(x,t;t,t)-v(x,t;t,t)=-\frac{\lambda}{6}(t-x)^3x^{\alpha}t^{-\alpha}\Xi_2\left(\alpha,1-\alpha;5/2;\sigma_1;\sigma_2\right),$$

$$v(x,t;t,t)=x^{\alpha}t^{-\alpha}(t-x)\,\Xi_2\left(\alpha,1-\alpha;3/2;\sigma_1;\sigma_2\right),$$

$$v_\xi(x,t;t,t)=\frac{x^{\alpha}\left(t^{\alpha}-\alpha t^{\alpha}+\alpha t^{\alpha-1}x\right)}{t^{2\alpha}},$$

$$v_\eta(x,t;\xi,\xi)=(\xi-x)\left(\frac{\xi}{x}\right)^{\alpha}\sum_{k=0}^{+\infty}\frac{(\alpha)_k(1-\alpha)_k}{(3/2)_k}\frac{\theta^k}{k!}R\left(2;\frac{3}{2}+k,1,2;\sigma_1;\sigma_2\right). \tag{44}$$

Substituting the defined expressions (43) and (44) into (42), we obtain the solution of problem (2) and (3) in the form

$$\begin{aligned}u(x,t)=&\frac{t^{\alpha}-\alpha t^{\alpha}+\alpha t^{\alpha-1}x}{x^{\alpha}}\psi_1(t)-\frac{1}{2}\left(\sqrt{2}\psi_2(t)+\psi'_1(t)\right)x^{-\alpha}t^{\alpha}(t-x)\,\Xi_2\left(\alpha,1-\alpha;3/2;\sigma_1;\sigma_2\right)\\&+\frac{\lambda}{6}(t-x)^3x^{-\alpha}t^{\alpha}\psi_1(t)\,\Xi_2\left(\alpha,1-\alpha;5/2;\sigma_1;\sigma_2\right)\\&+x^{-2\alpha}\int_x^t\Bigg\{(\xi-x)\left(\frac{\xi^3}{x}\right)^{\alpha}\sum_{k=0}^{+\infty}\frac{(\alpha)_k(1-\alpha)_k}{(3/2)_k}\frac{\bar{\sigma}_5^k}{k!}K_0\left(1;\frac{3}{2}+k,1,1;\sigma_1;\bar{\sigma}_2\right)\Phi_1(\xi)\\&+(\xi-x)\left(\frac{\xi^3}{x}\right)^{\alpha}\sum_{k=0}^{+\infty}\frac{(\alpha)_k(1-\alpha)_k}{(3/2)_k}\frac{\bar{\sigma}_5^k}{k!}K_0\left(2;\frac{3}{2}+k,1,2;\sigma_1;\bar{\sigma}_2\right)\Phi_2(\xi)\Bigg\}d\xi\\&-x^{-2\alpha}\iint_{\Omega_1}\xi^{2\alpha}v(x,t;\xi,\eta)\,f(\xi,\eta)\,d\xi d\eta,\end{aligned} \tag{45}$$

where $\sigma_1=-\frac{\lambda}{4}(\xi-x)^2$, $\bar{\sigma}_2=\xi-t$, $\bar{\sigma}_5=-\frac{(x-\xi)^2}{4x\xi}$,

$$\Phi_1(\xi)=\frac{\sqrt{2}}{2}\psi'_2(\xi)+\psi_3(\xi)+\left(\lambda-\frac{2\alpha}{\xi}\right)\psi_1(\xi)+\left(\frac{\alpha}{\xi}-1\right)\left(\sqrt{2}\psi_2(\xi)+\psi'_1(\xi)\right),$$

$$\Phi_2(\xi)=\frac{2\alpha}{\xi}\psi_1(\xi)+\frac{\sqrt{2}}{2}\psi_2(\xi)+\frac{1}{2}\psi'_1(\xi).$$

Theorem 2. *Let $\psi_j(x)\in C^{2-j}(\mathbb{R}^+)$, $j=\overline{0,2}$, and all corresponding derivatives of the initial functions vanish at $x=0$. Then, the function $u(x,t)$, defined by (45), is the unique solution to the Cauchy problem (2) and (3).*

6. CONCLUSIONS

Applying the Erd'alyi–Kober transmutation operator we constructed the Riemann function of the Cauchy problem for a pseudo-parabolic equation with singular coefficients. Based on this, we found the exact solution to the investigated problem. Despite the advancement of modern computational technology, constructing exact solutions for boundary value problems of partial differential equations remains an important and relevant task. These solutions allow for a deeper understanding of the qualitative features of described processes and phenomena, the properties of mathematical models, and can also be used as benchmark examples for asymptotic, approximate, and numerical methods.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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