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Discrete Equations and the Reduction Method

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Abstract—We consider linear bounded operators acting in Banach spaces with a basis, such operators can be represented by an infinite matrix. We prove that for an invertible operator there exists a sequence of invertible finite-dimensional operators so that the family of norms of their inverses is uniformly bounded. It leads to the fact that solutions of finite-dimensional equations converge to the solution of initial operator equation with infinite-dimensional matrix.

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1. INTRODUCTION

1.1. On Discrete Equations

Discrete equations is a very important mathematical object. This is related to computer calculations which help us to find numerical solution if we don't know its analytical expression. Discrete equations can appear via difference schemes [1] or difference potentials [2], or discrete convolutions [3]. The latter is more interesting for us because we try to develop discrete theory for pseudo-differential equations based on ideas and methods [4, 5]. Certain realization of these ideas and methods is presented in authors' papers [6, 8–13]. All mentioned papers are related to a solvability problem for discrete pseudo-differential equations. Such equations are roughly speaking infinite systems of linear algebraic equations, and for numerical solution we need to approximate these infinite systems by certain finite systems. In such cases, they used the reduction method.

This reduction method was developed in [3] for abstract situation and for different classes of operators. Some results were obtained in papers [14, 15] for general operators and discrete convolutions. But these papers don't give an answer to the question if arbitrary invertible operator admits the reduction method, assuming the operator is presented by infinite matrix. In this paper, we will prove this assertion.

1.2. Pre-History: Digital Operators, Discrete Equations and Discrete Boundary Value Problems

Here we will describe some problem which have been considered earlier, these problems are closely related to topic of the paper, and we will explain why we are interested in the infinite matrices and the reduction method.

The classical pseudo-differential operator in Euclidean space $\mathbb{R}^m$ is defined by the formula [4]

$$(Au)(x) = \int\limits_{\mathbb{R}^m} \tilde{A}(x,\xi) e^{-ix\cdot\xi} \tilde{u}(\xi) d\xi,$$

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where the sign $\sim$ over a function denotes its Fourier transform

$$\tilde{u}(\xi) = \int\limits_{\mathbb{R}^m} u(x) e^{ix\cdot\xi} dx,$$

and the function $\tilde{A}(x,\xi)$ is called a symbol of a pseudo-differential operator A.

Our main goal was describing a periodic variant of this definition and studying its certain properties related to solvability of corresponding equations in canonical domains of an Euclidean space. This problem is very large and in our opinion it should include the following aspects according to a lot of physical and technical applications of such operators and related equations:

1. infinite discrete Fourier transform as a natural technique for such equations;
2. choice of appropriate discrete functional spaces;
3. studying solvability for infinite discrete equations;
4. studying solvability of approximating finite discrete equations;
5. a comparison between continuous and infinite discrete equations;
6. a comparison between infinite discrete and finite discrete equations.

This is not completed list of questions for studying which we intend to consider. Some results in this direction were obtained for simplest pseudo-differential operators (Calderon–Zygmund operators [9]) and corresponding equations. Also certain results were related to approximate solutions.

There are few variants of the theory of discrete boundary value problems (see, for example, [1, 2]), but these theories are related especially to partial differential operators and do not use the harmonic analysis technique. Since the classical theory of pseudo-differential operators is based on the Fourier transform we will use the discrete Fourier transform and discrete analogue of pseudo-differential operators which include discrete analogues of partial differential and some integral convolution operators.

Given function u_d of a discrete variable $\tilde{x} \in h\mathbb{Z}^m$, $h > 0$, we define its discrete Fourier transform by the series

$$(F_d u_d)(\xi) \equiv \widetilde{u}_d(\xi) = \sum_{\tilde{x}\in\mathbb{Z}^m} e^{i\tilde{x}\cdot\xi} u_d(\tilde{x}), \quad \xi \in \hbar\mathbb{T}^m,$$

where $\mathbb{T}^m = [-\pi,\pi]^m$, $\hbar = h^{-1}$, and partial sums are taken over cubes

$$Q_N = \{\tilde{x} \in h\mathbb{Z}^m : \tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_m), \max_{1\le k\le m} |\tilde{x}_k| \le N\}.$$

We will remind here some definitions of functional spaces [6] and will consider discrete analogue of the Schwartz space $S(h\mathbb{Z}^m)$. Let us denote $\zeta^2 = h^{-2}\sum\limits_{k=1}^{m}(e^{-ih\cdot\xi_k} - 1)^2$ and introduce the following.

The space $H^s(h\mathbb{Z}^m)$ is a closure of the space $S(h\mathbb{Z}^m)$ with respect to the norm

$$||u_d||_s = \left(\int\limits_{\hbar\mathbb{T}^m} (1 + |\zeta^2|)^s |\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2}.$$

Fourier image of the space $H^s(h\mathbb{Z}^m)$ will be denoted by $\widetilde{H}^s(\hbar\mathbb{T}^m)$. One can define some discrete operators for such functions u_d.

If $\widetilde{A}_d(\xi)$ is a periodic function in $\mathbb{R}^m$ with the basic cube of periods $\hbar\mathbb{T}^m$, then we consider it as a symbol. We will introduce a digital pseudo-differential operator in the following way.

A digital pseudo-differential operator A_d in a discrete domain D_d is called the operator [6]

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbb{Z}^m} \int_{\hbar\mathbb{T}^m} \widetilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d,$$

We use the class $E_\alpha, \alpha \in \mathbb{R}$, [6] with the following condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \leq |A_d(\xi)| \leq c_2(1 + |\zeta^2|)^{\alpha/2}$$

and universal positive constants c_1, c_2.

Let $D \subset \mathbb{R}^m$ be a domain. We will study the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{1}$$

in the discrete domain $D_d \equiv D \cap h\mathbb{Z}^m$ and will seek a solution $u_d \in H^s(D_d)$, $v_d \in H_0^{s-\alpha}(D_d)$ [6–8].

Earlier some canonical domains [10–12] were considered, and some results on unique solvability of these equations and related discrete boundary value problems were described in these papers.

As we see, the equation (1), in general, is an infinite system of linear algebraic equations. To use computer calculations we need to approximate this system by a finite one. This is basic motive for this paper, and we will try to justificate this fact for a general situation.

2. INFINITE MATRICES

Let X be a Banach space with standard basis $\mathbf{e}_i = (0, \dots, 0, \underbrace{1}_{i}, 0, \dots)$, and we consider an infinite system of linear algebraic equations with the matrix $A = (a_{ij})_{i,j=1}^{\infty}$; this matrix is a representation of linear bounded operator A in the space X:

$$A : X \to X.$$

Let's introduce the following equation in the space X

$$A\mathbf{x} = \mathbf{y}, \quad \mathbf{y} \in X. \tag{2}$$

Let P_n be a projector on a linear span of vectors $\mathbf{e}_i, i = 1, \dots, n$; this linear span will be denoted by X_n. Then, we put $A_n = P_n A P_n$ so that $A_n : X_n \to X_n$ and we write the truncated equation

$$A_n \mathbf{x}_n = P_n \mathbf{y} \tag{3}$$

in the vector space X_n. Thus, the operator A_n is represented by the matrix $(a_{ij})_{i,j=1}^{n}$. Obviously, the sequence of operators A_n strongly converges to A, i.e., $\forall \mathbf{x} \in X$, $\lim\limits_{n\to\infty} A_n\mathbf{x} = A\mathbf{x}$.

We will give here one auxiliary result which will help us to obtain main theorem.

Lemma 1. *If a certain* $\mathbf{x} \in X$, $\lim\limits_{n\to\infty} A_n\mathbf{x} = A\mathbf{x}$, *and there is the sequence* $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset X$ *such that* $\lim\limits_{n\to\infty} \mathbf{x}_n = \mathbf{x}$, *then* $\lim\limits_{n\to\infty} A_n\mathbf{x}_n = A\mathbf{x}$.

Proof. Indeed, we have

$$||A\mathbf{x} - A_n\mathbf{x}_n|| \leq ||A\mathbf{x} - A_n\mathbf{x}|| + ||A_n\mathbf{x} - A_n\mathbf{x}_n|| \leq ||A\mathbf{x} - A_n\mathbf{x}|| + ||A||\,||\mathbf{x} - \mathbf{x}_n||.$$

Both summands $||A\mathbf{x} - A_n\mathbf{x}||$ and $||A|| \cdot ||\mathbf{x} - \mathbf{x}_n||$ tend to zero according to assumptions of Lemma, and the proof is completed. □

3. MAIN RESULT

The following assertion is called usually the "*reduction method*".

Theorem 1. *If the inverse bounded operator $A^{-1}: X \to X$ exists, then the following assertions are valid:*

1) starting from a certain $N, \forall n \geq N$, the operators $A_n : X_n \to X_n$ are invertible;

2) we have the estimate $||A_n^{-1}|| \leq C$, with constant C non-depending on n;

3) the solution $\mathbf{x}_n$ to the equation (3) *converges to the solution* $\mathbf{x}$ *of the equation* (2) *under $n \to \infty$.*

Proof. We use the proof by contradiction applying the theory of finite systems of linear algebraic equations. Namely, the Cramer's rule asserts that the operator A_n with the matrix $(a_{ij})_{i,j=1}^n$ will be invertible in the space X_n iff $\det A_n \neq 0$.

First step. Let's note that there are two possibilities for considered situation: either starting from a certain $N, \forall n \geq N$, the operators $A_n : X_n \to X_n$ are invertible or there is a subsequence A_{n_k} of non-invertible operators. If the first situation is valid, then we have the needed assertion. That's why we assume that the second situation is realized. So, we have a sequence of non-invertible operators A_{n_k}, $k \to \infty$. We will show that this assumption leads to a contradiction.

If operators A_{n_k} are invertible, then $\det A_{n_k} = 0$. But then there exists such a matrix $(a_{ij})_{i,j=1}^{n_k}$ with a certain non-zero minor so that all minor of bigger order are vanishing, otherwise all matrices A_{n_k} will be null-matrices. We will assume (without loss of generality) that this minor is related to the matrix

$$A_{m_k}=\begin{pmatrix} a_{11} & a_{12} & \cdots a_{1m_k} \\ a_{21} & a_{22} & \cdots a_{2m_k} \\ \cdots & \cdots & \cdots \\ a_{m_k1} & a_{m_k2} & \cdots a_{m_km_k} \end{pmatrix}, \quad m_k < n_k,$$

and $\det A_{m_k} \neq 0$. This implies that the homogeneous system $A_{n_k}\mathbf{x}_{n_k} = 0$ has non-trivial solutions. Now we will describe their structure.

Let's denote by Y_{m_k} the space $X_{n_k} \ominus X_{m_k}$ so that $X_{n_k} = X_{m_k} \oplus Y_{m_k}$, and the representation

$$\mathbf{x}_{n_k} = \mathbf{x}_{m_k} + \mathbf{y}_{m_k}, \quad \mathbf{x}_{m_k} \in X_{m_k}, \quad \mathbf{y}_{m_k} \in Y_{m_k},$$

is unique for arbitrary $\mathbf{x}_{n_k} \in X_{n_k}$.

We introduce the rectangular $(m_k \times n_k)$-matrix B_{m_k} of the following type

$$B_{m_k}=\begin{pmatrix} a_{1m_k+1} & a_{1m_k+2} & \cdots a_{1n_k} \\ a_{2m_k+1} & a_{2m_k+2} & \cdots a_{2n_k} \\ \cdots & \cdots & \cdots \\ a_{m_km_k+1} & a_{m_km_k+2} & \cdots a_{m_kn_k} \end{pmatrix}.$$

Such operator B_{m_k} is uniformly bounded as an operator $Y_{n_k-m_k} \to X_{m_k}$ because

$$||B_{m_k}y_{m_k}|| \leq ||Ay_{m_k}||.$$

Therefore, we have the following property

$$\mathbf{x}_{m_k} = -A_{m_k}^{-1}B_{m_k}\mathbf{y}_{m_k}.$$

If we will transfer to a limit, then we will see that the operator A has infinite-dimensional kernel, thus it is non-invertible, and we have contradiction.

Second step. According to the first step we have that starting from a certain $N, \forall n \geq N$, the operators $A_n : X_n \to X_n$ are invertible. Let's assume that the sequence $||A_n^{-1}||$ is unbounded. It means that there are sequences $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset X_n$ and $\{c_n\}_{n=1}^{\infty}$, $c_n > 0$ such that

$$||A_n^{-1}\mathbf{x}_n|| \geq c_n||\mathbf{x}_n||, \quad \lim_{n\to\infty} c_n = \infty.$$

If we put $\mathbf{x}'_n = \mathbf{x}_n/||\mathbf{x}_n||$ so that $||\mathbf{x}'_n|| = 1$, then we can write

$$||A_n^{-1}\mathbf{x}'_n|| \geq c_n. \tag{4}$$

Let $\mathbf{y}_n = A_n^{-1}\mathbf{x}'_n$. According to (4) we have $\mathbf{y}_n \to \infty$. Then,

$$A_n\mathbf{y}_n = \mathbf{x}'_n, \quad ||A_n\mathbf{y}_n|| = 1.$$

We put $\mathbf{y}'_n = \mathbf{y}_n/||\mathbf{y}_n||$, $||\mathbf{y}'_n|| = 1$ and then

$$||A_n\mathbf{y}'_n|| = 1/||\mathbf{y}_n|| \to 0,\ n \to \infty.$$

Thus, we have the sequence $\{\mathbf{y}'_n\}_{n=1}^{\infty}$, $||\mathbf{y}'_n|| = 1$ such that

$$||A_n\mathbf{y}'_n|| \to 0,\ n \to \infty. \tag{5}$$

Let's consider $A_n\mathbf{y}'_n - A\mathbf{y}'_n$. This is a vector of the following type

$$\begin{pmatrix} 0 \\ \dots \\ 0 \\ \sum\limits_{k=1}^{n} a_{n+1k}y'_k \\ \sum\limits_{k=1}^{n} a_{n+2k}y'_k \\ \dots \end{pmatrix},$$

where first n coordinates are zero. It seems That this vector tends to zero under $n \to \infty$. But there is a counterexample, the basis $\{\mathbf{e}_k\}_{k=1}^{\infty}$. Nevertheless, we will find a contradiction using another way.

Let's consider the sequence of operators A_n more carefully. Obviously, according to (5), we have

$$\inf_{||\mathbf{y}||=1} ||A_n\mathbf{y}|| = \alpha_n, \quad \lim_{n\to\infty} \alpha_n = 0.$$

Further, for an arbitrary $\mathbf{y} \in X$, $||\mathbf{y}|| = 1$ we have

$$|||A_n\mathbf{y}|| - ||A\mathbf{y}||| \leq ||A_n\mathbf{y} - A\mathbf{y}||,$$

and then

$$||A\mathbf{y}|| \leq ||A_n\mathbf{y}|| + ||A_n\mathbf{y} - A\mathbf{y}||.$$

Thus,

$$\inf_{||\mathbf{y}||=1} ||A\mathbf{y}|| \leq \inf_{||\mathbf{y}||=1} ||A_n\mathbf{y}|| + ||A_n\mathbf{y} - A\mathbf{y}|| \leq \alpha_n + ||A_n\mathbf{y} - A\mathbf{y}||$$

for all $\mathbf{y} \in X$. Fix $\mathbf{y} \in X$. Given $\varepsilon > 0$ we can find such $N \in \mathbb{N}$ that $\forall n \geq N$ we have

$$\alpha_n < \varepsilon/2, \quad ||A_n\mathbf{y} - A\mathbf{y}|| < \varepsilon/2$$

so that

$$\inf_{||\mathbf{y}||=1} ||A\mathbf{y}|| < \varepsilon,$$

and we conclude

$$\inf_{||\mathbf{y}||=1} ||A\mathbf{y}|| = 0. \tag{6}$$

The equality (6) implies that there is the sequence $\{\mathbf{z}_k\}_{k=1}^{\infty}$, $||\mathbf{z}||_k = 1$ such that

$$\lim_{k\to\infty} A\mathbf{z}_k = 0.$$

But the operator A is invertible, and then $\lim\limits_{k\to\infty} \mathbf{z}_k = 0$. The latter assertion is a contradiction.

Third step. This step is related to a convergence. We have

$$x = A^{-1}y$$

and

$$x_n = A_n^{-1}P_n y$$

We will denote $P_n y = y_n$ and estimate $x - x_n$. Let's write

$$x - x_n = A^{-1}y - A_n^{-1}y_n = \left(A^{-1}y - A^{-1}y_n\right) + \left(A^{-1}y_n - A_n^{-1}y_n\right).$$

and estimate summands separately in view of the equality

$$||x - x_n|| = ||A^{-1}y - A_n^{-1}y_n||.$$

Now we can apply Lemma with A^{-1} and A_n^{-1} instead of A and A_n, ant Theorem is proved. □

Remark. May be such a result exists in mathematical literature but the authors have no appropriate information.

4. CONCLUSIONS

This studying is very important for our studying discrete pseudo-differential equations and related discrete boundary value problems. As a rule such problems lead to infinite systems of linear algebraic equations, and we need a verification for change the infinite system by finite one. Moreover, pseudo-differential operators are defined in Fourier images, and now is not clear what approach is more effective from computational point of view, original space or its Fourier image.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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