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Cauchy Problem for Abstract Singular Equations Containing the Bessel Operator for Negative Values of the Parameter

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Abstract—Euler–Poisson–Darboux equation is considered, containing the generator of the Bessel operator function. Sufficient conditions for the unique solvability of the Cauchy problem for negative values of the equation parameter are obtained, in this case, instead of the initial condition on the first derivative, a condition on the derivative of a higher order is specified, the order of which depends on the parameter of the equation. The question of the unique solvability of the Cauchy problem for the factorized Euler–Poisson–Darboux equation is investigated. Examples are given.

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INTRODUCTION

Study of differential equations with unbounded operator coefficients, acting in the Banach space E, stimulates the development of the theory of resolving operators of the corresponding initial tasks. As a result of studies of first-order evolutionary equations $u'(t) = Au(t)$ semigroups of linear operators $T(t)$ arose, and when studying the second-order equation (abstract wave equation) $u''(t) = Au(t)$—operator cosine functions $C(t)$. Relaxation of requirements on resolving operators of the Cauchy problem for abstract differential equations of the first and second orders led to the concept of an integrated semigroup and an integrated cosine operator function. For terminology and literature sources, see monographs [1, 2], and review papers [3, 4].

The Bessel operator function (BOF) was introduced into consideration in [5, 6] as a resolving operator Cauchy problem for the Euler–Poisson–Darboux equation (EPD). But, just as in the theory of semigroups and operator cosine functions, the family of Bessel operator functions can be introduced (see [7]) independently from the EPD differential equation with which it is ultimately connected, and further in this section we will recall the process of constructing the BOF.

An important role in constructing the family is played by the operator depending on the parameter $k > 0$ generalized shift T_s^t, defined by the equality (see [8])

$$T_s^t Y(s) = \frac{\Gamma(k/2+1/2)}{\sqrt{\pi}\,\Gamma(k/2)} \int\limits_0^\pi Y\left(\sqrt{s^2+t^2-2st\cos\varphi}\right)\sin^{k-1}\varphi\, d\varphi, \quad s,t \ge 0,$$

where $\Gamma(\cdot)$ is the Euler gamma function. The generalized shift operator depends on the parameter $k > 0$, but, following [8], we will not note this fact in his entry.

Let E be a Banach space, parameter $k > 0$ and $Y_k(\cdot) : [0,\infty) \to B(E)$ be an operator function, acting in the space of linear bounded operators $B(E)$.

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Definition 1. *A strongly continuous family of linear bounded operators depending on the parameter* $k>0$ $Y_k(t):[0,\infty)\to B(E)$ *is called the Bessel operator function if*

$$Y_k(0)=I,\quad Y_k(t)Y_k(s)=T_s^tY_k(s),\quad s,t\geq 0$$

and there exist constants $\Upsilon\geq 1$, $\omega\geq 0$ *such that*

$$||Y_k(t)||\leq \Upsilon e^{\omega t},\ t\geq 0.$$

Associated with the BOF family is the Bessel differential operator

$$\frac{d^2}{dt^2}+\frac{k}{t}\frac{d}{dt},$$

which often occurs in differential equations with axial symmetry.

Definition 2. *BOF* $Y_k(t)$ *generator is the operator* A *with domain* $D(A)$*, consisting of those* $x\in E$ *for which the function* $Y_k(t)x$ *is twice differentiable at the point* $t=0$*, and which is defined by the equality*

$$Ax=\lim_{t\to 0+}\left(\frac{d^2Y_k(t)x}{dt^2}+\frac{k}{t}\frac{dY_k(t)x}{dt}\right).$$

Statements 1)–5) given later in this section were proven in [7].

1) If the operator A is a generator of the BOF $Y_k(t)$, then it is closed and its domain of definition $D(A)$ is dense in E. Moreover, E contains a dense set of elements on which all powers of the operator A are defined.

2) For any $t,s\geq 0$ and $x\in D(A)$ the following equalities hold

$$Y_k(t)Y_k(s)=Y_k(s)Y_k(t),\quad AY_k(t)x=Y_k(t)Ax.$$

3) Let $x\in D(A)$ and $t>0$, then $Y_k(t)x\in D(A)$ and

$$AY_k(t)x=\frac{d^2Y_k(t)x}{dt^2}+\frac{k}{t}\frac{dY_k(t)x}{dt}.$$

4) If $u_0\in D(A)$, then the function $Y_k(t)u_0$ is the only solution to the following Cauchy problem for the EPD equation

$$u''(t)+\frac{k}{t}u'(t)=Au(t)\quad (t>0),\quad u(0)=u_0,\quad u'(0)=0. \tag{1}$$

A literature review of publications related to the abstract EPD equation can be found in [5–7]. And in [9, 10] there is an extensive list of publications on equations in partial derivatives with the Bessel operator.

In what follows, it is convenient to use the symbol $Y_0(t)$ to denote the cosine operator function $C(t)$ with generator A, and the class of operators that are the generator of some BOF $Y_k(t),\ k\geq 0$ will be denoted by G_k. Criteria for membership of $A\in G_k$ or, which is the same, criteria for uniform correctness of the initial problem (1), are given in [5, 6].

5) Let $0\leq k<m$ and the operator A be a generator of the BOF $Y_k(t)$. Then, A will also be a generator of $Y_m(t)$, in this case the corresponding BOF $Y_m(t)$ has the form

$$Y_m(t)=\frac{2\Gamma(m/2+1/2)}{\Gamma(k/2+1/2)\Gamma(m/2-k/2)}\int\limits_0^1 s^k(1-s^2)^{(m-k)/2-1}Y_k(ts)\,ds. \tag{2}$$

The equality (2), from which the embedding $G_k\subset G_m,\ m>k$ follows, is called the formula for the shift of the BOF by parameter.

If the operator A is a generator of the cosine operator function $Y_0(t)=C(t)$, then from the equality (2) for $k=0$ it follows that the BOF $Y_m(t)$ is an operator cosine function integrated in a special way. For more details on this, see [11].

We also point out that in this work we make do with the concept of an integral of a continuous function, but if necessary, we can use the Bochner integral of a function with a value in Banach space.

1. CAUCHY PROBLEM FOR THE EULER–POISSON–DARBOUX EQUATION WITH NEGATIVE VALUES OF THE PARAMETER EQUATIONS

In the Banach space E, consider the Cauchy problem of the Euler–Poisson–Darboux equation

$$u''(t) + \frac{\mu}{t}u'(t) = Au(t), \quad t > 0, \tag{3}$$

$$u(0) = u_0, \quad u'(0) = 0. \tag{4}$$

As was already noted in the introduction, for $\mu \geq 0$ and $A \in G_\mu$ this problem is uniformly correct. At value of the parameter $\mu < 0$ the task of finding a solution to the equation (3) satisfying the conditions (4) is not correct due to the loss of uniqueness of the solution. Issues of non-uniqueness of solutions were discussed previously in [12].

The work [13] describes the process of constructing one of the operator functions $Y_\mu(t)$, which allows for $Y_k(t)$, $k \geq 0$ find a solution to the equation (3) for $\mu < k$, including for $\mu < 0$, satisfying the conditions (4). Further, in Theorems 1–3, we present the result of this process in the edition we need. In the following theorems, the assumption $A \in G_k$ for some $k \geq 0$ means, in particular, that with the operator A the Cauchy problem is correctly solvable (1) and $Y_k(t)$ is the resolving operator for this problem.

Theorem 1 [13]. *Suppose that $A \in G_k$ for some $k \geq 0$, $\mu < k$, $\mu \neq 1 - 2N$, $N \in \mathbb{N}$, and let l be the smallest natural number such that $q = 2l + \mu \geq k$, $u_0 \in D\left(A^{[l/2]+2}\right)$. Then, the function*

$$Y_\mu(t)u_0 = \frac{t^{1-\mu}}{(q-1)(q-3)\cdots(\mu+1)}\left(\frac{1}{t}\frac{d}{dt}\right)^l \left(t^{q-1}Y_q(t)u_0\right) \tag{5}$$

is a solution to the problem (3), (4).

Theorem 2 [13]. *Let the conditions of Theorem 1 be satisfied for $\mu = 0$. Then, the function*

$$Z_1(t)u_0 = \frac{2}{\pi}\int\limits_0^1 \left(1 - s^2\right)^{-1/2} \ln\left(t\left(1 - s^2\right)\right) Y_0(ts)u_0\, ds \tag{6}$$

is a solution to the equation (3) for $\mu = 1$ and satisfies the condition

$$\lim_{t \to 0} tZ_1'(t)u_0 = u_0.$$

Theorem 3 [13]. *Suppose that $A \in G_k$ for some $k \geq 0$, $\mu = 1 - 2N$, $N \in \mathbb{N}$, and let l be the smallest natural number such that $2l \geq k$, $u_0 \in D\left(A^{[N/2+l/2]+2}\right)$. Then, the function*

$$Y_\mu(t)u_0 = \frac{t^{2N}}{(-2)^{N-1}(N-1)!}\left(\frac{1}{t}\frac{d}{dt}\right)^N Z_1(t)u_0 \tag{7}$$

is a solution to the problem (3), (4).

Thus, the formulas (5)–(7) for $\mu < k$ define the function $Y_\mu(t)u_0$, which is a solution to the problem (3), (4), where $A \in G_k$ for some $k \geq 0$. The operator function $Y_\mu(t)$ is defined on the domain of definition $D\left(A^{n(\mu)}\right)$ of the operator $A^{n(\mu)}$, and the exponent $n(\mu)$ depending on μ is indicated in Theorems 1 and 3.

Note also that this function will have a unique solution only for $\mu \geq 0$, and for $\mu < 0$ the uniqueness is violated due to the presence of a nonzero solution $t^{1-\mu}Y_{2-\mu}(t)u_1$ for the equation (3), satisfying conditions $u(0) = u'(0) = 0$.

As established in [14], in the case $\mu < 0$ for the equation (3) the initial value problem is correct special type with conditions

$$u(0) = u_0, \quad \lim_{t \to 0} t^\mu \left(u(t) - Y_\mu(t)u_0\right)' = u_1, \tag{8}$$

the only solution to which is the function

$$u(t) = Y_\mu(t)u_0 + \frac{t^{1-\mu}}{1-\mu}Y_{2-\mu}(t)u_1. \tag{9}$$

If $\mu \neq 1-2N,\ N \in \mathbb{N}$, then instead of the initial problem of a special form with conditions (8) we can also consider the classical formulation of initial conditions. In this paper, we will show that to identify a unique solution EPD equation (3) for $\mu < 0$, it is necessary in (4) instead of the initial condition for the first derivative $u'(0) = 0$ set a condition on the derivative at zero of a higher order. The order of this derivative will depend from $\mu < 0$. This setting of initial conditions excludes the appearance in (9) of solutions of the form $t^{1-\mu} Y_{2-\mu}(t) u_1$, but, naturally, requires increased smoothness of the initial element u_0.

Theorem 4. *Suppose that $A \in G_k$ for some $k \geq 0$ and let the parameter $\mu < 0$ belongs either to the interval $-2m+1 < \mu < -2m+2$ or to the interval $-2m \leq \mu < -2m+1$, where $m \in \mathbb{N}$. If $u_0 \in D\left(A^{n(\mu)+[m/2]+1}\right)$, then defined by the equalities (5)–(7) the function $u(t) = Y_\mu(t) u_0$ is the only solution to the EPD equation (3), satisfying the conditions*

$$u(0) = u_0, \quad u^{(2m+1)}(0) = 0. \tag{10}$$

Proof. As stated in Theorems 1 and 3, the function $Y_\mu(t) u_0$ is a solution equation (3) and satisfies the first condition in (10). Let's show that she also satisfies the second condition in (10).

To do this, we represent the function $Y_\mu(t) u_0$ defined by the equality (5) in the form

$$Y_\mu(t) u_0 = Y_q(t) u_0 + b_1 t Y_q'(t) u_0 + b_2 t^2 Y_q''(t) u_0 + b_3 t^3 Y_q^{(3)}(t) u_0 + \ldots + b_l t^l Y_q^{(l)}(t) u_0 \tag{11}$$

with some constants $b_j,\ j = 1, ..., l$.

For the BOF $Y_q(t)$ the relation is valid (see (5)–(7))

$$Y_q'(t) u_0 = \frac{t}{q+1} Y_{q+2}(t) A u_0,$$

therefore, each term in the sum from the formula (11) for $t \to 0$ has an even order in t, in particular, the first term is of order t^0, the second is t^2, the third is t^2, the fourth is t^4, etc. Consequently, if for $t = 0$ derivatives of odd order $2m+1$ of the function are defined $Y_\mu(t) u_0$, $\mu \neq 1-2N$, $N \in \mathbb{N}$, then they all vanish.

To prove the theorem, it remains to establish the uniqueness of the solution to the problem (3), (10), which we will lead from the contrary. Let $u_1(t)$ and $u_2(t)$ be two solutions to this problem. Consider a function of two variables

$$w(t,y) = f\left(Y_k(y)\left(u_1(t) - u_2(t)\right)\right),$$

where $f \in E^*$ (E^* is the dual space), $t, y \geq 0$. She's obviously satisfies the following equation and conditions

$$\frac{\partial^2 w(t,y)}{\partial t^2} + \frac{\mu}{t}\frac{\partial w(t,y)}{\partial t} = \frac{\partial^2 w(t,y)}{\partial y^2} + \frac{k}{y}\frac{\partial w(t,y)}{\partial y}, \quad t, y > 0, \tag{12}$$

$$w(0,y) = 0, \quad w^{(2m+1)}(0,y) = 0. \tag{13}$$

We interpret the function $w(t,y)$ as a generalized function and to the problem (12), (13) We apply the Fourier–Bessel transform to the variable y. For the image $\hat{w}(t,\lambda)$ in the space of regular generalized functions we obtain the following problem

$$\frac{\partial^2 \hat{w}(t,\lambda)}{\partial t^2} + \frac{\mu}{t}\frac{\partial \hat{w}(t,\lambda)}{\partial t} = -\lambda^2\, \hat{w}(t,\lambda), \quad t > 0, \tag{14}$$

$$\hat{w}(0,\lambda) = 0, \quad \frac{\partial^{2m+1}\hat{w}(0,\lambda)}{\partial t^{2m+1}} = 0. \tag{15}$$

The general solution to the equation (14) has the form

$$\hat{w}(t,\lambda) = t^{-\nu}\left(d_1(\lambda)\, J_\nu(\lambda t) + d_2(\lambda)\, N_\nu(\lambda t)\right),$$

where $\nu = \mu/2 - 1/2$, $J_\nu(\cdot)$ is the Bessel function, $N_\nu(\cdot)$ is the Neumann function.

Taking into account the behavior of the Bessel and Neumann functions at zero, from the first condition in (15) we obtain $d_1(\lambda) = 0$. The order at zero with respect to t of the second term of the function $\hat{w}(t,\lambda)$ is equal to $-2\nu = 1-\mu$ and since $1-\mu-(2m+1) = -2m-\mu \le 0$, then from the second condition in (15) it follows $d_2(\lambda) = 0$. Therefore, $\hat{w}(t,\lambda) = w(t,y) = 0$ for any $y \ge 0$. Due to the arbitrariness of the functional $f \in E^*$ for $y = 0$, we obtain the equality $u_1(t) \equiv u_2(t)$, and the uniqueness of the solution to the problem (3), (10) installed. The theorem has been proven.

In what follows, we will repeatedly use the formula connecting the operator functions $Y_\mu(t)$ and $Y_{2+\mu}(t)$ for $\mu < 0$, $\mu \ne 1-2N$, $N \in \mathbb{N}$, which has the form

$$Y_\mu(t)u_0 = Y_{2+\mu}(t)u_0 + \frac{t}{1+\mu} Y'_{2+\mu}(t)u_0, \quad u_0 \in D\left(A^{l/2+2}\right). \tag{16}$$

For $-1 < \mu < 0$ the equality (16) follows from the representation (5). In general direct verification shows that the functions on the left and right sides of the equality (16) are solutions to the same Cauchy problem (3), (10). By virtue of Theorem 4, they must coincide.

Note again that the case $\mu = 1-2N$, $N \in \mathbb{N}$ in Theorem 4 has not yet been studied and will be considered further.

Example 1. For $-1 < \mu < 0$, $A \in G_k$, $0 \le k \le 2+\mu$ and $u_0 \in D\left(A^3\right)$. Let's consider the EPD equation (3). In this case, under the conditions of Theorem 1 we have $l = 1$, $q = 2+\mu \ge k$ and using the formula (5), we define the function

$$Y_\mu(t)u_0 = Y_{2+\mu}(t)u_0 + \frac{t}{1+\mu} Y'_{2+\mu}(t)u_0.$$

By Theorem 4, the function $u(t) = Y_\mu(t)u_0$ is the only solution to the equation (3), satisfying conditions

$$u(0) = u_0, \quad u'''(0) = 0, \tag{17}$$

which is easy to verify directly by calculating the derivatives of this function up to the third order:

$$u'(t) = Y'_\mu(t)u_0 = \frac{t}{1+\mu} AY_{2+\mu}(t)u_0, \quad u''(t) = Y''_\mu(t)u_0 = \frac{1}{1+\mu}\left(Y_{2+\mu}(t)Au_0 + tY'_{2+\mu}(t)Au_0\right),$$

$$u'''(t) = Y'''_\mu(t)u_0 = \frac{1}{1+\mu}\left(-\mu Y'_{2+\mu}(t)Au_0 + tY_{2+\mu}(t)A^2u_0\right).$$

In particular, if $A \in \mathbb{C}$ is an operator of multiplication by a number, then

$$Y_0(t) = C(t) = \operatorname{ch}\left(t\sqrt{A}\right), \quad Y_{2+\mu}(t) = \Gamma(3/2+\mu/2)\left(\frac{2}{t\sqrt{A}}\right)^{1/2+\mu/2} I_{1/2+\mu/2}\left(t\sqrt{A}\right),$$

$$u(t) = Y_\mu(t)u_0 = \Gamma(3/2+\mu/2)\left(\frac{2}{t\sqrt{A}}\right)^{1/2+\mu/2}\left(I_{1/2+\mu/2}\left(t\sqrt{A}\right) + \frac{t\sqrt{A}}{1+\mu} I_{3/2+\ mu/2}\left(t\sqrt{A}\right)\right)u_0,$$

where $I_\nu(\cdot)$ is the modified Bessel function.

Example 2. Let, in contrast to example 1, $-2 \le \mu < -1$ and, as before, $A \in G_k$, $0 \le k \le 2+\mu$, $u_0 \in D\left(A^2\right)$. Then, under the conditions of Theorem 1, we have $l = 1$, $q = 2+\mu \ge k$ and defined by the equality (16) function $u(t) = Y_\mu(t)u_0$ is still the only solution to the problem (3), (17).

It should be noted that as μ decreases, the range of possible values for the parameter k narrows. For example, if $\mu = -2$, then only the case $k = 0$, $A \in G_0$ is suitable, and then

$$u(t) = Y_{-2}(t)u_0 = C(t)u_0 - tC'(t)u_0.$$

In particular, if

$$E = L_p, \quad p > 1, \quad A = \frac{d^2}{dy^2},\ Y_0(t)u_0(y) = C(t)u_0(y) = \frac{1}{2}\left(u_0(y+t) + u_0(y-t)\right),$$

then, using the formula (16), we define the function

$$u(t,y) = \frac{1}{2}\left(u_0(y+t) + u_0(y-t)\right) - \frac{t}{2}\left(u_0'(y+t) - u_0'(y-t)\right),$$

which is the only solution to the problem

$$\frac{\partial^2 u(t,y)}{\partial t^2} - \frac{2}{t}\frac{\partial u(t,y)}{\partial t} = \frac{\partial^2 u(t,y)}{\partial y^2}, \quad u(0,y) = u_0(y), \quad \frac{\partial^3 u(0,y)}{\partial t^3} = 0.$$

Example 3. Let $-3 < \mu < -2$, $A \in G_k$, $0 \le k \le 4+\mu$ and $u_0 \in D\left(A^5\right)$. In this case, under the conditions of Theorem 1 we have $l = 2$, $q = 4+\mu \ge k$ and, using the formula (5), we determine function

$$Y_\mu(t)u_0 = Y_{4+\mu}(t)u_0 + \frac{t^2}{(1+\mu)(3+\mu)} Y_{4+\mu}(t)Au_0 + \frac{t}{3+\mu} Y'_{4+\mu}(t)u_0, \tag{18}$$

where for $0 \le k < 4+\mu$ the function $Y_{4+\mu}(t)u_0$ is expressed in terms of $Y_k(t)u_0$ using the shift formula solutions by parameter (2).

By Theorem 4, the function $u(t) = Y_\mu(t)u_0$ is the only solution to the equation (3), satisfying conditions

$$u(0) = u_0, \quad u^{(5)}(0) = 0. \tag{19}$$

Example 4. Let, in contrast to example 3, $-4 \le \mu < -3$ and, as before, $A \in G_k$, $0 \le k \le 4+\mu$, $u_0 \in D\left(A^5\right)$. Then, under the conditions of Theorem 1, we have $l = 2$, $q = 4+\mu \ge k$ and defined by the equality (18) the function $u(t) = Y_\mu(t)u_0$ is still the only solution to the problem (3), (19).

Example 5. Let $\mu = -6$, $A \in \mathbb{C}$, $u_0 \in \mathbb{C}$. In this case, under the conditions of Theorem 1 we have $l = 3$, $q = 0$ and, using the formula (5), we determine function

$$u(t) = Y_{-6}(t)u_0 = \left(\cosh\left(t\sqrt{A}\right) + \frac{2}{5}At^2\cosh\left(t\sqrt{A}\right) - At\sinh\left(t\sqrt{A}\right) + \frac{1}{15}A^{3/2}t^3\sinh\left(t\sqrt{A}\right)\right)u_0,$$

which is the only solution to the problem

$$u''(t) - \frac{6}{t}u'(t) = Au(t), \quad u(0) = u_0, \quad u^{(7)}(0) = 0.$$

Returning to the exceptional case $\mu = 1-2N$, $N \in \mathbb{N}$, we point out that the problem (3), (10) is incorrect, since the second derivative of the function $Y_\mu(t)u_0$ at $t = 0$ is not defined, as can be seen from the following example.

Example 6. Let $\mu = -1$, $A \in G_0$ and $u_0 \in D\left(A^3\right)$. In this case, under the conditions of Theorem 3 we have $N = 1$ and, using the formula (7), we define the function

$$Y_{-1}(t)u_0 = tZ'_1(t)u_0,$$

where the function $Z_1(t)u_0$ is given by the equality (6). Calculating the derivatives, we get

$$Y'_{-1}(t)u_0 = tZ_1(t)Au_0, \quad Y''_{-1}(t)u_0 = Z_1(t)Au_0 + tZ'_1(t)Au_0.$$

and, as follows from (6), the term $Z_1(t)Au_0$ has a logarithmic singularity at $t = 0$.

In addition to the previously indicated initial problem (3), (8) of a special form, for $\mu < 0$ in [15], a criterion for the uniform correctness of the following weighted initial problem was obtained

$$u''(t) + \frac{\mu}{t}u'(t) = Au(t), \quad u(0) = 0, \quad \lim_{t\to 0} t^\mu u'(t) = u_1, \tag{20}$$

which we will present below.

Theorem 5. *Let $\mu < 0$ and $u_1 \in D(A)$. In order for the problem (20) to be uniformly correct, it is necessary and sufficient that $A \in G_{2-\mu}$, and the only solution to this problem has the form*

$$u(t) = \frac{t^{1-\mu}}{1-\mu} Y_{2-\mu}(t)u_1,$$

where $Y_{2-\mu}(t)$—OFB, the generator of which is the operator A.

Note 1. The second initial condition in the problem (20) can have the form

$$\lim_{t\to 0} D^{-\mu}_{0+}u'(t) = u_1,$$

where $\mu < 0$, $D^{-\mu}_{0+}$ is the fractional derivative of Riemann–Liouville, and the correctness of the problem will not be violated.

2. CAUCHY PROBLEM FOR NEGATIVE VALUES OF THE EQUATION PARAMETER FOR A FACTORED EQUATION EULER–POISSON–DARBOUX WITH TWO FACTORS

Let us apply Theorem 4 to the study of the Cauchy problem for the factorized Euler–Poisson–Darboux equation of the form

$$\left(\frac{d^2}{dt^2}+\frac{\mu}{t}\frac{d}{dt}-A\right)\left(u''(t)+\frac{\mu}{t}u'(t)-Au(t)\right)=0, \quad t>0 \tag{21}$$

for $\mu<0$ and $A\in G_k,\ k\geq 0$.

Interest in factorized equations arose after publications [16, 17] . If $\mu<0$ and $A=B^2$, where B is a group generator, then a number of results on the representation of solutions to the equation (21) was obtained in [18, 19], and issues of non-uniqueness were studied in [20]. We'll consider equation (21) with the operator $A\in G_k$ of a more general form.

If $\mu\geq 0$, $A\in G_\mu$, then questions of unique solvability of factored singular equations were previously considered in [21].

Let's denote

$$u''(t)+\frac{\mu}{t}u'(t)-Au(t)=v(t), \tag{22}$$

and let the parameter μ belong either to the interval $-2m+1<\mu<-2m+2$, or the interval $-2m\leq \mu<-2m+1$, where $m\in\mathbb{N}$.

Then, the equation (21) will be written in the form

$$v''(t)+\frac{\mu}{t}v'(t)=Av(t). \tag{23}$$

According to Theorem 4, for the equation (23) the problem with the conditions is correct

$$v(0)=v_0, \quad v^{(2m+1)}(0)=0, \tag{24}$$

and at the same time $v(t)=Y_\mu(t)v_0$, and for the inhomogeneous equation (22) the problem with the conditions is correct

$$u(0)=u_0, \quad u^{(2m+1)}(0)=0. \tag{25}$$

Direct inspection shows that the function

$$\omega(t)=\frac{t^2}{2(\mu+1)}Y_{\mu+2}(t)v_0$$

is a particular solution to the equation (22). Indeed, we find the derivatives

$$\omega'(t)=\frac{t}{\mu+1}Y_{\mu+2}(t)v_0+\frac{t^2}{2(\mu+1)}Y'_{mu+2}(t)v_0,$$

$$\omega''(t)=\frac{1}{\mu+1}\left(Y_{\mu+2}(t)+2tY'_{\mu+2}(t)+\frac{t^2}{2}Y''_{\mu+2}(t)\right)v_0$$
$$=\frac{1}{\mu+1}\left(Y_{\mu+2}(t)v_0+\frac{t^2}{2}Y_{\mu+2}(t)Av_0+\left(1-\frac{\mu}{2}\right)tY'_{\mu+2}(t)v_0\right),$$

and, by virtue of equality (16), we will have

$$\omega''(t)+\frac{\mu}{t}\omega'(t)-A\omega(t)=Y_{\mu+2}(t)v_0+\frac{t}{\mu+1}Y'_{\mu+2}(t)v_0=Y_\mu(t)v_0=v(t),$$

which proves the assertion.

Consequently, the solution to the Cauchy problem for an inhomogeneous equation (22) has the form

$$u(t)=Y_\mu(t)u_0+\omega(t)=Y_\mu(t)u_0+\frac{t^2}{2(\mu+1)}Y_{\mu+2}(t)v_0. \tag{26}$$

Taking into account the equality proved in [17]

$$\frac{d^j}{dt^j}\left(u''(t)+\frac{\mu}{t}u'(t)\right)\bigg|_{t=0}=\frac{\mu+1}{j+1}u^{(j+2)}(0), \tag{27}$$

from the conditions (24) we get

$$v(0)=(\mu+1)u''(0)-Au(0)=v_0, \quad v^{(2m+1)}(0)=\frac{\mu+1}{2m+2}u^{(2m+3)}(0)-Au^{(2m+1)}(0)=0. \tag{28}$$

Choosing in (28) $v_0=(\mu+1)u_2-Au_0$, from (24), (25) we obtain the initial conditions in terms of the function $u(t)$, which have the form

$$u(0)=u_0, \quad u''(0)=u_2, \quad u^{(2m+1)}(0)=0, \quad u^{(2m+3)}(0)=0. \tag{29}$$

Thus, by virtue of the established representation (26)

Theorem 6. *Let the conditions of Theorem 4 be satisfied. If*

$$u_0\in D\left(A^{n(\mu)+[m/2]+1}\right), \quad u_2\in D\left(A^{n(\mu)+[m/2]}\right),$$

then the function

$$u(t)=Y_\mu(t)u_0+\frac{t^2}{2}Y_{\mu+2}(t)\left(u_2-\frac{1}{\mu+1}Au_0\right) \tag{30}$$

is the only solution to the factorized EPD equation (21), satisfying the conditions (29).

Remark 1. It is easy to verify that for $\mu\geq 0$ and $A\in G_\mu$, defined equality (30) the function $u(t)$ will be the only solution to the equation (21), satisfying the conditions

$$u(0)=u_0\in D(A^3), \quad u'(0)=0, \quad u''(0)=u_2\in D(A^2), \quad u'''(0)=0.$$

Example 7. Let in the problem (21) and (29)

$$E=L_p,\ p>1, \quad \mu=-2, \quad A=\frac{d^2}{dy^2}, \quad Y_0(t)u_0(y)=C(t)u_0(y)=\frac{1}{2}\left(u_0(y+t)+u_0(y-t)\right),$$

the function $u_0(y)$ has continuous derivatives up to the sixth order, and the function $u_0(y)$ has continuous derivatives up to the fourth. Then (see Example 2),

$$Y_{-2}(t)u_0(y)=\frac{1}{2}\left(u_0(y+t)+u_0(y-t)\right)-\frac{t}{2}\left(u_0'(y+t)-u_0'(y-t)\right),$$

and according to the formula (30) the only solution to the problem (21), (29) for $\mu=-2$ has the form

$$\begin{aligned} u(t)&=Y_{-2}(t)u_0(y)+\frac{t^2}{2}Y_0(t)\left(u_2(y)+Au_0(y)\right)\\ &=\frac{1}{2}\left(u_0(y+t)+u_0(y-t)\right)-\frac{t}{2}\left(u_0'(y+t)-u_0'(y-t)\right)\\ &+\frac{t^2}{4}\left(u_2(y+t)+u_2(y-t)+u_0''(y+t)+u_0''(y-t)\right).\end{aligned}$$

Example 8. If $-2\leq\mu<0,\ \mu\neq -1$ and $A\in\mathbb{C}$ is the operator of multiplication by a number, then (see Examples 1 and 2)

$$Y_\mu(t)=\Gamma(3/2+\mu/2)\left(\frac{2}{t\sqrt{A}}\right)^{1/2+\mu/2}\left(I_{1/2+\mu/2}\left(t\sqrt{A}\right)+\frac{t\sqrt{A}}{1+\mu}I_{3/2+\ mu/2}\left(t\sqrt{A}\right)\right),$$

$$Y_{2+\mu}(t)=\Gamma(3/2+\mu/2)\left(\frac{2}{t\sqrt{A}}\right)^{1/2+\mu/2}I_{1/2+\mu/2}\left(t\sqrt{A}\right),$$

and according to the formula (30) the only solution to the problem (21) and (29) in this case has the form

$$u(t) = Y_\mu(t)u_0 + \frac{t^2}{2(\mu+1)}Y_{\mu+2}(t)\left((\mu+1)u_2 - Au_0\right).$$

In particular, for $\mu = -2$, using the formula (16), we obtain

$$u(t) = Y_{-2}(t)u_0 + \frac{t^2}{2}Y_0(t)(Au_0 + u_2) = Y_0(t)u_0 - tY_0'(t)u_0 + \frac{t^2}{2}Y_0(t)(Au_0 + u_2)$$
$$= \left(u_0 + \frac{t^2}{2}Au_0 + \frac{t^2}{2}u_2\right)\cosh\left(t\sqrt{A}\right) - t\sqrt{A}\sinh\left(t\sqrt{A}\right)u_0.$$

3. CAUCHY PROBLEM FOR NEGATIVE PARAMETER VALUES FOR A FACTORED EQUATION EULER–POISSON–DARBOUX WITH THREE FACTORS

Next, we apply Theorems 4 and 6 to the study of the Cauchy problem for the factorized Euler–Poisson–Darboux equation with three factors

$$\left(\frac{d^2}{dt^2} + \frac{\mu}{t}\frac{d}{dt} - A\right)\left(\frac{d^2}{dt^2} + \frac{\mu}{t}\frac{d}{dt} - A\right)\left(u''(t) + \frac{\mu}{t}u'(t) - Au(t)\right) = 0, \quad t > 0 \tag{31}$$

for $\mu < 0$ and $A \in G_k$, $k \geq 0$.

Let's denote

$$\left(u''(t) + \frac{\mu}{t}u'(t) - Au(t)\right) = w(t), \tag{32}$$

and let, as before, the parameter μ belong either to the interval $-2m+1 < \mu < -2m+2$, or the interval $-2m \leq \mu < -2m+1$, where $m \in \mathbb{N}$. Then, the equation (31) will be written in the form

$$\left(\frac{d^2}{dt^2} + \frac{\mu}{t}\frac{d}{dt} - A\right)\left(w''(t) + \frac{\mu}{t}w'(t) - Aw(t)\right) = 0. \tag{33}$$

According to Theorem 6, for a homogeneous equation (33) of fourth order the Cauchy problem with the conditions is correct

$$w(0) = w_0, \quad w''(0) = w_2, \quad w^{(2m+1)}(0) = w^{(2m+3)}(0) = 0, \tag{34}$$

and wherein

$$w(t) = Y_\mu(t)w_0 + \frac{t^2}{2(\mu+1)}Y_{\mu+2}(t)\left((\mu+1)w_2 - Aw_0\right)$$
$$= Y_{\mu+2}(t)w_0 + \frac{t}{\mu+1}Y_{\mu+2}'(t)w_0 + \frac{t^2}{2(\mu+1)}Y_{\mu+2}(t)\left((\mu+1)w_2 - Aw_0\right).$$

By virtue of Theorem 4, for a second-order inhomogeneous equation (32) the Cauchy problem with the conditions is correct

$$u(0) = u_0, \quad u^{(2m+1)}(0) = 0. \tag{35}$$

To solve the Cauchy problem (32) and (35) for an inhomogeneous equation, we select a particular solution $\tilde{\omega}(t)$ of the equation (32) in the form

$$\tilde{\omega}(t) = \varphi(t)Y_{\mu+2}(t)v_1 + \psi(t)Y_{\mu+2}'(t)v_2, \tag{36}$$

where the scalar functions $\varphi(t)$, $\psi(t)$ and $v_1, v_2 \in E$ are subject to selection.

Note that in the proof of Theorem 6 the function $\omega(t)$, was defined similarly and $\psi(t) \equiv 0$ was chosen.

Let us substitute the function $\tilde{\omega}(t)$ into the left side of the equation (32) and into the equality obtained after this we equate the elements, containing $Y_{\mu+2}(t)$ and $Y_{\mu+2}'(t)$ on the left and right sides, respectively. As a result, we obtain the following two equations

$$\left(\varphi''(t) + \frac{\mu}{t}\varphi'(t)\right)v_1 + \left(2\psi'(t) - \frac{2}{t}\,\psi(t)\right)Av_2 = w_0 + \frac{t^2}{2(\mu+1)}\left((\mu+1)w_2 - Aw_0\right), \tag{37}$$

$$\left(2\varphi'(t) - \frac{2}{t}\,\varphi(t)\right) v_1 + \left(\psi''(t) - \frac{\mu+4}{t}\psi'(t) + \frac{3\mu+6}{t^2}\psi(t)\right) v_2 = \frac{t}{\mu+1} w_0. \tag{38}$$

Since the right-hand sides of the equations (37) and (38) are polynomials in t, then the functions $\varphi(t)$ and $\psi(t)$ should also be sought in the form of polynomials of the form

$$\varphi(t) = \alpha_1 t^4 + \beta_1 t^2 + \gamma_1, \quad \psi(t) = \alpha_2 t^3 + \beta_2 t^2 + \gamma_2 t, \tag{39}$$

substituting which into the equations (37) and (38), we obtain

$$\alpha_1 = \gamma_1 = \beta_2 = \gamma_2 = 0, \tag{40}$$

$$\alpha_2 A v_2 = \frac{(\mu+1)w_2 - Aw_0}{8(\mu+1)}, \tag{41}$$

$$\beta_1 v_1 = \frac{w_0}{2(\mu+1)}. \tag{42}$$

If we require the existence of the operator A^{-1}, then, taking into account the equalities (36) and (39)–(42), we define a particular solution $\tilde{\omega}(t)$ inhomogeneous equation (32) in the form

$$\tilde{\omega}(t) = \frac{t^2}{2(\mu+1)} Y_{\mu+2}(t) w_0 + \frac{t^3}{8(\mu+1)} Y'_{\mu+2}(t) \left((\mu+1)A^{-1}w_2 - w_0\right).$$

Therefore, by Theorem 4, the solution to the Cauchy problem for an inhomogeneous equation (32) has the form

$$u(t) = Y_\mu(t)u_0 + \tilde{\omega}(t)$$
$$= Y_\mu(t)u_0 + \frac{t^2}{2(\mu+1)} Y_{\mu+2}(t) w_0 + \frac{t^3}{8(\mu+1)} Y'_{\mu+2}(t) \left((\mu+1)A^{-1}w_2 - w_0\right). \tag{43}$$

Given equality (27) and choosing in (43)

$$w_0 = (\mu+1)u_2 - Au_0, \quad w_2 = \frac{\mu+1}{3} u_4 - Au_2,$$

we write the initial conditions (34) in terms of the solution $u(t)$, which together with (35) leads to the conditions

$$u(0) = u_0, \quad u''(0) = u_2, \quad u^{(4)}(0) = u_4, \quad u^{(2m+1)}(0) = u^{(2m+3)}(0) = u^{(2m+5)}(0) = 0. \tag{44}$$

Thus, it is fair

Theorem 7. *Let the conditions of Theorem 4 be satisfied. If A^{-1} exists and*

$$u_0 \in D\left(A^{n(\mu)+[m/2]+2}\right), \quad u_2 \in D\left(A^{n(\mu)+[m/2]+1}\right), \quad u_4 \in D\left(A^{n(\mu)+[m/2]}\right),$$

then the function

$$u(t) = Y_\mu(t)u_0 + \frac{t^2}{2} Y_{\mu+2}(t) \left(u_2 - \frac{1}{\mu+1} Au_0\right) + \frac{t^3}{8} Y'_{\mu+2}(t) \left(\frac{1}{\mu+1} Au_0 - 2u_2 + \frac{\mu+1}{3} A^{-1} u_4\right)$$

is the only solution to the factorized EPD equation (31), satisfying the conditions (44).

Remark 2. If in the conditions (44) $u_4 = 0$, then in Theorem 7 the existence no inverse operator A^{-1} is required.

Remark 3. It is easy to verify that for $\mu \geq 0$ and $A \in G_\mu$, defined equality (43) the function $u(t)$ will be the only solution to the equation (31), satisfying the conditions

$$u(0) = u_0 \in D(A^4),\ u'(0) = 0,\ u''(0) = u_2 \in D(A^3),\ u'''(0) = 0,\ u^{(4)}(0) = u_4 \in D(A^2),\ u^{(5)}(0) = 0.$$

Example 9. If $\mu = -2$ and $A \in \mathbb{C}$ is an operator of multiplication by the number $A \neq 0$, then, using the results of Example 8, by Theorem 7 we will find a solution to the problem (31), (44), which has the form

$$u(t) = \left(u_0 + \frac{t^2}{2}Au_0 + \frac{t^2}{2}u_2\right)\cosh\left(t\sqrt{A}\right)$$
$$-\left(t\sqrt{A}u_0 + \frac{t^3}{8}\left(A^{3/2}u_0 + 2\sqrt{A}u_2 + \frac{1}{3\sqrt{A}}u_4\right)\right)\sinh\left(t\sqrt{A}\right).$$

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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