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FAMILY OF BESSEL OPERATOR FUNCTIONS

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Abstract. In this paper, we introduce the notion of Bessel operator functions and their generator and examine their properties. A criterion of the uniform well-posedness of the Cauchy problem for the Euler–Poisson–Darboux equation is obtained, and relationships of this family with a series of other resolving operators are found.

***Keywords and phrases*:** Bessel operator function, generator, Euler–Poisson–Darboux equation, Cauchy problem, uniform well-posedness criterion.

***AMS Subject Classification*:** 34G10

1. Introduction. The study of differential equations with unbounded operator coefficients that act in a Banach space E plays an important role in the development of the theory of resolving operators of the corresponding initial-value problems. As a result of the study of first-order evolution equations

$$u'(t) = Au(t),$$

semigroups of linear operators $T(t)$ appeared, whereas the second-order equation (the abstract wave equation)

$$u''(t) = Au(t)$$

is related to cosine operator functions $C(t)$. Weakening of the conditions on the resolving operators of the Cauchy problem for abstract differential equations of the first and second orders led to the concept of an integrated semigroup and an integrated cosine operator function. For the terminology and bibliography, see [16, 17] and the review papers [20, 23].

Later, in the works [6, 9, 11, 12, 15], the Bessel, Struve, and Legendre operator functions were introduced as resolving operators for the Euler–Poisson–Darboux (EPD), Bessel–Struve, and Legendre equations, respectively.

In this paper, we present another approach to constructing a family of Bessel operator functions. Just as in the theory of semigroups and cosine operator functions, the family of Bessel operator functions is initially investigated independently of the differential equation with which it will ultimately be associated. An important role in constructing the family is played by the generalized shift operator T_s^t, which depends on the parameter $k > 0$ and is defined by the equality (see [19])

$$T_s^t Y(s) = \frac{\Gamma(k/2 + 1/2)}{\Gamma(1/2)\Gamma(k/2)} \int_0^{\pi} Y\left(\sqrt{s^2 + t^2 - 2st\cos\varphi}\right) \sin^{k-1}\varphi \, d\varphi,$$

where $\Gamma(\cdot)$ is the Euler gamma function and $s, t \geq 0$. The generalized shift operator depends on the parameter $k > 0$, but, following [19], we omit this dependence in the notation.

We also indicate that in this paper we use the concept of the integral of a continuous function, but if necessary, we can use the Bochner integral of a function with a value in a Banach space.

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2. Family of Bessel operator functions. Let $k > 0$ and $Y_k(\cdot) : [0, \infty) \to B(E)$ be an operator-valued function acting in the space of linear bounded operators $B(E)$.

Definition 1. A strongly continuous family of linear bounded operators $Y_k(t) : [0, \infty) \to B(E)$ depending on a parameter $k > 0$ is called a *Bessel operator function* (BOF) if the following conditions are fulfilled:

(a) $Y_k(0) = I$;
(b) $Y_k(t)Y_k(s) = T_s^t Y_k(s)$, $s, t \geq 0$;
(c) there exist $M \geq 1$ and $\omega \geq 0$ such that $\|Y_k(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

The family of BOFs is closely related to the Bessel differential operator

$$B_k = \frac{d^2}{dt^2} + \frac{k}{t}\frac{d}{dt};$$

if necessary, we will indicate the variable of its action: $B_k = B_{k,t}$. Below, we also use the notation $Y_k'(t)x = (Y_k(t)x)'$.

Definition 2. The generator of a BOF $Y_k(t)$ is the operator A acting by the rule

$$Ax = \lim_{t \to 0+} B_k Y_k(t)x,$$

whose domain $D(A)$ consists of all $x \in E$ for which the function $Y_k(t)x$ is twice differentiable at the point $t = 0$.

Theorem 1. *If an operator A is the generator of a BOF $Y_k(t)$, then the domain $D(A)$ is dense in E. Moreover, the set of elements on which all powers of the operator A are defined is dense in E.*

Proof. Let V be the set of numerical functions $v(t)$ that are smooth and finite on $(0, \infty)$. Consider the set $E(V) \subset E$ of elements of the form

$$y = \int_0^\infty v(s)Y_k(s)x ds, \quad x \in E,$$

and prove that $y \in D(A)$. Taking into account the condition (b) of Definition 1 and the properties of the operator T_s^t and integrating by parts we obtain

$$B_{k,t}Y_k(t)y = \int_0^\infty v(s)B_{k,t}T_s^t Y_k(s)x\, ds = \int_0^\infty v(s)B_{k,s}T_s^t Y_k(s)x\, ds$$

$$= \int_0^\infty \left(v''(s) - k\left(\frac{v(s)}{s}\right)'\right) T_s^t Y_k(s)x\, ds.$$

Passing to the limit as $t \to 0+$, we see that $y \in D(A)$ and

$$Ay = \int_0^\infty \left(v''(s) - k\left(\frac{v(s)}{s}\right)'\right) Y_k(s)x ds.$$

Now we prove that $E(V)$ is dense in E. Indeed, in the opposite case, there exists a linear bounded functional $f \in E^*$, $f \neq 0$, such that $f(E(V)) = 0$. Then

$$\int_0^\infty v(s)f(Y_k(s)x)ds = 0, \quad x \in E,$$

and, therefore, due to the arbitrariness of $v(s) \in V$, we conclude that $f(Y_k(s)x) \equiv 0$ for $s > 0$. Tending $s \to 0$, we see that $f(x) = 0$ for any $x \in E$ and hence $f = 0$, a contradiction. Therefore, $E(V)$ is dense in E. Since $E(V) \subset D(A)$, we also conclude that $D(A)$ is dense in E. The denseness of the set of elements on which all powers of the operator A are defined is proved similarly. □

Theorem 2. *Let $Y_k(t)$ be a Bessel operator function and A be its generator. Then for any $t, s \geq 0$ and $x \in D(A)$, the following equalities hold*:

$$Y_k(t)Y_k(s) = Y_k(s)Y_k(t), \tag{1}$$

$$AY_k(t)x = Y_k(t)Ax. \tag{2}$$

Proof. Equality (1) follows from the definition of the family of BOFs and the properties of the generalized shift operator T_s^t. We prove that

$$B_{k,s}Y_k(s)Y_k(t)x = Y_k(t)B_{k,s}Y_k(s)x, \quad x \in D(A). \tag{3}$$

Indeed, due to Theorem 1 and the characteristic property of the generalized shift operator (see [16])

$$\lim_{h\to 0+} \frac{T_s^h Y_k(s)x - Y_k(s)x}{h^2} = \frac{1}{2(k+1)} B_{k,s}Y_k(s)x,$$

we have

$$B_{k,s}Y_k(s)Y_k(t)x = 2(k+1) \lim_{h\to 0+} \frac{T_s^h Y_k(s)Y_k(t)x - Y_k(s)Y_k(t)x}{h^2}$$

$$= 2(k+1) \lim_{h\to 0+} \frac{Y_k(t)\left(Y_k(h)Y_k(s)x - Y_k(s)x\right)}{h^2} = 2(k+1)Y_k(t) \lim_{h\to 0+} \frac{T_s^h Y_k(s)x - Y_k(s)x}{h^2}$$

$$= Y_k(t)B_{k,s}Y_k(s)x;$$

thus, Eq. (3) is proved. Passing in (3) to the limit as $s \to 0+$, we obtain Eq. (2). The theorem is proved. □

Theorem 3. *Let $Y_k(t)$ be a Bessel operator function and A be its generator. If $x \in D(A)$ and $t > 0$, then $Y_k(t)x \in D(A)$ and*

$$AY_k(t)x = B_{k,t}Y_k(t)x. \tag{4}$$

Proof. For $s > 0$ the following equalities hold:

$$B_{k,s}Y_k(s)Y_k(t)x = B_{k,s}T_t^s Y_k(t)x = B_{k,t}T_t^s Y_k(t)x = B_{k,t}Y_k(s)Y_k(t)x = Y_k(s)B_{k,t}Y_k(t)x.$$

Passing to the limit as $s \to 0+$, we obtain the required assertions. □

3. Cauchy problem for the Euler–Poisson–Darboux equation.

Now we discuss the Cauchy problem related to the Bessel operator function $Y_k(t)$.

Definition 3. A solution of a second-order linear differential equation with the operator coefficient A is a function $u(t)$, which is twice continuously differentiable for $t \geq 0$, takes values from $D(A)$ for $t > 0$, i.e., $u(t) \in C^2(R_+, E) \cap C(R_+, D(A))$, and satisfies this equation.

Theorem 4. *Let $Y_k(t)$ be a Bessel operator function, A be its generator, and $u_0 \in D(A)$. Then the function $Y_k(t)u_0$ is a solution of the Euler–Poisson–Darboux equation*

$$B_k u(t) \equiv u''(t) + \frac{k}{t}u'(t) = Au(t), \quad t > 0, \tag{5}$$

and satisfies the initial conditions

$$u(0) = u_0, \quad u'(0) = 0. \tag{6}$$

Proof. If $u_0 \in D(A)$, then, due to Eq. (4) proved in Theorem 3, the function $Y_k(t)u_0$ is a solution of Eq. (5) and, obviously, satisfies the first initial condition in (6). To verify the second condition in (6), we rewrite Eq. (5) in the form

$$(t^k u'(t))' = t^k Au(t). \tag{7}$$

Integrating (7), we obtain for the derivative of the function $Y_k(t)u_0$ the representation

$$Y_k'(t)u_0 = t^{-k}\int\limits_0^t s^k Y_k(s)Au_0\,ds,$$

which implies the required assertion. ☐

Theorem 5. *If $Y_k(t)$ is a Bessel operator function, then its generator A is closed.*

Proof. Let $x_n \in D(A)$, $x_n \to x_0$, and $Ax_n \to y_0$. Then, due to Theorems 2 and 4, the following equalities are fulfilled:

$$\lim_{t\to 0+} B_{k,t}Y_k(t)x_0 = \lim_{t\to 0+}\lim_{n\to\infty} B_{k,t}Y_k(t)x_n = \lim_{t\to 0+}\lim_{n\to\infty} AY_k(t)x_n = \lim_{t\to 0+} Y_k(t)y_0 = y_0.$$

Therefore, $x_0 \in D(A)$ and $Ax_0 = y_0$. The theorem is proved. ☐

Definition 4. The Cauchy problem (5), (6) is said to be uniformly well posed if for any $u_0 \in D(A)$, there exist an operator function $\widetilde{Y}_k(t)$ (resolving operator), which is defined on E and commutes with A, and a number $M \geq 1$, $\omega \geq 0$, such that for any $u_0 \in D(A)$, the function $u(t) = \widetilde{Y}_k(t)u_0$ is a unique solution of this problem satisfying the estimates

$$\left\|\widetilde{Y}_k(t)\right\| \leq M\exp(\omega t), \tag{8}$$

$$\left\|\widetilde{Y}_k'(t)u_0\right\| \leq Mt\exp(\omega t)\left\|Au_0\right\|. \tag{9}$$

We denote by G_k the set of operators A for which the problem (5), (6) is uniformly well posed; note that G_0 is the set of generators of the cosine operator function and $C(t) = Y_0(t)$.

Note that the case of the abstract wave equation ($k = 0$ in (5)) was considered in detail in [2, 3, 22]. It was proved that the problem (5), (6) for $k = 0$ is uniformly well posed if and only if the operator A is the generator of the cosine operator function $C(t)$. Also, necessary and sufficient conditions for an operator A to be a generator of the cosine operator function were stated in terms of an estimate of the norm of the resolvent $R(\lambda) = (\lambda I - A)^{-1}$ of the operator A and its derivatives.

Assume that the Cauchy problem (5), (6) is uniformly weel posed, i.e., $A \in G_k$. Introduce the resolving operator $\widetilde{Y}_k(t)$, which assigns the value of the solution $u(t)$ of this problem at a moment $t > 0$ to an element $u_0 \in D(A)$. Due to Definition 4, this operator is linear and continuous; it can be prolonged by continuity to a linear bounded operator defined on the whole Banach space E. We prove below that the resolving operator $\widetilde{Y}_k(t)$ is a Bessel operator function, i.e., satisfies the conditions (a)–(c) of Definition 1 and the operator A is its generator.

Detailed proofs of Theorems 6–12 stated below can be found in [15].

Theorem 6. *If the problem (5), (6) is uniformly well posed and* $\operatorname{Re}\lambda > \omega$, *then λ^2 belongs to the regular set $\rho(A)$ and for any $x \in E$ the following representation holds*:

$$\lambda^{(1-k)/2}R(\lambda^2)x = \frac{2^{(1-k)/2}}{\Gamma(k/2+1/2)}\int\limits_0^\infty K_\nu(\lambda t)t^{(k+1)/2}\widetilde{Y}_k(t)x\,dt, \tag{10}$$

where $K_\nu(\cdot)$ is the Macdonald function (modified Bessel function) with $\nu = (k-1)/2$.

Thus, the spectrum $\sigma(A)$ of the operator $A \in G_k$ always lies to the left of a certain parabola.

Theorem 7. *Let the problem* (5), (6) *be uniformly well pose and let* $\widetilde{Y}_k(t)$ *be the resolving operator for this problem. Then the operator* A *is the generator of the* C_0*-semigroup* $T(t)$ *and the following representation holds*:

$$T(t)x = \frac{1}{2^k\ \Gamma(k/2+1/2)\ t^{k/2+1/2}} \int_0^\infty s^k \exp\left(-\frac{s^2}{4t}\right) \widetilde{Y}_k(s)x\,ds, \quad x \in E. \tag{11}$$

The representation (11) implies that the semigroup $T(t)$ can be prolonged to an operator function, which is analytic in some sector. Therefore, for the search for a criterion of the uniform well-posedness of the problem (5), (6), we can restrict ourselves by the class G of operators that are the generators of analytic C_0-semigroups $T(t)$. Criteria of the fact $A \in G$ can be found in [23].

It was proved in [4] that if $A \in G$, then for $\operatorname{Re}\lambda > \omega$, there exist the fractional power of the resolvent $R(\lambda)$:

$$R^\alpha(\lambda)x = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \exp(-\lambda t) T(t)x\,dt, \quad x \in E, \quad \alpha > 0.$$

In addition to Theorem 6, we also state another necessary condition of the uniform well-posedness of the problem (5), (6).

Theorem 8. *If the problem* (5), (6) *is uniformly well posed and* $\operatorname{Re}\lambda > \omega$, *then* λ^2 *belongs to the resolvent set* $\rho(A)$ *of the operator* A, *the fractional powers of the resolvent can be represented in the form*

$$R^{1+k/2}(\lambda^2) = \frac{1}{\Gamma(k+1)\,\lambda} \int_0^\infty t^k \exp(-\lambda t) \widetilde{Y}_k(t)\,dt, \tag{12}$$

and the following estimates hold:

$$\left\| \frac{d^n}{d\lambda^n}\left(\lambda R^{1+k/2}(\lambda^2)\right) \right\| \le \frac{M\,\Gamma(k+n+1)}{(\operatorname{Re}\lambda - \omega)^{k+n+1}}, \quad n = 0, 1, 2, \ldots. \tag{13}$$

In fact, the estimates (13) are also sufficient conditions of the uniform well-posedness of the problem (5), (6).

Theorem 9. *Let* $A \in G$, *let the estimates* (13) *be fulfilled, and let* $F_k(\lambda) = \Gamma(k+1)\lambda R^{1+k/2}(\lambda^2)$. *Then the problem* (5), (6) *is uniformly well-posed and the resolving operator* $\widetilde{Y}_k(t)$ *for this problem is defined by the equality*

$$\widetilde{Y}_k(t) = \lim_{n\to\infty} e^{-nt} \left(I + \sum_{m=0}^\infty (-1)^m \frac{n^{k+2m+2}}{m!\,\Gamma(k+m+2)}\, t^{m+1} F_k^{(m)}(n) \right).$$

In particular, on elements of the domain of the operator A *it has the form*

$$\widetilde{Y}_k(t)u_0 = \frac{\Gamma(k+1)}{2\pi i\, t^k} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} \lambda R^{1+k/2}(\lambda^2) u_0\,d\lambda, \quad u_0 \in D(A), \quad \sigma > \omega.$$

Theorems 8 and 9 can be combined in the following criterion.

Theorem 10 (criterion of uniform well-posedness). *Let an operator* A *be the generator of an analytic* C_0*-semigroup. The problem* (5), (6) *is uniformly well posed if and only if for some constants* $M \ge 1$ *and* $\omega \ge 0$, *the number* λ^2 *such that* $\operatorname{Re}\lambda > \omega$ *belongs to the resolvent set of the operator* A *and the fractional powers of the resolvent of the operator* A *satisfies the estimates* (13).

Theorem 11. *Let the Cauchy problem* (5), (6) *be uniformly well posed and let* $\widetilde{Y}_k(t)$ *be the resolving operator of this problem. Then* $\widetilde{Y}_k(t)$ *satisfies the conditions* (a)–(c) *of Definition* 1 *and, therefore,* $\widetilde{Y}_k(t)$ *is a BOF, i.e.,* $\widetilde{Y}_k(t) = Y_k(t)$.

Thus, if in the study of the Cauchy problem (5), (6) we restrict ourselves to closed operators A with dense domains, then the class of equations for which this problem is uniformly well posed coincides with the class of equations for which the operator A is the generator of the BOF $Y_k(t)$, and the BOF itself is the resolving operator $\widetilde{Y}_k(t) = Y_k(t)$ for the problem considered. Examples of BOFs and the corresponding generators are given in [15].

The family of BOFs $Y_k(t)$ depends on the parameter $k > 0$. The specified dependence is involved in the following formula for the shift by this parameter.

Theorem 12. *Let* $0 \le m < k$ *and* $A \in G_m$ *be the generator of a BOF* $Y_m(t)$. *Then* $A \in G_k \supset G_m$ *and, moreover, the corresponding BOF* $Y_k(t)$ *has the form*

$$Y_k(t) = \frac{2\Gamma(k/2+1/2)}{\Gamma(m/2+1/2)\Gamma(k/2-m/2)} \int\limits_0^1 s^m(1-s^2)^{(k-m)/2-1} Y_m(ts)\, ds. \tag{14}$$

In particular, if $A \in G_0 \subset G_k$ is the generator of the cosine operator function $C(t)$, then

$$Y_k(t) = \frac{2\Gamma(k/2+1/2)}{\Gamma(1/2)\Gamma(k/2)} \int\limits_0^1 (1-s^2)^{k/2-1} C(ts)\, ds, \quad k > 0; \tag{15}$$

moreover (see [7]),

$$\lim_{k\to 0} Y_k(t)x = C(t)x, \quad x \in E,$$

uniformly with respect to $t \in [0, t_0]$, $t_0 > 0$.

Earlier (see [14, 21]), Eq. (15) was used for constructing BOFs interpreted as resolving operators for differential equations considered in these works. Further generalization of the parameter shift formula can be found [13].

Equality (14) implies the following formulas for the derivatives of the BOFs:

$$Y_k'(t)u_0 = \frac{t}{k+1} Y_{k+2}(t)Au_0, \quad \lim_{t\to 0} Y_k''(t)u_0 = \frac{1}{k+1} Au_0, \quad u_0 \in D(A).$$

Concluding this section, we note that the case of perturbation of an operator $A \in G_k$ by a bounded operator was studied in [5], and if $A \in G_k$ and $B \in G_m$, $m < k$, then the question of whether the operator $A + B$ belongs to a certain correctness class was considered in [8].

4. Conclusion. In this section, we discuss the relationships of the BOF $Y_k(t)$ with a number of other resolving operators.

For $k < 0$ and $A \in G_{2-k}$, the operator function

$$Z_k(t) = \frac{t^{1-k}}{1-k} Y_{2-k}(t)$$

defines a solution of the Euler–Poisson–Darboux equation (5) satisfying the conditions

$$u(0) = 0, \quad \lim_{t\to 0+} t^k u'(t) = u_1. \tag{16}$$

For this reason, it is called the Bessel operator function with negative index. It was proved in [12] that the set G_{2-k} is the well-posedness class of the weighted problem (5), (16) and the function $Z_k(t)u_1$, $u_1 \in D(A)$, is the unique solution of it.

Note that for $k < 0$, the Cauchy problem for the Euler–Poisson–Darboux equation (5) with the conditions

$$u(0) = 0, \quad u'(0) = u_1$$

is not well posed due to loss of uniqueness (see [1]).

In [11], the Struve operator function $L_k(t)$, $k > 0$, was introduced. It related to the BOF $Y_k(t)$ by the formula

$$L_k(t)x = \frac{\sqrt{\pi}\Gamma(k/2+1)}{\Gamma(k/2+1/2)} \int\limits_0^t {}_2F_1\left(\frac{1}{2}, \frac{k}{2}; 1; 1 - \frac{t^2}{\tau^2}\right) Y_k(\tau)x\, d\tau, \quad A \in G_k, \quad x \in E,$$

where ${}_2F_1(a, b; c; z)$ in the Gauss hypergeometric function.

If $k > 0$, $A \in G_k$, and $u_0, u_1 \in D(A)$, then the function $u(t) = Y_k(t)u_0 + L_k(t)u_1$ is a unique solution of the Bessel–Struve equation

$$u''(t) + \frac{k}{t}\left(u'(t) - u'(0)\right) = Au(t), \quad t > 0, \tag{17}$$

satisfying the conditions

$$u(0) = u_0, \quad u'(0) = u_1. \tag{18}$$

Note that the presence of the load $u'(0)$ specified at $t = 0$ in Eq. (17) changes the formulation of the initial-value problem for $k > 0$. Instead of the well-posed weighted problem for the Euler–Poisson–Darboux equation (5) with the condition (16), one should consider the ordinary Cauchy problem with condition (18) for the Bessel–Struve equation (17).

Next, we indicate the relationship between the BOF $Y_k(t)$ and the resolving operator of the Cauchy problem for the Legendre equation

$$u''(t) + k \coth t\, u'(t) + \left(\frac{k}{2}\right)^2 u(t) = Au(t), \quad t > 0, \quad k > 0. \tag{19}$$

The differential operator in the left-hand side of Eq. (19) appears in the solution of the Laplace equation in the coordinates of an elongated ellipsoid of revolution. The well-posed statement of initial conditions for the abstract Legendre equation (19) and for the Euler–Poisson–Darboux equation (5) requires the following initial conditions at the point $t = 0$:

$$u(0) = u_0, \quad u'(0) = 0 \tag{20}$$

(see [9]).

It was proved in [9] that the set of operators A such that the problem (19), (20) is uniformly well posed coincides with the set G_k. Formulas that relate the resolving operator of this problem and the BOF $Y_k(t)$ can also be found in [9].

Finally, we recall the relationship between the BOF $Y_k(t)$ and the integrated cosine operator function found in [10]. Let $k = 2\alpha > 0$ and let an operator A be the generator of α times integrated cosine operator function $C_\alpha(t)$, $u_0 \in D(A)$. Then the problem (5), (6) is uniformly weel posed, i.e., $A \in G_k$ and the corresponding BOF can be represented in the form

$$Y_k(t)u_0 = \frac{2^\alpha \Gamma(\alpha + 1/2)}{\sqrt{\pi}\, t^\alpha} \left(C_\alpha(t)u_0 - \int\limits_0^1 P'_{\alpha-1}(\tau) C_\alpha(t\tau) u_0\, d\tau \right),$$

where $P_\nu(t)$ is the spherical Legendre function (see [18]).

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