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FREDHOLM OPERATOR MANIFOLDS

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Abstract. We consider special classes of operators acting in functional spaces on manifolds. We can say that our approach is an operator-geometric treatment of the well-known locality principle. We described the Fredholm property in an abstract form and show how these results can be applied to the study of elliptic pseudodifferential operators on manifolds with nonsmooth boundaries.

Keywords and phrases: local operator, operator manifold, Fredholm property, index.

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1. Introduction. The works of I. B. Simonenko (see, e.g., [15]) devoted to the local principle, which the author became acquainted with many years ago, had a strong influence on all his subsequent research. Almost immediately, algebraic versions of this principle appeared (see [6]), and all further work in this direction was somehow connected with algebraic constructions (see [1–4, 7–14]). In our opinion, the operator-geometric constructions proposed in the present paper more clearly and accurately reflect the nature of a pseudodifferential operator acting in function spaces on a manifold. In their simplest form, these ideas have already appeared in the author's works [16, 17]. We present here two main results on the representation of such an operator: a theorem that states that the operator admits a "decomposition" into several component operators, and an index theorem that states that the index of an operator is the sum of the indices of the component operators. Some results are described in [18].

2. Archetypes of basic constructions. The results obtained in this work can be conventionally called a "geometric interpretation of the local principle." The author's main idea is to define a manifold by means of a local operator acting in a local functional space on this manifold.

2.1. Local functional spaces. We recall the definitions of "local" Sobolev–Slobodetskii spaces related to pseudodifferential operators of variable order (see [19]); below we will use them as "model" spaces.

Let $D \subset \mathbb{R}^m$ be a domain. In fact, we will deal with the following types of "canonical" conical domains: the space $\mathbb{R}^m$, the half-space $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x', x_m),\ x_m > 0\}$, and the k-dimensional conical edge of the form $\mathbb{R}^k \times C^a_{m-k}$, where $C^a_{m-k} = \{x \in \mathbb{R}^{m-k} : x = (x'', x_{m-k}),\ x_{m-k} > a|x''|, a > 0\}$. All m-dimensional domains D_k are labeled with subscripts $k = 0, 1, \dots, m$, so that

$$D_m \equiv \mathbb{R}^m \times C^a_0 \equiv \mathbb{R}^m, \quad D_{m-1} \equiv \mathbb{R}^{m-1} \times C^a_1 \equiv \mathbb{R}^m_+,$$
$$D_k \equiv \mathbb{R}^k \times C^a_{m-k}, \quad k = 1, 2, \dots, m-2, \qquad D_0 \equiv \mathbb{R}^0 \times C^a_m \equiv \{x \in \mathbb{R}^m : x_m > a|x'|\}.$$

By definition (see [5, 19]), the local Sobolev–Slobodetskii space $H^{s(x_0)}(\mathbb{R}^m)$ consists of (generalized) functions u with a finite norm (the point $x_0 \in \mathbb{R}^m$ is fixed)

$$\|u\|_{s(x_0)} = \left(\int_{\mathbb{R}^m} |\tilde{u}(\xi)|^2 (1+|\xi|)^{2s(x_0)} d\xi\right)^{1/2}, \tag{1}$$

where $\tilde{u}$ means the Fourier transform of the function u:

$$\tilde{u}(\xi) \equiv \int_{\mathbb{R}^m} e^{ix\cdot\xi} u(x) dx.$$

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2.2. Local operators. Local operators act in local spaces. Roughly speaking, local operators act as follows:

$$u(x) \longmapsto \int_D \int_{\mathbb{R}^m} A(\xi) e^{i\xi\cdot(x-y)} u(y)\, dy\, d\xi, \quad x \in D,$$

where $A(\xi)$ is a function defined on $\mathbb{R}^m$, which is called the local symbol, and D is one of canonical conical domains (this notion will be clarified below).

3. Operator manifolds. Let M be an m-dimensional compact manifold with nonsmooth boundary, i.e., there exist points of the boundary whose neighborhoods are diffeomorphic to one of the canonical domains $D_k \subset \mathbb{R}^m$, $k = 0, 1, \ldots, m-1$, whereas neighborhoods of interior points of the manifold M are diffeomorphic to $D_m \equiv \mathbb{R}^m$. In other words, on the manifold M, some smooth compact submanifolds (sigularities) M_k, $k = 0, 1, \ldots, m-2$ are indicated, whose neighborhoods are diffeomorphic to D_k, whereas neighborhood of smoothness points of the boundary are diffeomorphic to D_{m-1}. To each point x_0 of the manifold M, we assign a local Sobolev–Slobodetskii space $H^{s(x_0)}(D_{x_0})$ (see [19]). Moreover, we assume that for all points $x_0 \in M_k$, the function $s(x_0)$ is constant and takes the same value s_k and the domain D_{x_0} also has the form D_k.

Definition 1. A manifold M is called an *operator manifold* if in a neighborhood of each point of the manifold, an operator-valued function $M \ni x \longmapsto A(x)$ is defined, where $A(x) : H^{s(x)}(D_x) \to H^{t(x)}(D_x)$ is a linear bounded operator.

Perhaps this definition seems too general, but below we will try to show that it is quite viable, even in cases different from the specific Sobolev–Slobodetskii spaces. The use of abstract results will be associated with Sobolev–Slobodetskii spaces.

4. Virtual operators. In this section, we develop an abstract version of the theory of local operators in functional spaces inspired by works of I. B. Simonenko (see [15]) devoted to local-type operators (for brevity, we call them local operators). The simplest models of such operators are pseudodifferential operator. There exist several approaches to the Freholm theory of such operators and related equations on nonsmooth manifolds or on smooth manifolds with nonsmooth boundaries (see [1–4, 8, 10, 11, 13, 14]); however, it is unlikely that these constructions can be fit into the described abstract scheme.

Recall some ideas and definitions from [15, 18]. Let B_1 and B_2 be Banach spaces of functions defined on a compact m-dimensional manifold M. We assume that compactly supported smooth functions are dense in these functional spaces. Further, let $A : B_1 \to B_2$ be a linear bounded operator.

Definition 2. An operator A is said to be *local* if the operator $f \cdot A \cdot g$ is compact for any two smooth functions f and g with nonintersecting supports.

Below we consider only functional spaces that contain smooth functions and the corresponding multipliers are bounded; moreover, all operators considered are defined up to compact operators.

In accordance with Sec. 4.3, we choose a finite covering of the manifold M, the corresponding partition of unity $\{U_j, f_j\}_{j=1}^n$, and a system of smooth functions $\{g_j\}_{j=1}^n$ such that $\operatorname{supp} g_j \subset V_j$, $\overline{U_j} \subset V_j$, and $g_j(x) \equiv 1$ for $x \in \operatorname{supp} f_j$, $\operatorname{supp} f_j \cap (1 - g_j) = \emptyset$.

Corollary 1. *An operator A on a compact manifold M can be represented in the form*

$$A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + T,$$

where $T : B_1 \to B_2$ is a compact operator.

Remark 1. It is easy to see that such operators are defined uniquely up to compact operators; this fact does not affect the index of the operator.

Taking into account this remark, for an arbitrary operator $A : B_1 \to B_2$ we introduce the essential norm (see [15])

$$|||A||| \equiv \inf \|A + T\|,$$

where the infimum is taken over all compact operators $T : B_1 \to B_2$.

Now we introduce Euclidean local models of operators. Let B_1' and B_2' be Banach spaces of functions defined on the Euclidean space $\mathbb{R}^m$ and $\widetilde{A} : B_1' \to B_2'$ be a linear bounded operator.

Due to the definition of the manifold M, each point $x \in M$ possesses a neighborhood $U \ni x$ and the corresponding diffeomorphism $\omega : U \to D \subset \mathbb{R}^m$, $\omega(x) \equiv y$. We introduce an appropriate operator S_ω acting from B_k to B_k', $k = 1, 2$. For a function $u \in B_k$ vanishing outside U, we set

$$(S_\omega u)(y) = u(\omega^{-1}(y)), \quad y \in D, \qquad (S_\omega u)(y) = 0, \quad y \notin D.$$

Definition 3. A local representative of an operator $A : B_1 \to B_2$ at a point $x \in M$ is an operator'[1] $\widetilde{A} : B_1' \to B_2'$ such that for any $\varepsilon > 0$, there exists a neighborhood U_j of the point $x \in U_j \subset M$ in which the following inequality holds:

$$|||g_j A f_j - S_{\omega_j^{-1}} g_j \widetilde{A} f_j S_{\omega_j}||| < \varepsilon.$$

4.1. Virtual operator. Again, let M be a compact manifold with boundary ∂M and $A(x)$ be an operator-valued function defined on M. On the boundary ∂M, we choose smooth k-dimensional submanifolds M_k, $k = 0, 1, \ldots, m-1$, such that $M_{m-1} \equiv \partial M$ by definition and M_0 is the set of isolated points of ∂M, $M_m \equiv M$. Finally, introduce the set of classes of operators $\mathfrak{T}_k$, $k = 0, 1, \ldots, m$, such that for $x \in M_k$, the operator $A(x) : H_k^{(1)} \to H_k^{(2)}$ is a linear bounded operator and $H_k^{(j)}$, $k = 0, 1, \ldots, m$, $j = 1, 2$, are Banach spaces. In fact, we assume here that M is an operator manifold of a more general type than in Definition 2. A submanifold M_k is called a singular k-submanifold if for all $x \in M_k$, the inclusion $A(x) \in \mathfrak{T}_k$ holds.

Theorem 1. *If a family $A(x)$ consists of local Fredholm operators and is continuous[2] on each component*

$$\overline{M_k \setminus \bigcup_{i=0}^{k-1} M_i}, \quad k = 0, 1, \ldots, m,$$

then it generates a unique Fredholm operator A' acting in the spaces of direct sums

$$\sum_{k=0}^{m} \oplus H_1^{(k)} \to \sum_{k=0}^{m} \oplus H_2^{(k)}.$$

Proof. The proof is performed according to the scheme proposed by the author in [18]. A covering of the manifold and a partition of unity are chosen in a special way, the family $A(x)$ is clustered, and each cluster generates a separate operator. □

Definition 4. The operator A' is called the virtual operator corresponding to the family $A(x)$. The virtual operator A' is said to be *elliptic* if the family $A(x)$ consists of Fredholm operators for all $x \in M$.

4.2. Index of the virtual operator. Below we present the theorem on the index of the virtual operator. Unfortunately, it does not give a recipe for calculating the index, but it shows which operators we must work with to obtain acceptable formulas for the index.

Theorem 2. *The operator A' acts on smooth functions as follows:*

$$Au = \sum_{k=0}^{m} A^{(k)} u_k + Tu,$$

[1] Note that it is *another* operator acting in *another* functional space.
[2] With respect to the essential norm $||| \cdot |||$.

where u_k is a component of the function u concentrated in a neighborhood of the submanifold

$$\overline{M_k \setminus \Big(\bigcup_{j=0}^{k-1} M_j\Big)},$$

so that

$$u = \sum_{k=0}^{m} u_k.$$

Remark 2. The operators $A^{(k)}$ have been constructed in [18]; each of the operators $A^{(k)}$ is related to the corresponding singular k-submanifold.

Theorem 3. *The index of an elliptic virtual operator*

$$A' : \sum_{k=0}^{m} \oplus H_1^{(k)} \to \sum_{k=0}^{m} \oplus H_2^{(k)}$$

can be represented as the sum of the corresponding indexes:

$$\operatorname{Ind} A' = \sum_{k=0}^{m} \operatorname{Ind} A^{(k)}. \tag{2}$$

Remark 3. For an elliptic pseudodifferential operator in the space $H^s(\mathbb{R}^m_+)$ with smooth symbol $A(x,\xi)$ (see [5]), we in fact have two components of the virtual operator, namely, the operator $A^{(m)}$ corresponding to the closure of interior points $\mathbb{R}^m_+$ and the operator $A^{(m-1)}$ corresponding to boundary points $\mathbb{R}^{m-1}$. The operator symbols have different nature for interior and boundary points. In the former case, the symbol is an integral operator over the whole space $\mathbb{R}^m$; in the latter case it is an integral operator over the half-space. The index of the operator $A^{(m)}$ vanishes due to the classical Atiah–Singer theorem, whereas the index of $A^{(m-1)}$ depends on the so-called factorization index of the elliptic symbol $A(x,\xi)$ at a boundary point $x \in \mathbb{R}^{m-1}$.

4.3. Example: Pseudodifferential operators on a compact manifold whose boundary has singularities. In this section, we demonstrate how the abstract scheme proposed above operates for a particular pseudodifferential operator A on an m-dimensional compact manifold M with boundary containing singularities. Such operators are usually defined by using their symbols $A(x,\xi)$, $(x,\xi) \in \mathbb{R}^{2m}$ (here we use local coordinates; usually, the symbols are defined on the (co)tangent bundles). Assume that on the boundary ∂M, there are smooth compact submanifolds M_k of dimensions $0 \le k \le m-1$, which are singularities of the boundary. These singularities of the boundary are described by special local representatives of A at a point $x_0 \in M$ on the chart $U \ni x_0$ as follows:

$$(A_{x_0}u)(x) = \int_{D_{x_0}} \int_{\mathbb{R}^m} e^{i\xi\cdot(x-y)} A(\varphi(x_0),\xi) u(y) d\xi dy, \quad x \in D_{x_0},$$

where $\varphi : U \to D_{x_0}$ is a diffeomorphism and the shape of the canonical domain D_{x_0} depends on the location of the point x_0 on the manifold M. We consider the following canonical domains D_{x_0}: $\mathbb{R}^m$, $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x', x_m),\ x_m > 0\}$, $W^k = \mathbb{R}^k \times C^{m-k}$, where C^{m-k} is a convex cone in $\mathbb{R}^{m-k}$. In other words, the boundary ∂M may contain conical points and edges of various dimensions.

We consider pseudodifferential operators A in the Sobolev–Slobodetskii spaces $H^s(M)$; as local versions of these spaces, we take the spaces $H^s(D_{x_0})$.

Definition 5. The *symbol* of a pseudodifferential operator A is an operator-valued function $A(x) : M \to \{A_x\}_{x\in M}$ defined by local representatives of the operator A.

Under some additional smoothness conditions for the function $A(x,\xi)$, the following theorem holds.

Theorem 4. *An operator A is a Fredholm operator if and only if its symbol consists of Fredholm operators.*

This theorem is called the locality principle; some applications can be found in [15, 16]. Taking into account the structure of local representatives, we state the following definition.

Definition 6. A pseudodifferential operator A is said to be *elliptic* if its symbol' consists of invertible operators.

Remark 4. If the ellipticity is violated on a submanifold M_k, then the operator can be made a Fredholm operator by modifying the corresponding local representatives of the operator A, namely, by adding boundary or coboundary operators (see [16]).

4.4. Index of Fredholm operators. As above, using the partition of unity on the manifold M and the theory of envelopes proposed by I. B. Simonenko (see [15, 18]), starting from an elliptic symbol $A(x)$, we can construct n virtual operators A_j (their number is equal to the number of singular submanifolds, including the boundary ∂M and the manifold itself M). The resulting virtual operator A' acts in the direct sums of the spaces:

$$A' : H^s(\mathbb{R}^m) \oplus H^s(\mathbb{R}^m_+) + \sum_{k=0}^{m-2} \oplus H^s(C^{m-k}) \to H^{s-\alpha}(\mathbb{R}^m) \oplus H^{s-\alpha}(\mathbb{R}^m_+) + \sum_{k=0}^{m-2} \oplus H^{s-\alpha}(C^{m-k})$$

if the symbol of the original operator A has order α,

$$c_1(1 + |\xi|)^\alpha \le |A(x, \xi)| \le c_2(1 + |\xi|)^\alpha,$$

with positive constants c_1, c_2 for all (x, ξ).

Since the symbols of the virtual and original operators are the same and homotopies in the class of symbols are equivalent to homotopies in the class of operators, we arrive at the following result.

Theorem 5. *The index of a Fredholm operator A is defined by the formula*

$$\operatorname{Ind} A = \sum_{j=1}^{n} \operatorname{Ind} A_j.$$

Conclusion. The constructions proposed above look rather abstract. However, we note that they all appeared as a result of the author's reflections on the theory of pseudodifferential equations and boundary-value problems on manifolds with nonsmooth boundaries. In general, such a theory fits into the scheme described above, and elements of this theory were already present in the author's earlier works. Unfortunately, it still contains many uncertainties, but at least it explains how a general boundary-value problem for an elliptic pseudodifferential equation should look like and in what direction one should move in search of its solvability.

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COMPLIANCE WITH ETHICAL STANDARDS

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