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FIRST-ORDER COVARIANT DIFFERENTIAL OPERATORS

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Abstract. An internal description of the class of all nonlinear differential operators of the first order on the space of collections consisting of continuously differentiable vector and scalar fields on $\mathbb{R}^3$ is given. Operators of this class are invariant with respect to translations of $\mathbb{R}^3$ and are transformed by the covariant way under rotations of $\mathbb{R}^3$.

Keywords and phrases: first-order differential operator, divergence differential operator, vector field, pseudovector field, covariance.

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1. Introduction. In theoretical physics, when solving problems that are formulated in terms of partial differential equations, the key issue is the problem of constructing adequate evolutionary equations based on physically justified provisions. The solution of such problems is ultimately based on the description of classes of equations that satisfy the stated physical conditions. With an internal description of each of these classes, physicists get the opportunity to choose an appropriate equation for solving specific problems by setting up experiments and comparing the results obtained with the predictions obtained as a result of solving mathematical problems that model the physical situation. This is how physical problems were solved and mathematical physics related to their solution developed. Along this path, the following equations appeared in mathematical physics: the heat conduction equation, the system of equations of hydrodynamics of Newtonian fluids, the system of Maxwell's equations, etc., which currently seem to be sufficiently justified from a physical point of view.

The present paper is devoted to the problem of constructing suitable evolutionary differential equations intended to describe the dynamics of condensed matter in terms of fields on $\mathbb{R}^3$. Specifically, the paper proposes a description of a certain class of suitable evolutionary partial differential equations of the first order with respect to spatial coordinates. These are the equations of mathematical physics in the construction of which the physical mechanisms of energy dissipation are neglected. In a situation where the medium is not subject to external influences, both stationary and nonstationary, the coefficients of such equations do not depend explicitly on either time or spatial variables. In addition, such equations have the property of independence of their form from the specific coordinate system in which the physical fields are described. Mathematically, this is expressed as their covariance property under the action of transformations of the group $\mathcal{O}_3$. We will call such equations *isotropic*.

The requirement of covariance leads to the fact that fields on $\mathbb{R}^3$ must be transformed under the action of transformations of the group $\mathcal{O}_3$ according to the representations of this group (see, e.g., [10]), and, similarly, the coefficients of the differential operators defining the equation must be also transformed according to the representations of this group. These requirements impose quite significant restrictions on the general form of evolutionary differential equations.

The problem of creating a general method for constructing such equations has been the subject of many works in mathematical physics (see, e.g., [1, 2, 4–6, 8, 9, 15–20]). The overwhelming majority of them are based on modifications of the Lagrange or Hamilton formalisms that are traditional in theoretical physics. However, apparently, such approaches are not adequate for constructing evolution equations for describing condensed media with internal degrees of freedom. For this reason, the question arises of a more general approach to constructing equations of nonequilibrium thermodynamics. In this regard, in this paper we propose a solution to the problem of describing a class of any systems

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of first-order dynamic equations in spatial coordinates that are isotropic in the above sense for vector and scalar fields on $\mathbb{R}^3$.

2. Covariant systems of equations. Consider a linear manifold of tuples $\mathsf{Y}(\boldsymbol{x},t) = \langle Y_a(\boldsymbol{x},t);\ a = 1,\dots,N\rangle$ of $\mathbb{R}$-valued functions on $\mathbb{R}^3$, i.e., functions $\mathsf{Y} : \mathbb{R}^3 \mapsto \mathbb{R}^N$. Moreover, we assume that functions in these tuples depend on the parameter $t \in \mathbb{R}_+$, whose physical meaning is time.

Next, we assume that the functions $Y_a(\boldsymbol{x},t)$, $a = 1,\dots,N$, are differentiable with respect to $\boldsymbol{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}_+$. This means that the manifold considered is a linear topological space $[C_{1,\mathrm{loc}}(\mathbb{R}^3)]^N$ with a countably normed topology. Within this space, it is possible to study solutions of first-order differential equations satisfied by elements $\mathsf{Y}(\boldsymbol{x},t)$ of this space. Our goal is to describe the variety of all admissible systems of evolution equations on the space $[C_{1,\mathrm{loc}}(\mathbb{R}^3)]^N$ that have the form

$$\dot{\mathsf{Y}}(\boldsymbol{x},t) = \big(\mathsf{L}[\mathsf{Y}]\big)(\boldsymbol{x},t) \tag{2.1}$$

and satisfy the covariance condition stated above. Here the dot means differentiation with respect to t and $\mathsf{L} : [C_{1,\mathrm{loc}}(\mathbb{R}^3)]^N \mapsto [C_{1,\mathrm{loc}}(\mathbb{R}^3)]^N$ is a differential operator whose order with respect to the components of the vector $\boldsymbol{x} \in \mathbb{R}^3$ is equal to 1; in the general case, this operator is nonlinear; it is defined by the formula

$$(\mathsf{L}[\mathsf{Y}])(\boldsymbol{x},t) = \big(\mathcal{A}_k(\mathsf{Y})\nabla_k\mathsf{Y} + \mathsf{H}(\mathsf{Y})\big)(\boldsymbol{x},t). \tag{2.2}$$

Here and below, ∇_k, $k = 1,2,3$, is the gradient operator in $\mathbb{R}^3$ and $\mathcal{A}_k(\mathsf{Y})$, $k = 1,2,3$, is the triple of matrix-valued functions of $\mathsf{Y} \in \mathbb{R}^N$ taking values in $\mathbb{R}^N \times \mathbb{R}^N$. They are independent of $\boldsymbol{x} \in \mathbb{R}^3$ and t and for each fixed $\mathsf{Y} \in \mathbb{R}^N$ and $k = 1,2,3$ the matrix $\mathcal{A}_k(\mathsf{Y})$ acts as follows:

$$\nabla_k\mathsf{Y}(\boldsymbol{x},t) = \Big\langle \nabla_k Y_a(\boldsymbol{x},t);\ a = 1,\dots,N\Big\rangle.$$

The tuple $\mathsf{H}(\mathsf{Y}) = \langle H_a(\mathsf{Y}); a = 1,\dots,N\rangle$ consists of continuous functions on $\mathbb{R}^N$ that do not depend explicitly on either $\boldsymbol{x} \in \mathbb{R}^3$ or t. In addition, here and below we adopt the algebraic convention of summation over doubly repeated vector indices. In this case, summation is performed over $k = 1,2,3$. The linear manifold of all such operators is denoted by $\mathfrak{K}_1$. Note also that the term *systems of evolution equations* used above only indicates that these systems have the form (2.1), and it does not have any physical meaning.

We assume that the linear space of tuples Y is transformed according to a reducible representation of the group $\mathcal{O}_3$ or its subgroup $\mathcal{O}_{3,+}$ of continuous rotations of the space $\mathbb{R}^3$ (see, e.g., [10]). In this paper, we consider the case where the space $\mathbb{R}^N$ of tuples Y is represented as the direct sum of linear spaces, which are transformed according to an irreducible vector representation with respect to continuous rotations $\mathbb{R}^3$ (in particular, a pseudovector representation) and a one-dimensional space of scalars, which is considered as a space transformed according to the trivial representation. Within this decomposition, each tuple Y can be represented as a pair $\mathbf{Y} = \langle \mathsf{W}, \mathsf{Z}\rangle$, where the tuple $\mathsf{W} = \langle Y_a; a = 1,\dots,3n\rangle$ consists only of components of vector representations; their number is equal to n, so that W is then the tuple $\langle \boldsymbol{W}^{(a)}; a = 1,\dots,n\rangle$ whose components are vectors (or pseudovectors). The tuple

$$\mathsf{Z} = \big\langle Z^{(a)};\ a = 1,\dots,r\big\rangle = \big\langle Y_a;\ a = 3n+1,\dots,N\big\rangle, \quad Z^{(a)} = Y_{a-2n},$$

consists of $r = N - 3n$ scalars.

The components of tuples $\mathsf{Y}(\boldsymbol{x},t)$ of functions are split so that their values ??are consistent with the splitting of the sets $\mathsf{Y} \in \mathbb{R}^N$ specified above. As a result, they are represented in the form $\mathsf{Y}(\boldsymbol{x},t) = \langle \mathsf{W}(\boldsymbol{x},t), \mathsf{Z}(\boldsymbol{x},t)\rangle$.

Within such splitting of the tuple Y, Eq. (2.1) with an operator $\mathsf{L}[\mathsf{Y}]$ of the form (2.2) can be written as the following system of two equations

$$\begin{aligned} \dot{\mathsf{W}}(\boldsymbol{x},t) &= \big(\mathcal{A}_k^{(w)}(\mathsf{W},\mathsf{Z})\nabla_k\mathsf{W} + \mathcal{A}_k^{(wz)}(\mathsf{W},\mathsf{Z})\nabla_k\mathsf{Z} + \mathsf{F}(\mathsf{W},\mathsf{Z})\big)(\boldsymbol{x},t), \\ \dot{\mathsf{Z}}(\boldsymbol{x},t) &= \big(\mathcal{A}_k^{(zw)}(\mathsf{W},\mathsf{Z})\nabla_k\mathsf{W} + \mathcal{A}_k^{(z)}(\mathsf{W},\mathsf{Z})\nabla_k\mathsf{Z} + \mathsf{G}(\mathsf{W},\mathsf{Z})\big)(\boldsymbol{x},t), \end{aligned} \tag{2.3}$$

where we use the decomposition $\mathsf{H}(\mathsf{Y}) = \langle \mathsf{F}(\mathsf{W},\mathsf{Z}), \mathsf{G}(\mathsf{W},\mathsf{Z})\rangle$ corresponding to the decomposition of the space $\mathbb{R}^N$. According to this decomposition, the values of the matrix-valued functions $\mathcal{A}_k(\mathsf{Y})$ are also split into the blocks $\mathcal{A}_k^{(w)}$, $\mathcal{A}_k^{(wz)}$, $\mathcal{A}_k^{(zw)}$, $\mathcal{A}_k^{(z)}$ of sized $n\times n$, $n\times r$, $r\times r$, $r\times n$, respectively:

$$\mathcal{A}_k(\mathsf{Y}) = \begin{pmatrix} \mathcal{A}_k^{(w)}(\mathsf{Y}) & \mathcal{A}_k^{(wz)}(\mathsf{Y}) \\ \mathcal{A}_k^{(zw)}(\mathsf{Y}) & \mathcal{A}_k^{(z)}(\mathsf{Y}) \end{pmatrix}. \tag{2.4}$$

Since each component of the tuple $\nabla_k \mathsf{W}(\boldsymbol{x},t)$, on which the matrices $\mathcal{A}_k^{(w)}$, $k=1,2,3$, act, is a vector in $\mathbb{R}^3$, each element $(\mathcal{A}_k^{(w)})_{a,b}$, $a,b=1,\dots,n$, of this matrix is a (3×3)-matrix, $((\mathcal{A}_k^{(w)})_{a,b})_{jl} \equiv T_{jkl}^{(a,b)}$, $j,l=1,2,3$. For the same reason, each element of the matrices $\mathcal{A}_k^{(wz)}$ and $\mathcal{A}_k^{(zw)}$ can be represented as a three-dimensional vector. Therefore, for any $k=1,2,3$, each element of the matrix $(\mathcal{A}_k^{(wz)})_{a,b}$, $a=1,\dots,n$, $b=n+1,\dots,n+r$, and each element of the matrix $(\mathcal{A}_k^{(zw)})_{a,b}$, $a=n+1,\dots,r$, $b=1,\dots,n$, is also a vector. We introduce the notation

$$\big((\mathcal{A}_k^{(wz)})_{a,b}\big)_j \equiv T_{jk}^{(a,b)}, \quad j=1,2,3; \qquad \big((\mathcal{A}_k^{(zw)})_{a,b}\big)_l \equiv \tilde{T}_{kl}^{(a,b)}, \quad l=1,2,3,$$

for the corresponding values of a and b. Finally, for any fixed $k=1,2,3$, the matrix element of $(\mathcal{A}_k^{(w)})_{a,b} \equiv T_k^{(a,b)}$, $a,b=n+1,\dots,n+r$, can be represented by numbers from $\mathbb{R}$ (here we omit the indication of the dependence of the matrix elements on the tuples W and Z. In terms of these matrix elements, restoring the dependence on W and$\mathsf{Z}$, we rewrite the system of equation (2.3) in the form

$$\begin{aligned} \dot{W}_j^{(a)}(\boldsymbol{x},t) &= \left(\sum_{b=1}^{n} T_{jkl}^{(a,b)}(\mathsf{W},\mathsf{Z})\nabla_k W_l^{(b)} + \sum_{b=n+1}^{n+r} T_{jk}^{(a,b)}(\mathsf{W},\mathsf{Z})\nabla_k Z^{(b)} + F_j^{(a)}(\mathsf{W},\mathsf{Z})\right)(\boldsymbol{x},t), \\ \dot{Z}^{(a)}(\boldsymbol{x},t) &= \left(\sum_{b=1}^{n} \tilde{T}_{kl}^{(a,b)}(\mathsf{W},\mathsf{Z})\nabla_k W_l^{(b)} + \sum_{b=n+1}^{n+r} T_{k}^{(a,b)}(\mathsf{W},\mathsf{Z})\nabla_k Z^{(b)} + G^{(a)}(\mathsf{W},\mathsf{Z})\right)(\boldsymbol{x},t), \end{aligned} \tag{2.5}$$

respectively, for $a=1,\dots,n$ and $a=n+1,\dots,n+r$. In the first equation corresponding to the $a=1,\dots,n$, we use the component-wise notation of three-dimensional vectors for the values of the vector fields $\boldsymbol{W}^{(a)}(\boldsymbol{x},t) = \langle W_j^{(a)}, j=1,2,3\rangle$.

To introduce the notion of covariance of the system (2.5), we additionally split the components of the tuple of vector W into two types according to their transformation properties: the components $\boldsymbol{W}^{(a)}$ with the indices $a=1,\dots,p$ are transformed as vectors under transformations of the group $\mathcal{O}_3$, whereas the components with the indices $a=p+1,\dots,p+q=n$ are transformed as vectors only under continuous rotations of the space $\mathbb{R}^3$ and remain unchanged under discrete transformations of the group $\mathcal{O}_3$, i.e., they are pseudovectors. According to this, we split the components of the vector fields from the tuple $\mathsf{W}(\boldsymbol{x},t)$ in such a way that the values of the fields $\boldsymbol{W}^{(a)}(\boldsymbol{x},t)$, $a=1,\dots,p$, are transformed as vectors, whereas the values of the fields $\boldsymbol{W}^{(a)}(\boldsymbol{x},t)$, $a=p+1,\dots,p+q$, are transformed as pseudovectors.

Definition 2.1. A tensor-valued function $T_{j_1\dots j_l}(\mathsf{W},\mathsf{Z})$ with values in the space of tensors (pseudotensors) of rank $l\in\mathbb{N}$ of the space $\mathbb{R}^3$ is said to be covariant[1] under transformations of the group $\mathcal{O}_3$ (or $\mathcal{O}_{3,+}$) if for any orthogonal matrix $\mathcal{Q}$ of this group, the following relation holds:

$$\mathcal{Q}_{j_1k_1}\dots\mathcal{Q}_{j_lk_l}T_{k_1\dots k_l}(\mathsf{W},\mathsf{Z}) = T_{j_1\dots j_l}(\mathcal{Q}\mathsf{W},\mathsf{Z}). \tag{2.6}$$

If a function takes pseudotensor values, then for any matrix $\mathcal{Q}$ that represents a total reflection of the space $\mathbb{R}^3$, the following relation holds:

$$\mathcal{Q}_{j_1k_1}\dots\mathcal{Q}_{j_lk_l}T_{k_1\dots k_l}(\mathsf{W},\mathsf{Z}) = (-1)^{l-1}T_{j_1\dots j_l}(\mathcal{Q}\mathsf{W},\mathsf{Z}). \tag{2.7}$$

[1]In the monographs [3, 7], covariant functions are called concomitants. Regarding the terminology of tensor analysis used in this paper, see, e.g., [11]. We do not make a distinction between covariant and contravariant tensors.

If $l = 0$, then such a function is said to be scalar (respectively, pseudoscalar).

Definition 2.2. A system of equations (2.5) is said to be covariant under transformations of the group $\mathcal{O}_3$ if for any tuples W and Z, the coefficients of the system $F_j^{(a)}(\mathsf{W},\mathsf{Z})$, $T_k^{(a,b)}$, $T_{jk}^{(a,b)}(\mathsf{W},\mathsf{Z})$, $T_{kl}^{(a,b)}(\mathsf{W},\mathsf{Z})$, and $T_{jkl}^{(a,b)}(\mathsf{W},\mathsf{Z})$ are covariant functions with $l = 1, 2, 3$, respectively, and

(1) $T_{jkl}^{(a,b)}(\mathsf{W},\mathsf{Z})$ are third-rank tensors with respect to the indices j, k, and l if $a, b = 1, \dots, p$ and $a, b = p+1, \dots, p+q$ and third-rank pseudotensors if $a = 1, \dots, p$ and $b = p+1, \dots, p+q$, or $a = p+1, \dots, p+q$ and $b = 1, \dots, p$;
(2) $T_{jk}^{(a,b)}(\mathsf{W},\mathsf{Z})$ are second-order tensors with respect to the indices j and k if $a = 1, \dots, p$ and second-order pseudotensors if $a = p+1, \dots, n$ and $b = n+1, \dots, n+r$;
(3) $\tilde{T}_{kl}^{(a,b)}(\mathsf{W},\mathsf{Z})$, $a = n+1, \dots, n+r$, are second-order tensors with respect to k and l if $b = 1, \dots, p$ and second-order pseudotensor if $b = p+1, \dots, n$;
(4) $T_k^{(a,b)}$ are vectors with components $k = 1, 2, 3$ for $a, b = n+1, \dots, n+r$;
(5) $F_j^{(a)}$ are vectors with components $j = 1, 2, 3$ for $a = 1, \dots, p$ and pseudovectors for $a = p+1, \dots, n$;
(6) $G^{(a)}$, $a = n+1, \dots, n+r$, are scalar functions.

As was said above, we will describe below the linear manifold of operators of the form (2.2) that are covariant under transformations of the group $\mathcal{O}_3$. The above definition implies that the solution of such a problem is reduced to describing the linear manifolds of covariant functions that represent the coefficients of the system (2.5), i.e., the linear manifolds of continuous (pseudo)tensor-valued functions $T_{jkl}^{(a,b)}(\mathsf{W},\mathsf{Z})$, $T_{jk}^{(a,b)}(\mathsf{W},\mathsf{Z})$, and $T_{kl}^{(a,b)}(\mathsf{W},\mathsf{Z})$ and the vector-valued functions $T_k^{(a,b)}(\mathsf{W},\mathsf{Z})$.

3. Covariant tensor-valued functions. In this section, we propose an approach to describing any covariant tensor-valued functions on $\mathbb{R}^3$ and, in particular, the tensor-valued functions $T_{jkl}^{(a,b)}(\mathsf{W},\mathsf{Z})$, $T_{jk}^{(a,b)}(\mathsf{W},\mathsf{Z})$, and $T_{kl}^{(a,b)}(\mathsf{W},\mathsf{Z})$ and the vector-valued functions $T_k^{(a,b)}(\mathsf{W},\mathsf{Z})$ and $F_k^{(a)}(\mathsf{W},\mathsf{Z})$. For simplicity, we restrict ourselves to the case where these functions are represented by polynomials in the components of the vectors of the tuples W. In this case, we are not interested in the nature of the dependence of these functions on the components of the set Z.

Definition 3.1. A covariant (pseudo)tensor-valued function $T_{j_1 \dots j_l}(\mathsf{W},\mathsf{Z})$ of rank l is said to be polynomial on $\mathbb{R}^N$ if it is defined by the formula

$$T_{j_1 \dots j_l}(\mathsf{W},\mathsf{Z}) = \sum_{d=0}^{D} \sum_{\langle a_1, \dots, a_d \rangle \in I_n^d} A_{j_1 \dots j_l k_1 \dots k_d}^{(a_1, \dots, a_d)}(\mathsf{Z}) W_{k_1}^{(a_1)} \dots W_{k_d}^{(a_d)}. \tag{3.1}$$

Now we introduce invariant tensors (pseudotensors) on $\mathbb{R}^3$.

Definition 3.2. A tensor (pseudotensor) $A_{j_1, \dots, j_m}^{(a_1, \dots, a_d)}$ of rank $m \in \mathbb{N}$ in the space $\mathbb{R}^3$ is said to be invariant under transformations of the group $\mathcal{O}_3$ (or $\mathcal{O}_{3,+}$) if for any orthogonal matrix from this group we have

$$\mathcal{Q}_{j_1 k_1} \dots \mathcal{Q}_{j_m k_m} A_{k_1, \dots, k_m}^{(a_1, \dots, a_d)} = A_{j_1, \dots, j_m}^{(a_1, \dots, a_d)}. \tag{3.2}$$

Lemma 3.1. *Let $T_{j_1 \dots j_l}(\mathsf{W},\mathsf{Z})$ be a covariant polynomial (pseudo)tensor-valued function of rank l. Then its coefficients in the formula (3.1) are invariant (pseudo)tensors for each fixed Z.*

Proof. Using the orthogonal property of the matrix $\mathcal{Q}$, from (3.1) we obtain:

$$\mathcal{Q}_{i_1 j_1}\dots\mathcal{Q}_{i_l j_l}T_{j_1\dots j_l}(\mathsf{W},\mathsf{Z}) = \sum_{d=0}^{D}\sum_{\langle a_1,\dots,a_d\rangle\in I_n^d}\mathcal{Q}_{i_1 j_1}\dots\mathcal{Q}_{i_l j_l}A^{(a_1,\dots,a_d)}_{j_1\dots j_l k_1\dots k_d}(\mathsf{Z})W^{(a_1)}_{k_1}\dots W^{(a_d)}_{k_d}$$

$$= \sum_{d=0}^{D}\sum_{\langle a_1,\dots,a_d\rangle\in I_n^d}\mathcal{Q}_{i_1 j_1}\dots\mathcal{Q}_{i_l j_l}(\mathcal{Q}_{s_1 k_1}\mathcal{Q}_{s_1 m_1})\dots(\mathcal{Q}_{s_d k_d}\mathcal{Q}_{s_d m_d})A^{(a_1,\dots,a_d)}_{j_1\dots j_l k_1\dots k_d}(\mathsf{Z})W^{(a_1)}_{m_1}\dots W^{(a_d)}_{m_d}$$

$$= \sum_{d=0}^{D}\sum_{\langle a_1,\dots,a_d\rangle\in I_n^d}\mathcal{Q}_{i_1 j_1}\dots\mathcal{Q}_{i_l j_l}\mathcal{Q}_{s_1 k_1}\dots\mathcal{Q}_{s_d k_d}A^{(a_1,\dots,a_d)}_{j_1\dots j_l k_1\dots k_d}(\mathsf{Z})(\mathcal{Q}_{s_1 m_1}W^{(a_1)}_{m_1})\dots(\mathcal{Q}_{s_d m_d}W^{(a_d)}_{m_d}).$$

On the other hand,

$$T_{i_1\dots i_l}(\mathcal{Q}\mathsf{W},\mathsf{Z}) = \sum_{d=0}^{D}\sum_{\langle a_1,\dots,a_d\rangle\in I_n^d}A^{(a_1,\dots,a_d)}_{i_1\dots i_l s_1\dots s_d}(\mathsf{Z})(\mathcal{Q}_{s_1 m_1}W^{(a_1)}_{m_1})\dots(\mathcal{Q}_{s_d m_d}W^{(a_d)}_{m_d}).$$

Substituting these decomposition in (2.6) and comparing the right- and left-hand sides taking into account the arbitrariness of the vectors (pseudovectors) $\boldsymbol{W}^{(a)}$, $a = 1,\dots,n$, that constitute the tuple W, we have

$$\mathcal{Q}_{i_1 j_1}\dots\mathcal{Q}_{i_l j_l}\mathcal{Q}_{s_1 k_1}\dots\mathcal{Q}_{s_d k_d}A^{(a_1,\dots,a_d)}_{j_1\dots j_l k_1\dots k_d}(\mathsf{Z}) = A^{(a_1,\dots,a_d)}_{i_1\dots i_l s_1\dots s_d}(\mathsf{Z}). \qquad \square$$

Due to Lemma 3.1, to solve the problem of description of covariant functions, it is necessary to examine the general structure of invariant tensors. The set of all invariant tensors in $\mathbb{R}^3$ is described by the following assertion.

Theorem 3.1 (see, e.g., [10, p. 198]). *Each tensor $A_{j_1,\dots,j_n}$ of even rank n invariant under transformations of the group $\mathcal{O}_3$ and, in particular, under transformations of the group $\mathcal{O}_{3,+}$, belongs to the linear span of the tensors $\delta_{j_{k_1}j_{k_2}}\dots\delta_{j_{k_{n-1}}j_{k_n}}$.*

Each tensor $A_{j_1,\dots,j_n}$ of odd rank n invariant under transformations of the group $\mathcal{O}_{3,+}$ belongs to the linear span of the tensors $\varepsilon_{j_{k_1}j_{k_2}j_{k_3}}\delta_{j_{k_4}j_{k_5}}\dots\delta_{j_{k_{n-1}}j_{k_n}}$. Here ε_{jkl} is the Levi-Civita symbol (totally skew-symmetric pseudotensor of rank 3), where $\langle k_1,\dots,k_n\rangle$ are permutations of the set $I_n = \{1,\dots,n\}$. There are no pseudotensor of even rank and tensors of odd rank that are invariant under transformations of the complete group $\mathcal{O}_3$.

We describe bases in each linear spans mentioned above. Let n be an even natural number. We denote by $\mathfrak{c}$ each of the tuples $\{k_i, k_{i+1}\}$, $i \in \{1,3,\dots,n-1\}$, consisting of nonintersecting pair of indices $k_j \in I_n$, $j = 1,\dots,n$, which are called *pairwise partitions* of I_n. Denote by $\mathfrak{C}(I_n)$ the family of all pairwise partitions of the set I_n. Based on Theorem 3.1, we can prove the following assertion.

Theorem 3.2. *The collection of tensors of the form*

$$A^{(+)}_{j_1\dots j_n}(\mathfrak{c}) = \prod_{\{p,q\}\in\mathfrak{c}}\delta_{j_p j_q}, \quad \mathfrak{c}\in\mathfrak{C}(I_n),$$

is a basis in the linear manifold of even-rank tensor that are invariant under the group $\mathcal{O}_3$.

The collection of pseudotensors of the form

$$A^{(-)}_{j_1\dots j_n}(\mathfrak{c};l_1,l_2,l_3) = \varepsilon_{j_{l_1},j_{l_2},j_{l_3}}\prod_{\{p,q\}\in\mathfrak{c}}\delta_{j_p j_q}, \quad \mathfrak{c}\in\mathfrak{C}(I_n\setminus\{l_1,l_2,l_3\}), \quad \{l_1,l_2,l_3\}\subset I_n,$$

where l_1, l_2, l_3 are selected in ascending order, is a basis in the linear manifold of odd-rank pseudotensors that are invariant under the group $\mathcal{O}_{3,+}$, so that each (pseudo)tensor $A_{j_1,\dots,j_n}$ invariant under the group $\mathcal{O}_{3,+}$ can be represented in the form

$$A_{j_1,\dots,j_n} = \sum_{\mathfrak{c}\in\mathfrak{C}(I_n)}\lambda^{(+)}(\mathfrak{c})A^{(+)}_{j_1,\dots j_n}(\mathfrak{c}) + \sum_{\{l_1,l_2,l_3\}\subset I_n}\ \sum_{\mathfrak{c}\in\mathfrak{C}(I_n\setminus\{l_1,l_2,l_3\})}\lambda^{(-)}(\mathfrak{c})A^{(-)}_{j_1\dots j_n}(\mathfrak{c};l_1,l_2,l_3), \qquad (3.3)$$

where the coefficients $\lambda^{(+)}(\mathfrak{c}) = 0$ *if* n *is odd and* $\lambda^{(-)}(\mathfrak{c}) = 0$ *if* n *is even.*

Proof. Due to Theorem 3.1, we must prove separately, the linear independence of all tensors $A^{(+)}_{j_1 \cdots j_n}$ for even n and the linear independence of all pseudotensors $A^{(-)}_{j_1 \cdots j_n}$ for odd n.

Consider the case of even rank n. We apply induction over $n/2$. Assume that the tensors $A^{(+)}_{j_1 \cdots j_{n-2}}$ are linearly independent. Assume that there exists a collection of coefficients $\lambda(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}(I_n)$, such that

$$\sum_{\mathfrak{c} \in \mathfrak{C}(I_n)} \lambda(\mathfrak{c}) A^{(+)}_{j_1 \cdots j_n}(\mathfrak{c}) = 0.$$

For each pair $\{s_1, s_2\} \subset I_n$, we extract in this equality the sum

$$D^{(s_1,s_2)}_{n-2} = \sum_{\mathfrak{c} \in \mathfrak{C}(I_n \setminus \{s_1,s_2\})} \lambda(\mathfrak{c}, \{s_1, s_2\}) \prod_{\{k,l\} \in \mathfrak{c}} \delta_{j_k, j_l}$$

and write the relation of the linear independence as follows:

$$\delta_{j_{s_1} j_{s_2}} D^{(s_1,s_2)}_{n-2} + \sum_{\{k_1,k_2\} \in I_n \setminus \{s_1,s_2\}} \sum_{\mathfrak{c} \in \mathfrak{C}(I_n \setminus \{s_1,s_2,k_1,k_2\})} \prod_{\{l,m\} \in \mathfrak{c}} \delta_{j_l j_m}$$
$$\times \Big[\delta_{j_{s_1} j_{k_1}} \delta_{j_{k_2} j_{s_2}} \lambda\big(\mathfrak{c} \cup \big\{\{s_1, k_1\}, \{s_2, k_2\}\big\}\big) + \delta_{j_{s_1} j_{k_2}} \delta_{j_{k_1} j_{s_2}} \lambda\big(\mathfrak{c} \cup \big\{\{s_1, k_1\}, \{s_2, k_2\}\big\}\big) \Big] = 0.$$

We perform the convolution in the left-hand side of this equality with respect to the indices j_{s_1} and j_{s_2}:

$$3 D^{(s_1,s_2)}_{n-2} + \sum_{\{k_1,k_2\} \in I_n \setminus \{s_1,s_2\}} \sum_{\mathfrak{c} \in \mathfrak{C}(I_n \setminus \{s_1,s_2,k_1,k_2\})} \delta_{j_{k_1} j_{k_2}} \prod_{\{l,m\} \in \mathfrak{c}} \delta_{j_l j_m}$$
$$\times \Big(\lambda\big(\mathfrak{c} \cup \big\{\{s_1, k_1\}, \{s_2, k_2\}\big\}\big) + \lambda\big(\mathfrak{c} \cup \big\{\{s_1, k_1\}, \{s_2, k_2\}\big\}\big) \Big) = 0.$$

Next, we change the order of summation and rewrite the second term as follows:

$$\sum_{\mathfrak{c} \in \mathfrak{C}(I_n \setminus \{s_1,s_2\})} \prod_{\{l,m\} \in \mathfrak{c}} \delta_{j_l j_m} \sum_{\{k_1,k_2\} \in I_n \setminus \{s_1,s_2\} : \{k_1,k_2\} \in \mathfrak{c}}$$
$$\times \Big(\lambda\Big(\big(\mathfrak{c} \setminus \{\{k_1, k_2\}\}\big) \cup \big\{\{s_1, k_1\}, \{s_2, k_2\}\big\} \Big) + \lambda\Big(\big(\mathfrak{c} \setminus \{\{k_1, k_2\}\}\big) \cup \big\{\{s_1, k_1\}, \{s_2, k_2\}\big\} \Big) \Big).$$

Due to the inductive hypothesis on the linear independence of the collection of all tensors $A^{(+)}_{j_1, \ldots, j_{n-2}}$, equating to zero the coefficients of tensors in this relation, we obtain the following system of linear equations for $(n-1)!!$ coefficients $\lambda(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}(I_n)$:

$$\big[(3 \cdot \mathbf{1} + \mathsf{R})\lambda\big](\mathfrak{c}'; s) = 3\lambda(\mathfrak{c}'; s) + \sum_{\substack{\{k_1,k_2\} \subset I_n \setminus \{1,s\}: \\ \{k_1,k_2\} \in \mathfrak{c}'}} \Big(\lambda(\mathfrak{c}'; k_1)\big|_{k_2 = s} + + \lambda(\mathfrak{c}'; k_2)\big|_{k_1 = s} \Big) = 0. \qquad (3.4)$$

Here the coefficients $\lambda(\mathfrak{c}'; s) \equiv \lambda(\mathfrak{c}' \cup \{\{1, s\}\})$ are marked by $(n-3)!!$ partitions $\mathfrak{c}' \in \mathfrak{C}(I_n \setminus \{1, s\})$ and $s = 2, \ldots, n$. Therefore, the system consists of $(n-3)!!(n-1)$ equations; the number of these equations coincides with the number of unknown coefficients $\lambda(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}(I_n)$.

The system of equations (3.4) is represented as the equality to zero of the image of the transformation by the operator $(3 \cdot \mathbf{1} + \mathsf{R})$ of a given set of coefficients $\{\lambda(\mathfrak{c}'; s)\}$, which acts in the space $\mathbb{R}^{(n-1)!!}$ of tuples $\{\lambda(\mathfrak{c}'; s); \mathfrak{c}' \in \mathfrak{C}(I_n \setminus \{1, s\}), s = 2, \ldots, n\}$. Naturally, this operator consists of two terms. The first term is the multiplication of a set of coefficients by 3. The matrix elements of the second term R are nonzero if and only if the tuple $\mathfrak{c}'$ does not contain the pair $\{1, s\}$; in this case the matrix element is equal to 1. Therefore, the sum of all matrix elements in a fixed column of the matrix of the operator $3 \cdot \mathbf{1} + \mathsf{R}$ is equal to $(n+1)$, i.e., is independent of the row of the matrix. Then, if nonzero components of the set $\{\lambda(\mathfrak{c}'); \mathfrak{c}' \in \mathfrak{C}(I_n \setminus \{1, s\})\}$ are only the components containing the pair $\{1, s\}$ with fixed

$s \in \{2, \dots, n\}$ in the partition $\mathfrak{c}'$, then this set is transformed by the operator $\mathbf{1} + \mathsf{R}$ into the set $\{\lambda(\mathfrak{c}'; s)\}$ with the same property. In this case, all these new nonzero coefficients are equal to

$$\sum_{\mathfrak{c}' \in \mathfrak{C}(I_n \setminus \{1,s\})} \lambda(\mathfrak{c}'; s).$$

This means that the operator $(\mathbf{1}+\mathsf{R})/(n-1)$ is idempotent, $(\mathbf{1}+\mathsf{R})^2 = \mathbf{1}+\mathsf{R}$ and hence its eigenvalues are 0 and/or 1. Therefore, $\det(3 \cdot \mathbf{1} + \mathsf{R}) \neq 0$. Then Eq. (3.4) has only trivial solution $\lambda(\mathfrak{c}'; s) = 0$.

The proof for the case of odd n is similar and we omit it. □

4. Coefficients of covariant differential operators. In this section, we formulate the main results of the work. We present a description of the general tensor structure of the coefficients of first-order covariant differential operators.

For fixed $\mathfrak{c} \in \mathfrak{C}(I_{l+d})$, consider the convolutions $W_{k_1}^{(a_1)} \dots W_{k_d}^{(a_d)}$ with the tensor $A^{(+)}_{j_1,\dots j_l k_1 \dots k_d}(\mathfrak{c})$:

$$A^{(+)}_{j_1,\dots j_l k_1 \dots k_d}(\mathfrak{c}) W_{k_1}^{(a_1)} \dots W_{k_d}^{(a_d)} = \Bigl(\prod_{\substack{\{p,q\}\in\mathfrak{c}:\\ \{p,q\}\subset I_l}} \delta_{j_p j_q}\Bigr)\Bigl(\prod_{\substack{\{p,q\}\in\mathfrak{c}:\\ \{p,q\}\subset I_{l+d}\setminus I_l}} (\boldsymbol{W}^{(a_p)}, \boldsymbol{W}^{(a_q)})\Bigr)\Bigl(\prod_{\substack{\{p,q\}\in\mathfrak{c}:\\ p\in I_l, q\in I_{l+d}\setminus I_l}} W_{j_p}^{(a_q)}\Bigr), \quad (4.1)$$

and the tensor $A^{(-)}_{j_1,\dots j_l k_1 \dots k_d}(\mathfrak{c})$:

$$A^{(-)}_{j_1,\dots j_l k_1 \dots k_d}(\mathfrak{c}) W_{k_1}^{(a_1)} \dots W_{k_d}^{(a_d)} = \varepsilon_{l_1,l_2,l_3}(\mathsf{W})\Bigl(\prod_{\substack{\{p,q\}\in\mathfrak{c}:\\ \{p,q\}\subset I_l\setminus\{l_2,l_2,l_3\}}} \delta_{j_p j_q}\Bigr)$$
$$\times \Bigl(\prod_{\substack{\{p,q\}\in\mathfrak{c}:\\ \{p,q\}\subset I_{l+d}\setminus(I_l\cup\{l_1,l_2,l_3\})}} (\boldsymbol{W}^{(a_p)}, \boldsymbol{W}^{(a_q)})\Bigr)\Bigl(\prod_{\substack{\{p,q\}\in\mathfrak{c}:\\ p\in I_l\setminus\{l_1,l_2,l_3\},\\ q\in I_{l+d}\setminus(I_l\cup\{l_1,l_2,l_3\})}} W_{j_p}^{(a_q)}\Bigr). \quad (4.2)$$

In the latter case, there exist for possibilities, depending on the number of elements s in the intersection $E(l_1, l_2, l_3) = (I_{l+d} \setminus I_l) \cap \{l_1, l_2, l_3\}$, so that

(1) $\varepsilon_{l_1,l_2,l_3}(\mathsf{W}) = \varepsilon_{j_{l_1} j_{l_2} j_{l_3}}$ for $s = 0$;

(2) $\varepsilon_{l_1,l_2,l_3}(\mathsf{W}) = \varepsilon_{j_{l_1} j_{l_2} k_{l_3}} W_{k_{l_3}}^{(a_{l_3})}$ for $s = 1$ and $l_3 \in E(l_1, l_2, l_3)$;

(3) $\varepsilon_{l_1,l_2,l_3}(\mathsf{W}) = \varepsilon_{j_{l_1} k_{l_2} k_{l_3}} W_{k_{l_2}}^{(a_{l_2})} W_{k_{l_3}}^{(a_{l_3})}$ for $s = 2$ with the set $\{l_2, l_3\} \subset E(l_1, l_2, l_3)$;

(4) $\varepsilon_{l_1,l_2,l_3}(\mathsf{W}) = \varepsilon_{k_{l_1} k_{l_2} k_{l_3}} W_{k_{l_1}}^{(a_{l_1})} W_{k_{l_2}}^{(a_{l_2})} W_{k_{l_3}}^{(a_{l_3})}$ for $s = 3$ with $\{l_1, l_2, l_3\} = E(l_1, l_2, l_3)$.

Now we apply the formulas obtained to the calculation of the tensor-valued functions $T_k^{(a,b)}$, $T_{jk}^{(a,b)}$, $\tilde{T}_{kl}^{(a,b)}$, and $T_{jkl}^{(a,b)}$. We represent $T_k^{(a,b)}$ by the decomposition (3.1) with $l = 1$ for $a, b = n+1, \dots, n+r$,

$$T_k^{(a,b)}(\mathsf{W}, \mathsf{Z}) = \sum_{d=0}^{D} \sum_{\langle a_1,\dots,a_d\rangle \in I_n^d} A_{kk_1\dots k_d}^{(a,b;a_1,\dots,a_d)}(\mathsf{Z}) W_{k_1}^{(a_1)} \dots W_{k_d}^{(a_d)}, \quad (4.3)$$

where the coefficients are invariant tensors defined by the formula (3.3) and the coefficients $\lambda^{(\pm)}_{a,b;a_1,\dots,a_d}(\mathfrak{c}, \mathsf{Z})$ are functions of the tuple Z. Therefore, we have

$$A_{kk_1,\dots,k_d}^{(a,b;a_1,\dots,a_d)}(\mathsf{Z}) = \sum_{\mathfrak{c}\in\mathfrak{C}(I_{d+1})} \lambda^{(+)}_{a,b;a_1,\dots,a_d}(\mathfrak{c}, \mathsf{Z}) A^{(+)}_{kk_1,\dots k_d}(\mathfrak{c}), \ d \text{ is odd}; \quad (4.4)$$

$$A_{kk_1,\dots,k_d}^{(a,b;a_1,\dots,a_d)}(\mathsf{Z}) = \sum_{\{l_1,l_2,l_3\}\subset I_{d+1}} \sum_{\mathfrak{c}\in\mathfrak{C}(I_{d+1}\setminus\{l_1,l_2,l_3\})} \lambda^{(-)}_{a,b;a_1,\dots,a_d}(\mathfrak{c}, \mathsf{Z}) A^{(-)}_{kk_1\dots k_d}(\mathfrak{c}; l_1, l_2, l_3), \ d \text{ is even}. \quad (4.5)$$

Transforming the sums in (4.3) by the formulas (4.1) and (4.2), we arrive at the expressions

$$T_k^{(a,b)}(\mathsf{W},\mathsf{Z}) = \sum_{c=1}^{n} W_k^{(c)} T_+^{(a,b;c)}(\mathsf{W},\mathsf{Z}) + \sum_{a',b'=1}^{n} [\boldsymbol{W}^{(a')},\boldsymbol{W}^{(b')}]_k T_-^{(a,b;a'b')}(\mathsf{W},\mathsf{Z}), \tag{4.6}$$

where we used the notation for the vector product of a pair vectors in $\mathbb{R}^3$. In the formula (4.6), the coefficients $T_+^{(a,b;c)}(\mathsf{W},\mathsf{Z})$ and $T_-^{(a,b;a',b')}(\mathsf{W},\mathsf{Z})$ are functions depending on a tuple of scalars Z and a tuple of invariants[2] consisting of a tuple of vectors W. We present their explicit polynomial dependence on the vectors $\boldsymbol{W}^{(a)}$, $a=1,\ldots,n$. The function $T_+^{(a,b;c)}(\mathsf{W},\mathsf{Z})$ has the form

$$\begin{aligned} T_+^{(a,b;c)}(\mathsf{W},\mathsf{Z}) = \sum_{d=0}^{D} \sum_{\langle a_1,\ldots,a_d\rangle \in I_n^d} \sum_{s=1}^{d} \delta_{ca_s} \Bigg[&\sum_{\substack{\mathfrak{c}\in\mathfrak{C}(I_{d+1}):\\ \{0,s\}\in\mathfrak{c}}} \lambda^{(+)}_{a,b;a_1,\ldots,a_d}(\mathfrak{c},\mathsf{Z}) \prod_{\substack{\{p,q\}\in\mathfrak{c};\\ \{p,q\}\in I_d}} \left(\boldsymbol{W}^{(a_p)},\boldsymbol{W}^{(a_q)}\right) \\ &+ \sum_{\{l_1,l_2,l_3\}\subset I_d} \left(\boldsymbol{W}^{(a_{l_1})},\left[\boldsymbol{W}^{(a_{l_2})},\boldsymbol{W}^{(a_{l_3})}\right]\right) \\ &\times \sum_{\substack{\mathfrak{c}\in\mathfrak{C}(I_{d+1}\setminus\{l_1,l_2,l_3\}):\\ \{0,s\}\in\mathfrak{c}}} \lambda^{(-)}_{a,b;a_1,\ldots,a_d}(\mathfrak{c},\mathsf{Z}) \prod_{\substack{\{p,q\}\in\mathfrak{c};\\ \{p,q\}\in I_d\setminus\{l_1,l_2,l_3\}}} \left(\boldsymbol{W}^{(a_p)},\boldsymbol{W}^{(a_q)}\right)\Bigg], \end{aligned} \tag{4.7}$$

where the zero label is assigned to the index k in the original expression and the notation for the scalar and mixed products of vectors in $\mathbb{R}^3$ are used. Since $a,b=n+1,\ldots,n+r$, this function must be scalar for $c=1,\ldots,p$ and pseudoscalar for $c=p+1,\ldots,p+q$. The latter fact can only take place in the case where the tuple of arguments of this function contains pseudoscalars. For this it is necessary that $n>1$, for $n=2$ the tuple W consists of a vector $\boldsymbol{W}^{(1)}$ and a pseudovector $\boldsymbol{W}^{(2)}$, whereas for $n\ge 3$, the tuple W satisfies the condition $p\neq 0$, i.e., contains at least one vector. The function is defined by the formula

$$\begin{aligned} T_-^{(a,b;a',b')}(\mathsf{W},\mathsf{Z}) = \sum_{d\ \text{even}}^{D} \sum_{\langle a_1,\ldots,a_d\rangle\in I_n^d} \sum_{\{l_2,l_3\}\subset I_d} \delta_{a_{l_2}a'}\delta_{a_{l_3}b'} \\ \times \sum_{\substack{\mathfrak{c}\in\mathfrak{C}(I_{d+1}\setminus\{l_2,l_3\}):\\ \{0,s\}\in\mathfrak{c}}} \lambda^{(-)}_{a,b;a_1,\ldots,a_d}(\mathfrak{c},\mathsf{Z}) \prod_{\substack{\{p,q\}\in\mathfrak{c};\\ \{p,q\}\in I_d\setminus\{l_2,l_3\}}} \left(\boldsymbol{W}^{(a_p)},\boldsymbol{W}^{(a_q)}\right); \end{aligned} \tag{4.8}$$

it can be scalar or pseudoscalar.

Obviously, for the vector-valued function $F_k^{(a)}(\mathsf{W},\mathsf{Z})$, $a=1,\ldots,n$, we have the formula

$$F_k^{(a)}(\mathsf{W},\mathsf{Z}) = \sum_{b=1}^{n} W_k^{(b)} F_+^{(a;c)}(\mathsf{W},\mathsf{Z}) + \sum_{b,c=1}^{n} [\boldsymbol{W}^{(b)},\boldsymbol{W}^{(c)}]_k F_-^{(a;b,c)}(\mathsf{W},\mathsf{Z})\,, \tag{4.9}$$

which is similar to (4.6); the coefficients $F_+^{(a;b)}$, $F_-^{(a;b,c)}$ have the same properties with respect to total reflections from the group $\mathcal{O}_3$. Their explicit expressions are similar to (4.7)–(4.8).

Formulas for the tensor-valued functions $T_{jk}(\mathsf{W},\mathsf{Z})$, $\bar{T}_{kl}(\mathsf{W},\mathsf{Z})$, and $T_{jkl}(\mathsf{W},\mathsf{Z})$ are similar to (4.3). They can be obtained by the same method based on the substitution in the formulas

$$T_{jk}^{(a,b)}(\mathsf{W},\mathsf{Z}) = \sum_{d=0}^{D} \sum_{\langle a_1,\ldots,a_d\rangle\in I_n^d} A_{jkk_1\ldots k_d}^{(a,b;a_1,\ldots,a_d)}(\mathsf{Z}) W_{k_1}^{(a_1)}\ldots W_{k_d}^{(a_d)}, \tag{4.10}$$

[2]Here and below, we do not explicitly specify the set of invariants, i.e., we do not discuss the question of the nature of the dependence of scalar functions on the elements of the *whole rational basis* (see, e.g., [13]).

$$\tilde{T}^{(a,b)}_{kl}(\mathsf{W},\mathsf{Z}) = \sum_{d=0}^{D} \sum_{\langle a_1,\dots,a_d\rangle \in I^d_n} \tilde{A}^{(a,b;a_1,\dots,a_d)}_{klk_1\dots k_d}(\mathsf{Z}) W^{(a_1)}_{k_1} \dots W^{(a_d)}_{k_d}, \tag{4.11}$$

$$T^{(a,b)}_{jkl}(\mathsf{W},\mathsf{Z}) = \sum_{d=0}^{D} \sum_{\langle a_1,\dots,a_d\rangle \in I^d_n} A^{(a,b;a_1,\dots,a_d)}_{jklk_1\dots k_d}(\mathsf{Z}) W^{(a_1)}_{k_1} \dots W^{(a_d)}_{k_d}. \tag{4.12}$$

We present their final form, omitting explicit expressions for the coefficients of the decomposition in terms of the elements of the basis of the corresponding tensor representation. These formulas are obtained by substituting into (4.10), (4.11), (4.12) the expressions for the coefficients $A^{(a,b;a_1,\dots,a_d)}_{jkk_1\dots k_d}(\mathsf{Z})$, $\tilde{A}^{(a,b;a_1,\dots,a_d)}_{klk_1\dots k_d}(\mathsf{Z})$, and $A^{(a,b;a_1,\dots,a_d)}_{jklk_1\dots k_d}(\mathsf{Z})$, which are similar to (4.4) and (4.5), with their subsequent transformation based on (4.1) and (4.2). As a result, we obtain the following formulas:

$$\begin{aligned} T^{(a,b)}_{jk}(\mathsf{W},\mathsf{Z}) &= \delta_{jk} S^{+}_{a,b}(\mathsf{W},\mathsf{Z}) + \sum_{a',b'=1}^{n} W^{(a')}_j W^{(b')}_k S^{(00)}_{a,b;a',b'}(\mathsf{W},\mathsf{Z}) + \varepsilon_{jkm} \sum_{c=1}^{n} W^{(c)}_m S^{-}_{a,b;c}(\mathsf{W},\mathsf{Z}) \\ &\quad + \sum_{a',b',c=1}^{n} \Big[W^{(c)}_j \varepsilon_{kpq} W^{(a')}_p W^{(b')}_q S^{(01)}_{a,b;c}(\mathsf{W},\mathsf{Z}) + W^{(c)}_k \varepsilon_{jpq} W^{(a')}_p W^{(b')}_q S^{(10)}_{a,b;c}(\mathsf{W},\mathsf{Z}) \Big]; \end{aligned} \tag{4.13}$$

$$\begin{aligned} \tilde{T}^{(a,b)}_{kl}(\mathsf{W},\mathsf{Z}) &= \delta_{kl} \tilde{S}^{+}_{a,b}(\mathsf{W},\mathsf{Z}) + \sum_{a',b'=1}^{n} W^{(a')}_k W^{(b')}_l \tilde{S}^{(00)}_{a,b;a',b'}(\mathsf{W},\mathsf{Z}) + \varepsilon_{klm} \sum_{c=1}^{n} W^{(c)}_m \tilde{S}^{-}_{a,b;c}(\mathsf{W},\mathsf{Z}) \\ &\quad + \sum_{a',b',c=1}^{n} \Big[W^{(c)}_k \varepsilon_{lpq} W^{(a')}_p W^{(b')}_q \tilde{S}^{(01)}_{a,b;c}(\mathsf{W},\mathsf{Z}) + W^{(c)}_l \varepsilon_{kpq} W^{(a')}_p W^{(b')}_q \tilde{S}^{(10)}_{a,b;c}(\mathsf{W},\mathsf{Z}) \Big]; \end{aligned} \tag{4.14}$$

$$\begin{aligned} T^{(a,b)}_{jkl}(\mathsf{W},\mathsf{Z}) &= \sum_{c=1}^{n} \Big[\delta_{jk} W^{(c)}_i U^{(1)}_{a,b;c}(\mathsf{W},\mathsf{Z}) + \delta_{jl} W^{(c)}_k U^{(2)}_{a,b;c}(\mathsf{W},\mathsf{Z}) + \delta_{kl} W^{(c)}_j U^{(3)}_{a,b;c}(\mathsf{W},\mathsf{Z}) \Big] \\ &\quad + \sum_{a',b',c'=1}^{n} W^{(a')}_j W^{(b')}_k W^{(c')}_l U^{+}_{a,b;a',b',c'}(\mathsf{W},\mathsf{Z}) + \varepsilon_{jkl} U^{-}_{a,b}(\mathsf{W},\mathsf{Z}) \\ &\quad + \sum_{a',b'=1}^{n} \Big[\delta_{jk} \varepsilon_{lpq} W^{(a')}_p W^{(b')}_q U^{(01)}_{a,b;a',b'}(\mathsf{W},\mathsf{Z}) + \delta_{jl} \varepsilon_{kpq} W^{(a')}_p W^{(b')}_q U^{(02)}_{a,b;a',b'}(\mathsf{W},\mathsf{Z}) \\ &\quad + \delta_{kl} \varepsilon_{kpq} W^{(a')}_p W^{(b')}_q U^{(03)}_{a,b;a',b'}(\mathsf{W},\mathsf{Z}) + W^{(a')}_j \varepsilon_{klm} W^{(b')}_m U^{(11)}_{a,b;a',b'}(\mathsf{W},\mathsf{Z}) \\ &\quad + W^{(a')}_k \varepsilon_{ljm} W^{(b')}_m U^{(12)}_{a,b;a',b'}(\mathsf{W},\mathsf{Z}) + W^{(a')}_l \varepsilon_{jkm} W^{(b')}_m U^{(13)}_{a,b;a',b'}(\mathsf{W},\mathsf{Z}) \Big] \\ &\quad + \sum_{a_1,b_1,a_2,b_2=1}^{n} \Big[W^{(a_1)}_j W^{(b_1)}_k \varepsilon_{lpq} W^{(a_2)}_p W^{(b_2)}_q U^{(21)}_{a,b;a_1,b_1,a_2,b_2}(\mathsf{W},\mathsf{Z}) \\ &\quad + W^{(a_1)}_j W^{(b_1)}_l \varepsilon_{kpq} W^{(a_2)}_p W^{(b_2)}_q U^{(22)}_{a,b;a_1,b_1,a_2,b_2}(\mathsf{W},\mathsf{Z}) \\ &\quad + W^{(a_1)}_k W^{(b_1)}_l \varepsilon_{jpq} W^{(a_2)}_p W^{(b_2)}_q U^{(23)}_{a,b;a_1,b_1,a_2,b_2}(\mathsf{W},\mathsf{Z}) \Big]. \end{aligned} \tag{4.15}$$

Now we analyze the formulas obtained. The coefficients $T^{(a,b)}_{jk}$ are tensors for $a = 1,\dots,p$ and pseudotensors for $a = p+1,\dots,n$, where $b = n+1,\dots,n+r$. Similarly, the coefficients $\tilde{T}^{(a,b)}_{kl}$ are tensors for $b = 1,\dots,p$ and pseudotensors for $b = p+1,\dots,n$, where $a = n+1,\dots,n+r$. Moreover, the coefficients $T^{(a,b)}_{jkl}$ are tensors for $a,b = 1,\dots,p$ and $a,b = p+1,\dots,n$ and pseudotensors for $a = 1,\dots,p$

and $b = p+1, \dots, n$ or for $a = p+1, \dots, n$ and $b = 1, \dots, p$. In this connection, restrictions on the choice of scalar (pseudoscalar) coefficients (4.13)–(4.15) may arise. Such restrictions do not arise if $n \geq 3$, since for such number of vectors (pseudovectors), there are pseudoscalar invariants of the group $\mathcal{O}_{3,+}$. Restrictions arise precisely in the most important cases from the point of view of applications in physics: $n = 1$ and $n = 2$.

If $n = 1, 2$ and we consider vector fields, then the conditions mentioned will be fulfilled if $S_{-}^{(a,b;c)} = 0$ in the tensor $T_{jk}^{(a,b)}$ and, moreover, $S_{01}^{(a,b;c)} = S_{10}^{(a,b;c)} = 0$ for $n = 1$. The same condition appears for the tensor $\tilde{T}_{kl}^{(a,b)}$: $\tilde{S}_{-}^{(a,b;c)} = 0$ and, moreover, $\tilde{S}_{01}^{(a,b;c)} = \tilde{S}_{10}^{(a,b;c)} = 0$ for $n = 1$. The coefficients of the tensors $T_{jkl}^{(a,b)}$ for $n = 1, 2$ must satisfy the conditions $U_{a,b}^{-} = 0$ and $U_{\dots\dots}^{(p,q)} = 0$ for $p = 0, 1, 2$ and $q = 1, 2, 3$.

If $n = 1, 2$ and we consider pseudovector fields, then the conditions mentioned will be fulfilled if $T_{jk}^{(a,b)} = 0$ and $\tilde{T}_{kl}^{(a,b)} = 0$. The coefficients of the tensor $T_{jkl}^{(a,b)}$ for $n = 1, 2$ must satisfy the conditions $U_{\dots\dots}^{(p,q)} = 0$ for $p = 0, 1, 2$ and $q = 1, 2, 3$, since the group $\mathcal{O}_3$ has no pseudoscalar invariants.

Finally, we consider the case $n = 2$, where $\boldsymbol{W}^{(1)}$ is a vector field and $\boldsymbol{W}^{(2)}$ is a pseudovector field. In this case, $T_{jk}^{(1,b)}$ and $\tilde{T}_{kl}^{(a,1)}$ are tensors and $T^{(2,b)}$ and $\tilde{T}_{kl}^{(a,2)}$ are pseudotensors for $a, b = 3, \dots, 2+r$; $T_{jkl}^{(a,a)}$, $a = 1, 2$, are tensors, whereas $T_{jkl}^{(1,2)}$ and $T_{jkl}^{(1,2)}$ are pseudotensors. Then the coefficients of $T_{jk}^{(1,b)}$ and $\tilde{T}_{kl}^{(a,1)}$ do not involve terms that are identically equal to zero. The coefficients of $T_{jk}^{(2,b)}$ and $\tilde{T}_{kl}^{(a,2)}$ must satisfy the condition $S_{1,b}^{+} = \tilde{S}_{a,2}^{+} = 0$. For the tensors $T_{jkl}^{(a,a)}$, $a = 1, 2$, the condition $U_{a,a}^{-} = 0$ must be fulfilled, whereas the tensors $T_{jkl}^{(1,2)}$ and $T_{jkl}^{(2,1)}$ may contain all terms involved in (4.13) and (4.15).

Remark. The results presented in this paper give a general form of the first-order covariant differential operator $\mathsf{L}[\cdot]$ without imposing conditions on operators of this class that guarantee the hyperbolicity of systems of equations (2.5).

5. Conclusion. The formulas (4.6), (4.9), and (4.13)–(4.15) solve the problem of listing all possible covariant first-order differential operators for vector and scalar fields on $\mathbb{R}^3$. They provide a method for constructing physically adequate dissipative-free evolution equations, since scalar and vector fields are the main types of fields used in theoretical physics. The significance of the resulting description is that, on their basis, it is possible to compose translationally invariant and covariant differential operators in the form of sums of elementary differential operators with coefficients represented by some functions of field invariants, excluding from the presented list operators that do not satisfy the stated conditions. Although tensor fields are used to a much lesser extent in describing the dynamics of condensed media, it would be desirable to conduct a similar study for them. However, it is ore important to describe covariant second-order differential operators, which are more realistic from the point of view of physical applications, since they allow for dissipative processes to be taken into account. Such studies have been carried out in special cases, the results of which are presented in [12, 14].

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