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# HYPERBOLIC QUASILINEAR COVARIANT FIRST-ORDER EQUATIONS OF DIVERGENT TYPE FOR VECTOR FIELDS ON $\mathbb{R}^3$

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***Abstract.*** In this paper, we present a complete description of the class of first-order hyperbolic quasilinear equations of divergent type that describe the change in time $t \in \mathbb{R}$ of vector fields $\boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbb{R}^3$ that are invariant under translations in time $t \in \mathbb{R}$ and space $\mathbb{R}^3$ and transform covariantly under the action of the rotation group $\mathbb{O}_3$ of the space $\mathbb{R}^3$. This class is compared with the class of similar equations that are hyperbolic in the sense of Friedrichs.

***Keywords and phrases***: quasilinear system, equation of divergent type, hyperbolicity, translational invariance, vector field, covariance, flux density.

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**1. Introduction.** In [10, 12, 13], for solving fundamental problems of nonequilibrium thermodynamics of condensed media, the problem was posed of classifying systems of differential equations of divergent type describing the evolution of a fixed set of fields on Euclidean space on the basis of equations that satisfy certain physically natural conditions. Namely, equations of this kind, in the absence of external influences, must be invariant under the group of time shifts and the group of translations of the space $\mathbb{R}^3$. In addition, they must transform in a special way under rotations of $\mathbb{R}^3$. In the simplest spherically symmetric case, which is the object of study in the present work, they must transform covariantly under the action of the group $\mathbb{O}_3$ as its tensor representations (see, e.g., [6]). However, these works did not raise the question of whether the solutions of the constructed systems of differential equations possess physically "reasonable" properties necessary for the possibility of using them in modeling the evolution of those physical systems for which they are actually intended. The present work is devoted to finding such conditions for elements of the class $\mathfrak{K}_1(\mathbb{R}^3)$ consisting of first-order quasilinear equations of divergence type that describe the vector field $\boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbb{R}^3$, as $t \in \mathbb{R}$ changes, and such that satisfy the above conditions, namely, the coefficients of these equations do not depend explicitly on either $t$ or $\boldsymbol{x}$, and the equations themselves are transformed under rotations of $\mathbb{R}^3$ as vectors in $\mathbb{R}^3$.

First-order evolutionary equations, which are used to describe continuous media, as a rule, do not describe the influence of their inherent physical dissipative mechanisms on their dynamics. For this reason, systems of first-order hyperbolic equations seem physically reasonable (see, e.g., [3, 9]). From the physical point of view, the requirement of hyperbolicity of the system of first-order equations means that all solutions $\omega_j(\boldsymbol{k})$, $j = 1, 2, 3$, of the corresponding "dispersion" equation (i.e., the relationship between the frequency of the field $\boldsymbol{v}(\boldsymbol{x}, t)$ and the wave vector $\boldsymbol{k} \in \mathbb{R}^3$) are real. In this paper, we establish necessary and sufficient hyperbolicity conditions for systems of first-order equations of the type indicated above.

**2. Covariant vector equations.** Consider the following evolutionary equation of divergent type for the vector field $\boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$:

$$\dot{v}_j(\boldsymbol{x}, t) = (\nabla_k S_{jk}[\boldsymbol{v}])(\boldsymbol{x}, t), \quad j = 1, 2, 3 \tag{1}$$

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($\nabla_j \equiv \partial/\partial x_j$, $j = 1, 2, 3$), where $S_{jk}[\boldsymbol{v}]$ is a function of this field at the current spatiotemporal point $\langle \boldsymbol{x}, t\rangle$. In the formula (1) and below, we use the rule of tensor algebra according to which the presence of repeating vector indices (in this case, subscripts $k$) means summation over all their admissible values ??1, 2, and 3. Thus, Eq. (1) is a first-order equation and is a system of three quasilinear equations.

We formulate the requirement of covariance of Eq. (1) with respect to transformations of the rotation group $\mathbb{O}_3$. Under rotations of the space $\mathbb{R}^3$, described by an orthogonal matrix $\mathsf{U} \in \mathbb{O}_3$, each position vector $\boldsymbol{x}$ goes into the vector $\boldsymbol{x}' = \mathsf{U}\boldsymbol{x}$. In this case, the element of the matrix-valued function $(S_{jk}[\boldsymbol{v}])(\boldsymbol{x}, t)$, $j, k = 1, 2, 3$, goes into $(S_{jk}[\boldsymbol{v}'])(\boldsymbol{x}', t) = (S_{jk}[\mathsf{U}\boldsymbol{v}])(\mathsf{U}\boldsymbol{x}, t)$ with realizations of the vector field $\boldsymbol{v}'(\boldsymbol{x}', t) = \mathsf{U}\boldsymbol{v}(\mathsf{U}\boldsymbol{x}, t)$. Then after the rotation of space, Eq. (1) takes the form

$$\dot{v}'_j(\boldsymbol{x}, t) = (\nabla'_k S_{jk}[\boldsymbol{v}'])(\boldsymbol{x}', t), \quad j = 1, 2, 3, \tag{2}$$

where $\nabla'_k(\cdot) = (\nabla'_k x_l) \cdot \nabla_l(\cdot) = U_{kl}\nabla_l(\cdot)$, since $(\mathsf{U}^{-1})_{lk} = U_{kl} \equiv (\mathsf{U})_{kl}$ for orthogonal matrices. Substituting the explicit expressions for $\nabla'_k x_l = U_{kl}$ and the function $S_{jk}[\boldsymbol{v}']$ into Eq. (2), we obtain

$$\dot{v}_m(\boldsymbol{x}, t) = \nabla_l(U_{jm}U_{kl}S_{jk}[\mathsf{U}\boldsymbol{v}])(\boldsymbol{x}, t), \quad j = 1, 2, 3.$$

The covariance requirement is that Eq. (2) must coincide with Eq. (1), i.e., their sets of solution must be the same. This requirement leads to the following condition on the choice of the function $S_{jk}[\boldsymbol{v}]$:

$$U_{jm}U_{kl}S_{ml}[\boldsymbol{v}] = S_{jk}[\mathsf{U}\boldsymbol{v}]. \tag{3}$$

Denote by $\mathfrak{K}_1(\mathbb{R}^3)$ the class of Eqs. (1) with matrix-valued functions $S_{jk}[\boldsymbol{v}]$ that possess the property (3); this class is called the class of spherically symmetric equations. The following assertion describes the class $\mathfrak{K}_1(\mathbb{R}^3)$.

**Theorem 1.** *A function $S_{jk}[\boldsymbol{v}]$ satisfies Eq. (3) if and only if*

$$S_{jk}[\boldsymbol{v}] = f(\boldsymbol{v}^2)\delta_{jk} + g(\boldsymbol{v}^2)v_j v_k, \tag{4}$$

*where $f$ and $g$ are functions on $\mathbb{R}_+$.*

*Proof.* For $\mathsf{U} = \mathbf{1}$, the formula (4) becomes an identity. Then for arbitrary transformation $\mathsf{U} \in \mathbb{O}_3$, due to (4), the matrix $S_{jk}[\boldsymbol{v}]$ is transformed to $U_{jm}U_{kl}S_{ml}[\boldsymbol{v}]$. Therefore, $S_{jk}[\boldsymbol{v}]$ is transformed as a (covariant) tensor of rank 2. We decompose this tensor with respect to some basis $w^{(a)}_{jk}[\boldsymbol{v}]$, $a = 1, \ldots, 9$, of the tensor representation of the group $\mathbb{O}_3$ (see, e.g., [6]),

$$S_{jk}[\boldsymbol{v}] = \sum_{a=1}^{9} h^{(a)}(\boldsymbol{v}^2)w^{(a)}_{jk}[\boldsymbol{v}],$$

with the coefficients $h^{(a)}$, which are functions $h^{(a)} : \mathbb{R}_+ \mapsto \mathbb{R}$ of the unique invariant $\boldsymbol{v}^2$ of the vector $\boldsymbol{v}$, and elements of the basis $w^{(a)}_{jk}[\boldsymbol{v}]$, $a = 1, \ldots, 9$, are, in general, functions of $\boldsymbol{v}$. These basis elements, due to the arbitrariness of the functions $h^{(a)}$, $a = 1, \ldots, 9$, and the relation (4), can be chosen so that the relations $w^{(a)}_{jk}[\mathsf{U}\boldsymbol{v}] = U_{jm}U_{kl}w^{(a)}_{ml}[\boldsymbol{v}]$ are fulfilled. If an element of the basis is independent of the vector $\boldsymbol{v}$, then it is an invariant tensor of rank 2. There exists a unique such tensor, namely, the Kronecker symbol $\delta_{jk}$. If an element of the basis depends on $\boldsymbol{v}$, then it can be chosen as the tensor product of the vector $\boldsymbol{v}$ and another vector. There exists a unique tensor product $w^{(a)}_{jk} = v_j v_k$ satisfying the relation $w^{(a)}_{jk}[\mathsf{U}\boldsymbol{v}] = U_{jm}U_{kl}w^{(a)}_{ml}[\boldsymbol{v}]$. □

**Corollary.** *The class of all quasilinear systems of the class $\mathfrak{K}_1(\mathbb{R}^3)$ is described by the formula*

$$\dot{v}_j(\boldsymbol{x}, t) = (\nabla_k[f(\boldsymbol{v}^2)\delta_{jk} + g(\boldsymbol{v}^2)v_j v_k])(\boldsymbol{x}, t), \quad j = 1, 2, 3, \tag{5}$$

*where $f$ and $g$ are differentiable functions on $\mathbb{R}_+$.*

**3. Notion of hyperbolicity.** To each system (1) of first-order quasilinear equations, for each value of the field $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$, we assign the following linearized (tangential) system of first-order linear differential equations for the variation $\delta\boldsymbol{v}(\boldsymbol{x}, t)$ of the vector field $\boldsymbol{v}(\boldsymbol{x}, t)$:

$$\delta\dot{v}_j(\boldsymbol{x}, t) = A_{jl}^{(k)}[\boldsymbol{v}]\nabla_k \delta v_l, \tag{6}$$

where the collection of matrices $\mathsf{A}^{(k)}$, $A_{jl}^{(k)} = (\mathsf{A}^{(k)})_{jl}$, $k = 1, 2, 3$, is a tensor of rank 3:

$$A_{jl}^{(k)}[\boldsymbol{v}] = \frac{\partial S_{jk}[\boldsymbol{v}]}{\partial v_l} = \frac{\partial}{\partial v_l}[f(\boldsymbol{v}^2)\delta_{jk} + g(\boldsymbol{v}^2)v_j v_k] = 2[f'(\boldsymbol{v}^2)\delta_{jk} + g'(\boldsymbol{v}^2)v_j v_k]v_l + g(\boldsymbol{v}^2)(\delta_{jl}v_k + \delta_{kl}v_j).$$

To Eq. (6), we assign the following homogeneous system of linear algebraic equations for the components of the vector $\boldsymbol{v}^{(0)}$:

$$\left(\omega(\boldsymbol{k})\delta_{jl} + \sum_{m=1}^{3} k_m A_{jl}^{(m)}\right)v_l^{(0)} = 0, \quad j = 1, 2, 3;$$

this system is obtained by substituting $\delta v_j = v_j^{(0)} \exp(i\omega(\boldsymbol{k})t - i(\boldsymbol{k}, \boldsymbol{x}))$, $j = 1, 2, 3$, with a constant vector $\boldsymbol{v}^{(0)} = \langle v_1^{(0)}, v_2^{(0)}, v_3^{(0)}\rangle$ into Eq. (6). Here $\omega$ is a constant "frequency" of the field and $\boldsymbol{k}$ is a constant "wave vector." A nonzero vector $\boldsymbol{v}^{(0)}$ exists if

$$\det\left(\omega(\boldsymbol{k})\delta_{jl} + \sum_{m=1}^{3} k_m A_{jl}^{(m)}\right) = 0. \tag{7}$$

Equation (7) is called the *spectral equation* (in theoretical physics, it is called the *dispersion equation*). The solutions $\omega_j(\boldsymbol{k})$, $j = 1, 2, 3$, of this third-order algebraic equation are called the dispersion functions.

**Definition 1** (see [3]). A system of the form (1) is said to be hyperbolic (in the Petrovsky sense) if the functions $\omega_j(\boldsymbol{k})$, $j = 1, 2, 3$, are real-valued and the values of each pair of these functions do not coincide at any point $\boldsymbol{k} \in \mathbb{R}^3$.

**4. Criterion of the real-valuedness of roots.** In this section, we recall the well-known criterion of the real-valuedness of roots of a cubic equation. Consider the equation

$$z^3 + a_1 z^2 + a_2 z + a_3 = 0 \tag{8}$$

for the unknown $z \in \mathbb{C}$ with real-valued coefficients $a_1$, $a_2$, and $a_3$. The following theorem holds.

**Theorem 2** (see [8]). *All roots of Eq.* (8) *are real-valued and distinct if and only if its discriminant* $\Delta$ *is positive*:

$$\Delta(a_1, a_2, a_3) = a_1^2 a_2^2 - 4a_2^3 - 4a_1^3 a_3 - 27a_3^2 + 18a_1 a_2 a_3 > 0. \tag{9}$$

*Proof.* The substitution $z = y - a_1/3$ reduces Eq. (9) to the form $y^3 + py + q = 0$, where

$$p = a_2 - \frac{a_1^3}{3}, \quad q = 2\left(\frac{a_1}{3}\right)^3 - \frac{a_1 a_2}{3} + a_3.$$

The solutions of this equation have the form

$$y_1 = \alpha + \beta, \quad y_{2,3} = -\frac{1}{2}(\alpha + \beta) \pm \frac{\sqrt{3}}{2}(\alpha - \beta)$$

(see, e.g., [7]), where

$$\alpha = \left(-\frac{q}{2} + \sqrt{Q}\right)^{1/3}, \quad \beta = \left(-\frac{q}{2} - \sqrt{Q}\right)^{1/3}, \quad Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2.$$

Moreover, all these solutions are real-valued and distinct if $Q < 0$. Since the discriminant $\Delta$ is related to $Q$ by the formula $\Delta = -108Q$, we conclude that the criterion of the real-valuedness and distinction of the roots is the inequality $\Delta > 0$. Substituting the explicit expressions of the coefficients $p$ and $q$ into the expression for $Q$, we obtain the inequality (9). □

**5. Calculation of the discriminant.** In this section, we calculate the discriminant $\Delta(a_1, a_2, a_3)$ of the spectral equation (7) determined by the left-hand side of the inequality (9). We rewrite Eq. (7) in the form (8):

$$\omega^3 + a_1\omega^2 + a_2\omega + a_3 = 0, \tag{10}$$

where $\omega \equiv \omega(\boldsymbol{k})$ and

$$a_1 = \operatorname{Sp}\mathsf{A}, \quad a_2 = \frac{1}{2}[\operatorname{Sp}^2\mathsf{A} - \operatorname{Sp}\mathsf{A}^2], \quad a_3 = \frac{1}{6}[2\operatorname{Sp}\mathsf{A}^3 - 3\operatorname{Sp}\mathsf{A}\cdot\operatorname{Sp}\mathsf{A}^2 + \operatorname{Sp}^3\mathsf{A}] \tag{11}$$

with the matrix $\mathsf{A} = \mathsf{A}^{(l)}k_l$ (see, e.g., [2]). Using (6), we represent the matrix $\mathsf{A}$ as the combination $\mathsf{A} = 2\mathsf{B} + g\mathsf{C}$ of two matrices with the elements

$$B_{jm} = b_j v_m, \quad C_{jm} = \zeta\delta_{jm} + v_j k_m, \quad b_j = f' k_j + \zeta g' v_j, \quad \zeta = (\boldsymbol{k}, \boldsymbol{v}).$$

We calculate the traces of sequential powers of the matrix $\mathsf{C}$: $c_1 = \operatorname{Sp}\mathsf{C} = 4\zeta$,

$$(\mathsf{C}^2)_{jl} = \zeta(\zeta\delta_{jl} + 3v_j k_l), \quad \operatorname{Sp}\mathsf{C}^2 = 6\zeta^2, \tag{12}$$

$$(\mathsf{C}^3)_{jl} = \zeta(\zeta\delta_{jm} + 3v_j k_m)(\zeta\delta_{lm} + v_m k_l) = \zeta^2(\zeta\delta_{jl} + 7v_j k_l), \quad \operatorname{Sp}\mathsf{C}^3 = 10\zeta^3. \tag{13}$$

From (12) and (13) we obtain

$$c_2 \equiv \frac{1}{2}[\operatorname{Sp}^2\mathsf{C} - \operatorname{Sp}\mathsf{C}^2] = 5\zeta^2, \quad c_3 \equiv \frac{1}{6}[2\operatorname{Sp}\mathsf{C}^3 - 3\operatorname{Sp}\mathsf{C}\cdot\operatorname{Sp}\mathsf{C}^2 + \operatorname{Sp}^3\mathsf{C}] = 2\zeta^3. \tag{14}$$

Moreover, since

$$(\boldsymbol{v}, \boldsymbol{b}) = \zeta(f' + \boldsymbol{v}^2 g'), \quad (\boldsymbol{k}, \boldsymbol{b}) = f'\boldsymbol{k}^2 + g'\zeta^2, \tag{15}$$

where $\boldsymbol{b} = \langle b_1, b_2, b_3\rangle$, we have

$$(\mathsf{C}\boldsymbol{b})_j = C_{jl}b_l = \zeta b_j + v_j(\boldsymbol{k}, \boldsymbol{b}), \quad (\boldsymbol{v}, \mathsf{C}\boldsymbol{b}) = \zeta(\boldsymbol{v}, \boldsymbol{b}) + \boldsymbol{v}^2(\boldsymbol{k}, \boldsymbol{b}), \tag{16}$$

$$(\mathsf{C}^2\boldsymbol{b})_j = (\mathsf{C}^2)_{jl}b_l = \zeta(\zeta b_j + 3v_j(\boldsymbol{k}, \boldsymbol{b})), \quad (\boldsymbol{v}, \mathsf{C}^2\boldsymbol{b}) = \zeta(\zeta(\boldsymbol{v}, \boldsymbol{b}) + 3\boldsymbol{v}^2(\boldsymbol{k}, \boldsymbol{b})). \tag{17}$$

Now we can calculate the coefficients $a_1$, $a_2$, and $a_3$ of the spectral equation (10) by the formulas (11). Taking into account the decomposition of the matrix $\mathsf{A}$, we get

$$a_1 = \operatorname{Sp}\mathsf{A} = 2\operatorname{Sp}\mathsf{B} + c_1 g = 2d_0 + c_1 g, \quad d_0 \equiv (\boldsymbol{b}, \boldsymbol{v}). \tag{18}$$

Further, since

$$\mathsf{A}^2 = (2\mathsf{B} + g\mathsf{C})^2 = 4\mathsf{B}^2 + 2g(\mathsf{BC} + \mathsf{CB}) + g^2\mathsf{C}^2, \quad \mathsf{B}^2 = (\boldsymbol{b}, \boldsymbol{v})\mathsf{B}, \quad \operatorname{Sp}\mathsf{B}^2 = (\boldsymbol{b}, \boldsymbol{v})^2,$$

we have

$$(\mathsf{A}^2)_{jl} = 4(\boldsymbol{b}, \boldsymbol{v})\mathsf{B}_{jl} + 2g(b_j(\mathsf{C}^{\mathrm{T}}\boldsymbol{v})_l + (\mathsf{C}\boldsymbol{b})_j v_l) + g^2(\mathsf{C}^2)_{jl}, \tag{19}$$

$$\operatorname{Sp}\mathsf{A}^2 = 4d_0^2 + 4gd_1 + g^2\operatorname{Sp}\mathsf{C}^2, \quad d_1 \equiv (\boldsymbol{v}, \mathsf{C}\boldsymbol{b}). \tag{20}$$

Therefore, due to (11),

$$a_2 = \frac{1}{2}[(2d_0 + c_1 g)^2 - g^2\operatorname{Sp}\mathsf{C}^2] - 2(d_0^2 + gd_1) = g^2 c_2 + 2g(d_0 c_1 - d_1). \tag{21}$$

Finally, we calculate the coefficient $a_3$. Since

$$\mathsf{A}^3 = 8\mathsf{B}^3 + 4g(\mathsf{B}^2\mathsf{C} + \mathsf{BCB} + \mathsf{CB}^2) + 2g^2(\mathsf{BC}^2 + \mathsf{CBC} + \mathsf{C}^2\mathsf{B}) + g^3\mathsf{C}^3,$$

we have

$$(\mathsf{A}^3)_{jl} = 8(\boldsymbol{b}, \boldsymbol{v})^2 b_j v_l + 4g\Big[(\boldsymbol{b}, \boldsymbol{v})b_j(\mathsf{C}^{\mathrm{T}}\mathbf{a}_l) + (\boldsymbol{v}, \mathsf{C}\boldsymbol{b})b_j v_l + (\boldsymbol{b}, \boldsymbol{v})(\mathsf{C}\boldsymbol{b})_j v_l\Big]$$
$$+ 2g^2\Big[b_j(\mathsf{C}^{2\mathrm{T}}\boldsymbol{v})_l + (\mathsf{C}\boldsymbol{b})_j(\mathsf{C}^{\mathrm{T}}\boldsymbol{v})_l + (\mathsf{C}^2\boldsymbol{b})_j \boldsymbol{v}_l\Big] + g^3(\mathsf{C}^3)_{jl},$$

$$\operatorname{Sp} \mathsf{A}^3 = 8d_0^3 + 12gd_0d_1 + 6g^2d_2 + g^3 \operatorname{Sp} \mathsf{C}^3, \quad d_2 = (\boldsymbol{v}, \mathsf{C}^2\boldsymbol{b}). \tag{22}$$

Then, using the formulas (16), (19), and (21), we obtain the following expression for the coefficient $a_3$ based on its definition (11):

$$a_3 = \operatorname{Sp} \mathsf{A}^3 + \operatorname{Sp} \mathsf{A}[a_2 - \operatorname{Sp} \mathsf{A}^2] = \frac{1}{3}\Big[8d_0^3 + 12gd_0d_1 + 6g^2d_2 + g^3 \operatorname{Sp} \mathsf{C}^3\Big]$$
$$+ \frac{1}{3}(2d_0 + c_1g)\Big[g^2(c_2 - \operatorname{Sp} \mathsf{C}^2) + 2gd_0c_1 - 4d_0^2 - 6gd_1\Big] = g^3c_3 + 2g^2(d_2 + d_0c_2 - d_1c_1), \tag{23}$$

where we used the identity $\operatorname{Sp} \mathsf{C}^3 + c_1(c_2 - \operatorname{Sp} \mathsf{C}^2) = 3c_3$.

We express the coefficients $a_1$, $a_2$, and $a_3$ through the functions $f$ and $g$. Following the formula (18), we get $d_0 = \zeta(f' + \boldsymbol{v}^2g')$. Further, based on (20), (15), and (16), we obtain

$$d_1 = \zeta^2(f' + 2\boldsymbol{v}^2g') + \boldsymbol{k}^2\boldsymbol{v}^2f';$$

similarly, using the formulas (22) and (17), (15), we have

$$d_2 = \zeta\Big[\zeta^2(f' + 4\boldsymbol{v}^2g') + 3\boldsymbol{v}^2\boldsymbol{k}^2f'\Big].$$

Then

$$d_2 + d_0c_2 - d_1c_1 = \zeta\Big[2\zeta(\boldsymbol{b}, \boldsymbol{v}) - \boldsymbol{v}^2(\boldsymbol{k}, \boldsymbol{b})\Big] = \zeta\Big[\zeta^2(2f' + \boldsymbol{v}^2g') - \boldsymbol{k}^2\boldsymbol{v}^2f'\Big]$$

and hence, due to (23),

$$a_3 = 2\zeta g^2\Big[\zeta^2(2f' + \boldsymbol{v}^2g' + g) - \boldsymbol{k}^2\boldsymbol{v}^2f'\Big] \equiv |\boldsymbol{k}|^3|\boldsymbol{v}|^3(\eta^3e_3 - \eta r_3), \tag{24}$$

where the dependence on the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{v}$ ($\eta = \cos(\widehat{\boldsymbol{k}, \boldsymbol{v}})$) is obvious and $e_3 = 2g^2(2f' + \boldsymbol{v}^2g' + g)$ and $r_3 = 2g^2f'$ are the coefficients. By the formulas (14) and (21) and the expressions for $d_0$ and $d_1$ found earlier, we obtain the representations for the coefficients $a_2$ and $a_1$:

$$a_2 = g\big[\zeta^2(5g + 6f' + 4\boldsymbol{v}^2g') - 2\boldsymbol{k}^2\boldsymbol{v}^2f'\big] \equiv \boldsymbol{k}^2\boldsymbol{v}^2\big(\eta^2e_2 - r_2\big), \tag{25}$$

$$e_2 = g(5g + 6f' + 4\boldsymbol{v}^2g'), \quad r_2 = 2gf'; \tag{26}$$

$$a_1 = 2\zeta(f' + \boldsymbol{v}^2g' + 2g) = |\boldsymbol{k}||\boldsymbol{v}|\eta e_1, \quad e_1 = 2(2g + f' + \boldsymbol{v}^2g'). \tag{27}$$

Substituting the expressions (25)–(27) of the coefficient $a_1$, $a_2$, and $a_3$ into (9), we arrive at the following theorem.

**Theorem 3.** *The discriminant* $\Delta(a_1, a_2, a_3)$ *of Eq.* (8) *is equal to*

$$\Delta(a_1, a_2, a_3) = \boldsymbol{k}^6\boldsymbol{v}^6\Delta(\eta e_1, \eta^2e_2 - r_2, \eta^3e_3 - \eta r_3), \tag{28}$$

*where the functions* $e_1$, $e_2$, $r_2$, $e_3$, *and* $r_3$ *are determined by the formulas* (24), (26), *and* (27). *The discriminant* $\Delta$ *is a bicubic polynomial of the variable* $\eta = \cos(\widehat{\boldsymbol{k}, \boldsymbol{v}})$.

**6. Real-valuedness of the spectrum.** Taking into account the formula (28), we conclude that the necessary and sufficient condition of the hyperbolicity of Eq. (5), i.e., the real-valuedness of solutions $\omega_j(\boldsymbol{k})$, $j = 1, 2, 3$, of the spectral equation (10) and their nondegeneracy, is the validity of the inequality $\Delta(\eta e_1, \eta^2e_2 - r_2, \eta^3e_3 - \eta r_3) > 0$. We analyze this inequality for a bicubic polynomial of $\eta$.

For $u \in \mathbb{R}$, consider the cubic polynomial $P(u) = \alpha_0u^3 + \alpha_1u^2 + \alpha_2u + \alpha_3$ and analyze the conditions for the coefficients $\alpha_0$, $\alpha_1$, $\alpha_2$, $\alpha_3$ in the case where it satisfies the inequality $P(u) > 0$ on the interval $[0, 1]$. Obviously, in this case $P(0) > 0$ and $P(1) > 0$. These conditions are also sufficient in the case where the interval $(0, 1)$ doe not contain a point $u_*$ such that $P'(u_*) = 0$ and $P''(u_*) > 0$, where $P'(u) = 3\alpha_0u^2 + 2\alpha_1u + \alpha_2$ and $P''(u) = 6\alpha_0 + 2\alpha_1$. If such a point exists, then the inequality $P(u_*) > 0$ is necessary, and this condition, together with the inequalities $P(0) > 0$ and $P(1) > 0$, forms a complete set of necessary and sufficient conditions. We discuss the possibilities that arise in this case.

We distinguish between the following cases:

I. $\alpha_0 = 0$;
II. $\alpha_0 \neq 0$ and $\alpha_1^2 \leq 3\alpha_0\alpha_2$;
III. $\alpha_0 \neq 0$ and $\alpha_1^2 > 3\alpha_0\alpha_2$.

In the first case, $P(u) = 2\alpha_1$ and $u_* = -\alpha_2/2\alpha_1$. If $\alpha_1 = 0$, then the point $u_*$ does not exist. If $\alpha_1 \neq 0$ and the point $u_* \in (0,1)$ exists, then $-\alpha_2/2\alpha_1 \in (0,1)$. Due to the condition $P(u_*) > 0$, we obtain $\alpha_1 > 0$. Therefore, in this case $\alpha_2 < 0$ and $\alpha_2 + 2\alpha_1 > 0$. If the point $u_*$ does not belong to $(0,1)$, then either $\alpha_1 < 0$ and at least one of the inequalities $\alpha_2 \leq 0$ or $\alpha_2 + 2\alpha_1 \leq 0$ is fulfilled, either $\alpha_1 > 0$ and one of the inequalities $\alpha_2 \geq 0$ or $\alpha_2 + 2\alpha_1 \geq 0$ exists. Finally, if the point $u_*$ lies on the interval $(0,1)$, then $\alpha_1 > 0$ and, moreover, $\alpha_2 < 0$ and $\alpha_1 + 2\alpha_2 > 0$.

In the case II, the point $u_*$ does not exist (in the case $\alpha_1^2 \leq 3\alpha_0\alpha_2$, $u_* = -\alpha_1/3\alpha_0$, and $P''(u_*) = 0$, i.e., $u_*$ is not a minimum point).

In the case III, the existence of the point $u_*$ is possible, and it can take one of the values

$$u_\pm = \frac{1}{3\alpha_0}(-\alpha_1 \pm R), \qquad R = \sqrt{\alpha_1^2 - 3\alpha_2\alpha_0}.$$

If $u_* \in (0,1)$, then the condition $P(u_*) > 0$ must be satisfied only for $u_* = u_+$ since $P''(u_\pm) = \pm 2R$, and the inequality $P''(u_*) > 0$ is impossible for the point $u_-$. Thus, we must only analyze the possibility of the condition $u_+ \in (0,1)$, since if $u_+ \notin (0,1)$, then the conditions $P(0) > 0$ and $P(1) > 0$ are sufficient for the validity of the inequality $P(u) > 0$ for $u \in [0,1]$. Consider two cases $\mathrm{III}_\pm$, depending on the sign of the coefficient $\alpha_0$.

In the case $\mathrm{III}_+$ for $\alpha_0 > 0$, $u_+ \in (0,1)$ if and only if $\alpha_1 < R < \alpha_1 + 3\alpha_0$. The left inequality holds if and only if either $\alpha_1 \leq 0$ or $\alpha_1 > 0$ and $\alpha_2 > 0$. The right inequality holds if and only if $3\alpha_0 + \alpha_1 > 0$ and $\alpha_2 + 2\alpha_1 + 3\alpha_0 > 0$.

In the case $\mathrm{III}_-$, the inequality $0 < u_+ < 1$ holds if either $R < \alpha_1$ or $3\alpha_0 + \alpha_1 < R$. The first inequality yields $\alpha_1 > 0$ and $\alpha_2 < 0$. The second inequality holds if either $3\alpha_0 + \alpha_1 \leq 0$ or $3\alpha_0 + \alpha_1 > 0$ and $3\alpha_0 + 2\alpha_1 + \alpha_2 > 0$.

We analyze the validity of the inequality $P(u_+) > 0$ for $u_+ \in (0,1)$ in the case III. Since

$$3P(u) = (3u + \alpha_1)P'(u) + 2u(3\alpha_0\alpha_2 - \alpha_1^2) + 9\alpha_3 - \alpha_1\alpha_2,$$

due to the conditions $P(u_+) > 0$ and $P'(u_+) = 0$ we obtain

$$2u_+R^2 < 9\alpha_3 - \alpha_1\alpha_2.$$

The inequality $P(u_+) > 0$ holds for $u_+ \in (0,1)$ if and only if $2u_+R < 9\alpha_3 - \alpha_1\alpha_2$ and therefore $9\alpha_3 - \alpha_1\alpha_2 > 0$. This implies that the last inequality holds if and only if $2R^3 < S$ in the case $\mathrm{III}_+$, where $S \equiv 3\alpha_0(9\alpha_3 - \alpha_1\alpha_2) + 2\alpha_1R^2 = 27\alpha_0\alpha_3 - 9\alpha_0\alpha_1\alpha_2 + 2\alpha_1^3$, or $2R^3 > S$ in the case $\mathrm{III}_-$. We analyze these possibilities.

In the case $\mathrm{III}_+$, the inequality $S > 0$ must hold. We rewrite the inequality $2R^3 < S$ in the equivalent radical-free form $4R^6 < S^2$. Taking into account the inequality $\alpha_0 > 0$, we have

$$4\alpha_0^2\alpha_2^3 + \alpha_0(27\alpha_3^2 - 18\alpha_1\alpha_2\alpha_3 - \alpha_1^2\alpha_2^2) + 4\alpha_1^3\alpha_3 > 0. \tag{29}$$

In the case $\mathrm{III}_-$, the inequality $2R^3 > S$ holds for $S < 0$; for $S > 0$, it is equivalent to the radical-free inequality $4R^6 > S^2$, which takes the form (29) after dividing by $\alpha_0 < 0$. We summarize the results obtained in the following assertion.

**Lemma 1.** *The cubic polynomials* $P(u) = \alpha_0u^3 + \alpha_1u^2 + \alpha_2u + \alpha_3$ *takes strictly positive values on the segment* $[0,1]$ *if and only if*

$$P(0) = \alpha_3 > 0, \quad P(1) = \alpha_0 + \alpha_2 + \alpha_2 + \alpha_3 > 0,$$

*and at least one of the following conditions* I–III *is fulfilled*:

I. *If* $\alpha_0 = 0$, *then*

$$\begin{aligned}&\textit{either } \alpha_1 = 0;\\ &\textit{or } \alpha_1 < 0 \textit{ and } (\alpha_2 \le 0 \textit{ or } \alpha_2 + 2\alpha_1 \le 0);\\ &\textit{either } \alpha_1 > 0 \textit{ and } (\alpha_2 \ge 0 \textit{ or } \alpha_2 + 2\alpha_1 \ge 0);\\ &\textit{simultaneously } \alpha_1 > 0,\ \alpha_2 > 0, \textit{ and } \alpha_2 + 2\alpha_1 > 0.\end{aligned}$$

II. $\alpha_0 \ne 0$, $\alpha_1^2 \le 3\alpha_0\alpha_2$.

III. *If* $\alpha_0 \ne 0$ *and* $\alpha_1^2 > \alpha_0\alpha_2$, *then in the case* $\mathrm{III}_+$, *where* $\alpha_0 > 0$, *one of the following possibilities is implemented*:

(i) $\alpha_1 > 0$ *and* $\alpha_2 \le 0$, *or at least one of the following inequalities is violated*:

$$3\alpha_0 + \alpha_1 > 0, \quad \alpha_2 + 2\alpha_1 + 3\alpha_0 > 0.$$

(ii) $\alpha_1 > 0$, $\alpha_2 > 0$, *the following inequalities hold simultaneously*:

$$3\alpha_0 + \alpha_1 > 0, \quad \alpha_2 + 2\alpha_1 + 3\alpha_0 > 0, \quad 9\alpha_3 > \alpha_1\alpha_2, \quad 27\alpha_0\alpha_3 - 9\alpha_0\alpha_1\alpha_2 + 2\alpha_1^3 > 0,$$

*and the inequality* (29) *is fulfilled.*

*In the case* $\mathrm{III}_-$, *where* $\alpha_0 < 0$, *one of the following possibilities is implemented*:

(i) *At least one of the following inequalities holds:* $\alpha_1 \le 0$; *or* $\alpha_2 \ge 0$; *or*

$$3\alpha_0 + \alpha_1 > 0, \quad \alpha_2 + 2\alpha_1 + 3\alpha_0 \le 0.$$

(ii) *The conditions* $\alpha_1 > 0$ *and* $\alpha_2 > 0$ *hold simultaneously and also either* $3\alpha_0 + \alpha_1 \le 0$ *or*

$$3\alpha_0 + \alpha_1 > 0, \quad \alpha_2 + 2\alpha_1 + 3\alpha_0 > 0.$$

*Moreover, together with the inequality* (29), *the following inequalities are fulfilled*:

$$9\alpha_3 > \alpha_1\alpha_2, \quad 27\alpha_0\alpha_3 - 9\alpha_0\alpha_1\alpha_2 + 2\alpha_1^3 < 0.$$

Now we apply the results of analyzing the inequality $P(u) > 0$ on $[0, 1]$ to the description of the validity domain of the inequality

$$\Delta(\eta e_1, \eta^2 e_2 - r_2, \eta^3 e_3 - \eta r_3) > 0.$$

For this purpose, we expand $\Delta(\eta e_1, \eta^2 e_2 - r_2, \eta^3 e_3 - \eta r_3)$ by powers of $\eta$. Note that the discriminant (9) possesses the following property:

$$\Delta(\lambda a_1, \lambda^2 a_2, \lambda^3 a_3) = \lambda^6 \Delta(a_1, a_2, a_3) \quad \forall \lambda \in \mathbb{R}.$$

Then the following identity holds:

$$\Delta(\eta e_1, \eta^2 e_2 - r_2, \eta^3 e_3 - \eta r_3) = \eta^6 \Delta(e_1, e_2 - \eta^{-2} r_2, e_3 - \eta^{-2} r_3).$$

The required expansion is obtained from the Taylor expansion of the function

$$\Delta(e_1, e_2 - \eta^{-2} r_2, e_3 - \eta^{-2} r_3)$$

by powers of $\eta^{-2}$, which contains four terms. Then

$$\Delta\Big(\eta e_1, \eta^2 e_2 - r_2, \eta^3 e_3 - \eta r_3\Big) = \eta^6 \Big(\Delta(e_1, e_2, e_3) + \eta^{-2}\Delta_1 + \eta^{-4}\Delta_2 + \eta^{-6}\Delta_3\Big).$$

Using the explicit form of $\Delta(a_1, a_2.a_3)$ and the explicit expressions for the functions $r_2$ and $r_3$, we obtain

$$\begin{aligned}
\Delta_1 &= -r_2\left(\frac{\partial\Delta}{\partial a_2}\right)_{a_2=e_2} - r_3\left(\frac{\partial\Delta}{\partial a_3}\right)_{a_3=e_3} \\
&= 4gf'\Big[6e_2^2 - 9e_1e_3 - e_1^2e_2 + g(2e_1^3 + 27e_3 - 9e_1e_2)\Big], \\
\Delta_2 &= \frac{1}{2}\left(\frac{\partial^2\Delta}{\partial a_2^2}r_2^2 + 2\frac{\partial^2\Delta}{\partial a_2\partial a_3}r_2r_3 + \frac{\partial^2\Delta}{\partial a_3^2}r_3^2\right)_{a_2=e_2,a_3=r_3} \\
&= 4g^2f'^2\Big[e_1^2 - 12e_2 - 27g^2 + 18ge_1\Big] \\
&= 4g^2f'^2\Big[g^2 + 4(f'^2 + \boldsymbol{v}^2gg' + \boldsymbol{v}^4g'^2) + 8\boldsymbol{v}^2f'g' - 20gf'\Big], \\
\Delta_3 &= -\frac{1}{6}\left(\frac{\partial^3\Delta}{\partial a_2^3}\right)_{a_2=e_2} r_2^3 = 32g^3f'^3.
\end{aligned}$$

Substituting the explicit expressions of the functions $e_1$, $e_2$, and $e_3$, we obtain the following formulas for $\Delta_1$ and $\Delta_2$:

$$\begin{aligned}
\Delta_1 &= 8g^2f'^2\Big[(8f' - 4\boldsymbol{v}^2g' - g)g - 4(f' + \boldsymbol{v}^2g')^2\Big], \\
\Delta_2 &= 4g^2f'^2\Big[(g + 4\boldsymbol{v}^2g' - 20f')g + 4(f' + \boldsymbol{v}^2g')^2\Big].
\end{aligned}$$

Moreover,

$$\begin{aligned}
2\Delta_1 + \Delta_2 &= 12g^2f'^2\Big[(4f' - 4\boldsymbol{v}^2g' - g)g - 4(f' + \boldsymbol{v}^2g')^2\Big]. \\
\Delta_1 + \Delta_2 + \Delta_3 &= 4g^2f'^2\Big[(4f' - 4\boldsymbol{v}^2g' - g)g - 4(f' + \boldsymbol{v}^2g')^2\Big].
\end{aligned}$$

Now we set $u = \eta^2$, $\alpha_0 = \Delta(e_1, e_2, e_3)$, $\alpha_1 = \Delta_1$, $\alpha_2 = \Delta_2$, and $\alpha_3 = \Delta_3$ and apply the formulas

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \Delta(e_1, e_2 - r_2, e_3 - r_3) = \Delta(e_1, e_2 - 2gf', e_3 - 2g^2f').$$

Due to the lemma, we arrive at the following assertion, which is the main result of this paper.

**Theorem 4.** *Equation* (1) *with the tensor function* (4) *is a system of first-order hyperbolic* (*in the Petrovsky sense*) *quasilinear equations if and only if the following inequalities hold*:

$$gf' > 0, \quad \Delta(e_1, e_2 - 2gf', e_3 - 2g^2f') > 0,$$

*and also at least one of the following conditions* I–III *is fulfilled*:

I. *If* $\Delta(e_1, e_2, e_3) = 0$, *then*

*either* $(8f' - 4\boldsymbol{v}^2g' - g)g - 4(f' + \boldsymbol{v}^2g')^2 = 0$;

*or* $(8f' - 4\boldsymbol{v}^2g' - g)g - 4(f' + \boldsymbol{v}^2g')^2 < 0$

*and* $\Big((g + 4\boldsymbol{v}^2g' - 20f')g + 4(f' + \boldsymbol{v}^2g')^2 \le 0$ *or* $(4f' - 4\boldsymbol{v}^2g' - g)g - 4(f' + \boldsymbol{v}^2g')^2 \le 0\Big)$;

*or* $(8f' - 4\boldsymbol{v}^2g' - g)g - 4(f' + \boldsymbol{v}^2g')^2 > 0$

*and* $\Big((g + 4\boldsymbol{v}^2g' - 20f')g + 4(f' + \boldsymbol{v}^2g')^2 \ge 0$ *or* $(4f' - 4\boldsymbol{v}^2g' - g)g - 4(f' + \boldsymbol{v}^2g')^2 \ge 0\Big)$;

*or simultaneously*

$$\begin{aligned}
(8f' - 4\boldsymbol{v}^2g' - g)g - 4(f' + \boldsymbol{v}^2g')^2 &> 0, \\
(g + 4\boldsymbol{v}^2g' - 20f')g + 4(f' + \boldsymbol{v}^2g')^2 &> 0, \\
(4f' - 4\boldsymbol{v}^2g' - g)g - 4(f' + \boldsymbol{v}^2g')^2 &> 0.
\end{aligned}$$

II. *If* $\Delta(e_1,e_2,e_3)\neq 0$, *then*

$$16g^2f'^2\Big[(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]^2 \le 3\Delta(e_1,e_2,e_3)\Big[(g+4\boldsymbol{v}^2g'-20f')g+4(f'+\boldsymbol{v}^2g')^2\Big].$$

III. *If* $\Delta(e_1,e_2,e_3)\neq 0$ *and*

$$16g^2f'^2[(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2]^2 > 3\Delta(e_1,e_2,e_3)\Big[(g+4\boldsymbol{v}^2g'-20f')g+4(f'+\boldsymbol{v}^2g')^2\Big],$$

*then in the case* $III_+$, *where* $\Delta(e_1,e_2,e_3)>0$, *one of the following possibilities is implemented*:

(i) $(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2>0$ *and* $(g+4\boldsymbol{v}^2g'-20f')g+4(f'+\boldsymbol{v}^2g')^2\le 0$ *or at least one of the inequalities is violated*:

$$3\Delta(e_1,e_2,e_3)+8(gf')^2\Big[(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]>0,$$
$$3\Delta(e_1,e_2,e_3)+12(gf')^2\Big[(4f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]>0.$$

(ii) $(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2>0$ *and* $(g+4\boldsymbol{v}^2g'-20f')g+4(f'+\boldsymbol{v}^2g')^2>0$, *the following inequalities hold simultaneously*:

$$\begin{aligned}
&3\Delta(e_1,e_2,e_3)+8(gf')^2\Big[(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]>0,\\
&3\Delta(e_1,e_2,e_3)+12(gf')^2\Big[(4f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]>0,\\
&9>gf'\Big[(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]\Big[(g+4\boldsymbol{v}^2g'-20f')g+4(f'+\boldsymbol{v}^2g')^2\Big],\\
&9\Delta(e_1,e_2,e_3)\Big(3-(gf')\Big[(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]\\
&\qquad\times\Big[(g+4\boldsymbol{v}^2g'-20f')g+4(f'+\boldsymbol{v}^2g')^2\Big]\Big)\\
&\qquad+32(gf')^3\Big[(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]^3>0,
\end{aligned}\tag{30}$$

*and also the following inequality implied by* (29) *is fulfilled*:

$$\begin{aligned}
&\Delta^2(e_1,e_2,e_3)\Big[(g+4\boldsymbol{v}^2g'-20f')g+4(f'+\boldsymbol{v}^2g')^2\Big]^3+4\Delta(e_1,e_2,e_3)\Big(27\\
&-18(gf')\Big[(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]\Big[(g+4\boldsymbol{v}^2g'-20f')g+4(f'+\boldsymbol{v}^2g')^2\Big]\\
&-(gf')^2\Big[(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]^2\Big[(g+4\boldsymbol{v}^2g'-20f')g+4(f'+\boldsymbol{v}^2g')^2\Big]^2\Big)\\
&\qquad+4^4(gf')^3\Big[(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]^3>0.
\end{aligned}\tag{31}$$

*In the case* $\mathrm{III}_-$, *where* $\Delta(e_1,e_2,e_3)<0$, *one of the following possibilities is implemented*:

(1) *At least one of the following inequalities is fulfilled*:

$$(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\le 0;$$
$$(g+4\boldsymbol{v}^2g'-20f')g+4(f'+\boldsymbol{v}^2g')^2\ge 0;$$
$$3\Delta(e_1,e_2,e_3)+8(gf')^2\Big[(8f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]>0;$$
$$3\Delta(e_1,e_2,e_3)+12(gf')^2\Big[(4f'-4\boldsymbol{v}^2g'-g)g-4(f'+\boldsymbol{v}^2g')^2\Big]\le 0.$$

(2) $(8f' - 4\boldsymbol{v}^2 g' - g)g - 4(f' + \boldsymbol{v}^2 g')^2 > 0$ *and* $(g + 4\boldsymbol{v}^2 g' - 20f')g + 4(f' + \boldsymbol{v}^2 g')^2 > 0$, *and also either*

$$3\Delta(e_1, e_2, e_3) + 8(gf')^2 \Big[(8f' - 4\boldsymbol{v}^2 g' - g)g - 4(f' + \boldsymbol{v}^2 g')^2\Big] \leq 0$$

*or*

$$3\Delta(e_1, e_2, e_3) + 8(gf')^2 \Big[(8f' - 4\boldsymbol{v}^2 g' - g)g - 4(f' + \boldsymbol{v}^2 g')^2\Big] \leq 0,$$
$$3\Delta(e_1, e_2, e_3) + 12(gf')^2 \Big[(4f' - 4\boldsymbol{v}^2 g' - g)g - 4(f' + \boldsymbol{v}^2 g')^2\Big] > 0.$$

*Moreover, the inequalities* (30) *and the inequality*

$$9\Delta(e_1, e_2, e_3)\Big(3 - (gf')\Big[(8f' - 4\boldsymbol{v}^2 g' - g)g - 4(f' + \boldsymbol{v}^2 g')^2\Big] \times \Big[(g + 4\boldsymbol{v}^2 g' - 20f')g + 4(f' + \boldsymbol{v}^2 g')^2\Big]\Big) + 32(gf')^3\Big[(8f' - 4\boldsymbol{v}^2 g' - g)g - 4(f' + \boldsymbol{v}^2 g')^2\Big]^3 < 0$$

*together with the inequality* (31) *hold.*

**7. Hyperbolicity in the Friedrichs sense.** Due to the technical complexity of establishing necessary and sufficient conditions for hyperbolicity of systems of first-order quasilinear equations, even in the simple case examined above, the object of studying such systems is often a weaker property than hyperbolicity, namely, the so-called $t$-hyperbolicity (hyperbolicity in the Friedrichs sense; see [3, 7]). According to the definition of $t$-hyperbolic first-order quasilinear systems, for each of these systems there exists a positive definite symmetric matrix $\mathsf{U}^{(0)}$ for which the matrices $\mathsf{U}^{(0)}\mathsf{U}^{(l)}$, $l = 1, 2, 3$, are symmetric. This definition of the $t$-hyperbolicity is connected with the following statement (below we provide explanations concerning the formulated statement, since in the well-known book [3], in our opinion, a significant inaccuracy was allowed in its formulation).

**Theorem 5.** *Consider the system of first-order quasilinear equations*

$$\frac{\partial u_j}{\partial t} = \sum_{l=1}^{m}\sum_{k=1}^{n} A_{jk}^{(l)} \frac{\partial u_j}{\partial x_l}, \quad j = 1, \dots, n$$

*for the unknown function* $u_j(x_1, \dots, x_m, t)$, $j = 1, \dots, n$, *where all matrices* $\mathsf{A}^{(l)}$ *of the collection* $(\mathsf{A}^{(l)})_{jk} = A_{jk}^{(l)}$, $l = 1, \dots, m$, *depend on the values of these functions at the current point* $\langle x_1, \dots, x_m; t\rangle$. *Assume that there exists a positive definite symmetric matrix* $\mathsf{A}^{(0)}$, $(\mathsf{A}^{(0)})_{jk} = A_{jk}^{(0)}$ *such that the matrices* $(\mathsf{A}^{(0)}\mathsf{A}^{(l)})_{jk}$, $l = 1, \dots, m$, *are symmetric. Then the spectral equation*

$$\det\left(\lambda - \sum_{l=1}^{m} \xi_l A_{jk}^{(l)}\right) = 0 \tag{32}$$

*of the matrix* $\sum_{l=1}^{m} \xi_l A_{jk}^{(l)}$ *has only real-valued solutions for all real tuples* $\xi_l$, $l = 1, \dots, m$.

*Proof.* The matrix $\mathsf{A}^{(0)}\mathsf{A}$ is symmetric for any tuple of numbers $\xi_j$, $j = 1, \dots, m$,

$$\mathsf{A} = \sum_{l=1}^{m} \xi_l \mathsf{A}^{(l)}, \quad (\mathsf{A}^{(l)})_{jk} = A_{jk}^{(l)}.$$

Consider the matrix pencil $\lambda\mathsf{A}^{(0)} - \mathsf{A}^{(0)}\mathsf{A}$ with the positive definite matrix $\mathsf{A}^{(0)}$. Then for any choice of numbers $\xi_j$, $j = 1, \dots, m$, all solutions of the spectral equation $\det(\lambda\mathsf{A}^{(0)} - \mathsf{A}^{(0)}\mathsf{A}) = 0$ for the matrix pencil are real-valued (see, e.g., [2]). Since

$$\det \mathsf{A}^{(0)^{-1}} \cdot \det(\lambda \mathsf{A}^{(0)} - \mathsf{A}^{(0)}\mathsf{A}) = \det(\lambda - \mathsf{A}),$$

i.e., the spectral equation (32) if the pencil coincides with the equation $\det(\lambda - \mathsf{A}) = 0$, we conclude that all solutions of this equation are also real-valued. □

Let us find out what necessary and sufficient restrictions on the choice of functions $f$ and $g$ must be imposed so that the equation corresponding to the pair of these functions (5) is $t$-hyperbolic. For this purpose, we need to find a symmetric positive definite matrix $\mathsf{A}^{(0)}$ for which the matrix

$$A_{jk} = \sum_{l=1}^{3} k_l A^{(0)}_{jm} A^{(l)}_{mk}$$

is symmetric.

Assume that the matrix $(\mathsf{A}^{(0)})_{jm} = A^{(0)}_{jm}$ has the form $A^{(0)}_{jm} = f^{(0)}(\boldsymbol{v}^2)\delta_{jm} + g^{(0)}(\boldsymbol{v}^2) v_j v_m$, where $f^{(0)}$ and $g^{(0)}$ are differentiable functions of $\boldsymbol{v}^2$. Calculate the corresponding matrix products:

$$\begin{aligned} A^{(0)}_{jm} A^{(l)}_{mk} &= \Big(f^{(0)}\delta_{jm} + g^{(0)} v_j v_m\Big)\Big(2\Big[f'\delta_{ml} + g' v_m v_l\Big] v_k + g\Big(\delta_{mk} v_l + \delta_{kl} v_m\Big)\Big) \\ &= 2 f^{(0)} f' \delta_{jl} v_k + f^{(0)} g \Big(\delta_{jk} v_l + \delta_{kl} v_j\Big) + \boldsymbol{v}^2 g^{(0)} g v_j \delta_{kl} \\ &\quad + \Big[2(f^{(0)} g' + g^{(0)} f') + g^{(0)}(g + 2g'\boldsymbol{v}^2)\Big] v_j v_k v_l. \end{aligned}$$

The expression obtained implies that the matrix $A^{(0)}_{jm} A^{(l)}_{mk}$ is symmetric with respect to the indices $j$ and $k$ if and only if the tensor

$$2 f^{(0)} f' \delta_{jl} v_k + \Big(f^{(0)} + \boldsymbol{v}^2 g^{(0)}\Big) g v_j \delta_{lk}$$

is symmetric with respect to these indices, i.e., the coefficients of the linearly independent tensors $\delta_{jl} v_k$ and $\delta_{lk} v_j$ coincide for all $l = 1, 2, 3$. Thus, we get the equality

$$2 f^{(0)} f' = g\Big(f^{(0)} + g^{(0)} \boldsymbol{v}^2\Big).$$

Now we find conditions under which the symmetric $(3 \times 3)$-matrix

$$A^{(0)}_{jm} = f^{(0)}(\boldsymbol{v}^2)\delta_{jm} + g^{(0)}(\boldsymbol{v}^2) v_j v_m$$

is positive definite. The coefficients of the polynomial $\det(\lambda - \mathsf{A}^{(0)})$ are

$$\operatorname{Sp} \mathsf{A}^{(0)} = 3 f^{(0)} + \boldsymbol{v}^2 g^{(0)} > 0, \quad \frac{1}{2}\big(\operatorname{Sp}^2 \mathsf{A}^{(0)} - \operatorname{Sp} \mathsf{A}^{(0)^2}\big), \quad \det \mathsf{A}^{(0)} = f^{(0)^2}(f^{(0)} + \boldsymbol{v}^2 g^{(0)}) > 0. \qquad (33)$$

Since

$$\operatorname{Sp} \mathsf{A}^{(0)^2} = 2 f^{(0)^2} + \Big(f^{(0)} + \boldsymbol{v}^2 g^{(0)}\Big)^2,$$

we see due to (33) that the second coefficient is defined by the expression

$$\frac{1}{2}\Big(\operatorname{Sp}^2 \mathsf{A}^{(0)} - \operatorname{Sp} \mathsf{A}^{(0)^2}\Big) = f^{(0)}\Big(3 f^{(0)} + 2\boldsymbol{v}^2 g^{(0)}\Big),$$

and the equation for eigenvalues of the matrix $\mathsf{A}^{(0)}$ takes the form

$$\lambda^3 - \Big(3 f^{(0)} + \boldsymbol{v}^2 g^{(0)}\Big)\lambda^2 + f^{(0)}\Big(3 f^{(0)} + 2\boldsymbol{v}^2 g^{(0)}\Big)\lambda - f^{(0)^2}\Big(f^{(0)} + \boldsymbol{v}^2 g^{(0)}\Big) = 0.$$

We write it in the form

$$\Big(\lambda - f^{(0)}\Big)^3 - \boldsymbol{v}^2 g^{(0)}\Big(\lambda - f^{(0)}\Big)^2 = 0$$

and find the eigenvalues $\lambda = f^{(0)}, f^{(0)} + \boldsymbol{v}^2 g^{(0)}$ of the matrix $\mathsf{A}^{(0)}$. Thus, if the inequalities $f^{(0)} > 0$ and $f^{(0)} + \boldsymbol{v}^2 g^{(0)} > 0$ are fulfilled, then the matrix $\mathsf{A}^{(0)}$ is positive definite.

Now we write the condition (6) in the form

$$g = 2f' \left(1 + \frac{\boldsymbol{v}^2 g^{(0)}}{f^{(0)}}\right)^{-1}.$$

The functions $g^{(0)}$ and $f^{(0)}$ can be chosen arbitrarily (but must satisfy the restrictions specified for them); therefore, introducing an arbitrary strictly positive function $h(\boldsymbol{v}^2) = 2(1 + \boldsymbol{v}^2 g^{(0)}/f^{(0)})^{-1}$, we obtain a necessary and sufficient condition of the $t$-hyperbolicity of the vector equation (5). Thus, we obtain the following assertion.

**Theorem 6.** *A quasilinear equation of the class $\mathfrak{K}_1(\mathbb{R}^3)$ described by Eq.* (5) *with arbitrary functions $f(\boldsymbol{v}^2)$ and $g(\boldsymbol{v}^2)$ is $t$-hyperbolic if and only if $g = hf'$, where $h$ is an arbitrary differentiable, strictly positive function on $\mathbb{R}_+$.*

**8. Conclusion.** The paper provides a complete description of the class of all hyperbolic systems of first-order quasilinear equations for a vector field $\boldsymbol{v}(\boldsymbol{x}, t)$ that can be used to describe the time evolution of this field under conditions of neglecting physical mechanisms of dissipation, in particular, energy. The solution to the problem presented in the paper shows that establishing the hyperbolicity of a specific system of first-order quasilinear equations, despite the fundamental solvability of this problem, is a rather labor-intensive process. Moreover, despite the routine nature of the solution proposed in the work and the complexity of the result obtained, the solution process was completed without using any approximations. This is due to the fact that the solved problem is still quite simple from the entire set of such problems that can arise in the process of constructing physically adequate mathematical models, when the systems of differential equations being studied are not specified, but, on the contrary, depend on a large set of parameters, and the meaning of the analysis being carried out is precisely to determine the permissible ranges of variation of these parameters. In more complicated cases, it is inevitable to apply some approximations in the solution process. In this sense, the use of the weaker concept of $t$-hyperbolicity, which is used in most studies of problems of mathematical physics related to first-order quasilinear equations (see, e.g., [1, 4, 5]), may be more preferable. If we additionally examine the possibility of intersection of different branches $\omega_j(\boldsymbol{k})$ of the spectral equations, then, together with establishing the domains of parameters of the system in which it is $t$-hyperbolic, this provides a sufficient condition for the hyperbolicity of the system.

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