DOI 10.1007/s10958-025-08074-8
Journal of Mathematical Sciences, Vol. 294, No. 1, October, 2025

THE DIRICHLET PROBLEM FOR PSEUDODIFFERENTIAL EQUATIONS IN SPECIAL DOMAINS

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UDC 517.95

In the Sobolev–Slobodetsky space, we study the solvability of a model elliptic pseudodifferential equation in a multi-dimenional domain with cut. Using a special factorization of the elliptic symbol, we find a general solution to the equation in the domain outside. This solution involves an arbitrary function which can be found by reducing with the help of the Dirichlet conditions to solving a system of one-dimensional linear equations. We study the behavior of these equations in the case where the angle of the wedge tends to zero and the wedge is transformed to an $(m-1)$-dimensional half-space. It is shown that the unique solvability of the obtained integral equation is equivalent to the solvability of the Dirichlet problem in a multi-dimensional domain with cut-off $(m-1)$-dimensional half-space. Bibliography: 6 *titles.*

1 Introduction

In this paper, we develop the factorization method [1] to study the solvability of elliptic pseudodifferential equations in a multi-dimensional domain with cut in the model situation where the cone in the multi-dimensional space is degenerated to a cone of less dimension. We study whether it is possible to pass to the limit for solutions to the Dirichlet problem for elliptic pseudodifferential equation in a multi-dimensional edge of codimension 2. Some other problems with integral conditions were studied in [2]–[4].

A pseudodifferential operator A in the domain $C \subset \mathbb{R}^m$ with symbol $A(\xi)$ is defined by

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Translated from *Problemy Matematicheskogo Analiza* **130**, 2025, pp. 25-33.

$$(Au)(x) = \int\limits_C \int\limits_{\mathbb{R}^m} A(\xi)u(y)e^{i(y-x)\xi}dyd\xi, \quad x \in C,$$

where $A(\xi)$ satisfies the condition

$$c_1 < |A(\xi)(1+|\xi|)^{-\alpha}| < c_2, \quad \xi \in \mathbb{R}^m, \quad c_1, c_2 > 0, \tag{1.1}$$

where the real number α is the order of the operator A.

This operator is considered in the Sobolev–Slobodetsky space $H^s(\mathbb{R}^m)$ equipped with the norm

$$\|u\|_s^2 = \int\limits_{\mathbb{R}^m} |\widetilde{u}(\xi)|^2(1+|\xi|)^{2s}d\xi,$$

where $\widetilde{u}$ denotes the Fourier-image of u

$$\widetilde{u}(\xi) = \int\limits_{\mathbb{R}^m} e^{iy\xi}u(y)dy.$$

We recall [1, 5] that the space $H^s(C)$ consists of functions of the space $H^s(\mathbb{R}^m)$ with support in $\overline{C}$. The $H^s(C)$-norm is induced by the $H^s(\mathbb{R}^m)$-norm. We denote by $S(\mathbb{R}^m)$ the Schwarz space of infinitely differentiable functions rapidly decreasing at infinity, $S'(\mathbb{R}^m)$ is the space of generalized functions over $S(\mathbb{R}^m)$, and $S'(C)$ is the space of generalized functions in $S'(\mathbb{R}^m)$ with support in $\overline{C}$. The space $H_0^s(C)$ consists of generalized functions of the space $S'(C)$ admitting an extension to the entire space $H^s(\mathbb{R}^m)$. The $H_0^s(C)$-norm is defined by

$$||f||_s^+ = \inf ||\ell f||_s,$$

where the infimum is taken over all possible extensions of ℓf.

We need the notion of the conjugate $\overset{*}{C}$ of a convex cone C and a radial tubular domain $T(C)$ over the cone C. By definition,

$$\overset{*}{C} = \{x \in \mathbb{R}^m : (x,y) > 0 \forall y \in C\},$$

where (x,y) denotes the inner product of x and y, and $T(C)$ is a subset of $\mathbb{C}^m$ of the form $T(C) = \mathbb{R}^m + iC$.

Definition 1.1. For an operator A the *wave factorization* of the symbol $A(\xi)$ with respect to a convex cone C is defined by

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

where $A_{\neq}(\xi)$ and $A_{=}(\xi)$ factors satisfying the following conditions:

(1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined for all $\xi \in \mathbb{R}^m$ except, possibly, for points on the boundary $\overset{*}{C} \cup (-\overset{*}{C})$,

(2) $A_{\neq}, (\xi)A_{=}(\xi)$ admit an analytic extension to a radial tubular domain $T(C^*), T(-C^*)$ respectively and satisfy the estimates

$$c_1 \leqslant |A_{\neq}(\xi + i\tau)|(1+|\xi|+|\tau)^{-æ} \leqslant c_2,$$

$$c_3 \leqslant |A_{\neq}(\xi - i\tau)|(1+|\xi|+|\tau)^{æ-\alpha} \leqslant c_4 \quad \forall\, \tau \in \overset{*}{C},$$

where $c_1, c_2, c_3, c_4 \geqslant 0$ and the real number æ is the *index* of the wave factorization.

Here, we consider model pseudodifferential operator with symbol independent of the spatial variable and the equation with such an operator will be studied in a canonical domain, i.e., in a certain cone.

We study the following pseudodifferential equation:

$$(Au)(x) = g(x), \quad x \in \mathbb{R}^m \setminus \overline{W}_+^a \tag{1.2}$$

where $W_+^a = C_+^a \times \mathbb{R}^{m-2}$, $C_+^a = \{x \in \mathbb{R}^2 | x = (x_1, x_2), x_2 > a|x_1|, a > 0\}$, g is an arbitrary function in $H_0^{s-\alpha}(\mathbb{R}^m \setminus \overline{W}_+^a)$ and $A(\xi)$ admits the wave factorization the wave factorization with respect to $-W_+^a$ with index æ such that $æ - s = 1 + \delta$, $|\delta| < 1/2$. Combining the methods and results of [1, 2], we can assert that its general solution $u(\xi)$ in the Sobolev–Slobodetskii space $H^s(\mathbb{R}^m) \setminus W_+^a$ has the form

$$\widetilde{u}(\xi) = A_{\neq}^{-1}(\xi) \left(Q(\xi)(I - G'_{m-2})Q^{-1}(\xi)(A_{=}^{-1}\widetilde{\ell g})(\xi) + \frac{\widetilde{c}_0(\xi_1 + a\xi_2, \xi') + \widetilde{c}_0(\xi_1 - a\xi_2, \xi')}{2} \right)$$

$$+ A_{\neq}^{-1}(\xi) \left(\text{v. p.} \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\widetilde{c}_0(\eta, \xi')}{\xi_1 + a\xi_2 - \eta} d\eta - \text{v. p.} \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\widetilde{c}_0(\eta, \xi')}{\xi_1 - a\xi_2 - \eta} d\eta \right), \tag{1.3}$$

where I is the identity operator, $\xi \in \mathbb{R}^m$, $\xi' \in \mathbb{R}^{m-2}$, ℓg is an extension of g on $H^{s-\alpha}(\mathbb{R}^m)$, $Q(\xi)$ is an arbitrary polynomial the first order in the variables ξ_1, ξ_2 satisfying the condition (1.1) with $\alpha = 1$, c_0 is an arbitrary function in $H^{s-æ+\frac{1}{2}}(\mathbb{R}^{m-1})$ and the integral operator G'_{m-2} is given by

$$G'_{m-2}u(\xi) = \lim_{\tau \to 0} \int_{\mathbb{R}^2} \frac{u(\eta_1, \eta_2, \xi') d\eta_1 d\eta_2}{(\xi_1 - \eta_1)^2 - a^2(\xi_2 - \eta_2 + i\tau)^2}.$$

Remark 1.1. Since we study the external problem for W_+^a, we use the operator $I - G'_{m-2}$ instead of G'_{m-2}.

To determine uniquely the unknown function $\widetilde{c}_0$, we use the Dirichlet boundary condition

$$u|_{x_1 - ax_2 = 0} = g_1(ax_1 + x_2, x'), u|_{x_1 + ax_2 = 0} = g_2(ax_1 - x_2, x'), \tag{1.4}$$

$$x' = (x_3, x_4, ...x_m), \quad g_1, g_2 \in H^{s-\frac{1}{2}}(\mathbb{R}^{m-1}).$$

2 Homogeneous Equation

We first consider the homogeneous equation with zero right-hand side g in (1.2):

$$(Au)(x) = 0, \quad x \in \mathbb{R}^m \setminus \overline{W}_+^a \tag{2.1}$$

Making the change of variables $y_1 = \xi_1 + a\xi_2$, $y_2 = \xi_1 - a\xi_2$, we introduce the one-dimensional singular integral operator with parameter

$$(Sv)(y, \xi') = \text{v. p.} \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{v(\eta, \xi')}{y - \eta} d\eta.$$

Then we write (1.3) in the form

$$\widetilde{U}(y_1,y_2,\xi')=a_{\neq}^{-1}(y_1,y_2,\xi')\Big(\frac{\widetilde{c}_0(y_1,\xi')+S\widetilde{c}_0(y_1,\xi')}{2})+\frac{\widetilde{c}_0(y_2,\xi')-S\widetilde{c}_0(y_2,\xi')}{2}\Big)$$

where

$$\widetilde{U}(y_1,y_2,\xi')=\widetilde{u}\Big(\frac{y_1+y_2}{2},\frac{y_1-y_2}{2a},\xi'\Big),$$
$$a_{\neq}(y_1,y_2,\xi')=A_{\neq}\Big(\frac{y_1+y_2}{2},\frac{y_1-y_2}{2a},\xi'\Big).$$

Further, we introduce two additional operators (I is the identity operator)

$$P=\frac{1}{2}(I+S),\quad Q=\frac{1}{2}(I-S).$$

Then the formula for solutions takes the form

$$\widetilde{U}(y_1,y_2,\xi')=a_{\neq}^{-1}(y_1,y_2,\xi')(P\widetilde{c}_0(y_1,\xi')+Q\widetilde{c}_0(y_2,\xi')).$$

Denoting $P\widetilde{c}_0=\widetilde{C}_0$ and $Q\widetilde{c}_0=\widetilde{D}_0$, we have

$$\widetilde{U}(y_1,y_2,\xi')=a_{\neq}^{-1}(y_1,y_2,\xi')\left(\widetilde{C}_0(y_1,\xi')+\widetilde{D}_0(y_2,\xi')\right) \tag{2.2}$$

Taking into account the conditions (1.4) and their Fourier transforms, integrating Equation (2.2) with respect to y_1 and y_2, we obtain the following system of linear integral equations:

$$\int_{-\infty}^{\infty}K_1(y_1,y_2,\xi')\widetilde{C}_0(y_1,\xi')dy_1+\widetilde{D}_0(y_2,\xi')$$
$$=\widetilde{G}_1(y_2,\xi')\widetilde{C}_0(y_1,\xi')+\int_{-\infty}^{\infty}K_2(y_1,y_2,\xi')\widetilde{D}_0(y_2,\xi')dy_2+=\widetilde{G}_2(y_1,\xi'), \tag{2.3}$$

where

$$K_1(y_1,y_2,\xi')=a_{\neq}^{-1}(y_1,y_2,\xi')\widetilde{a}_0^{-1}(y_2,\xi'),\quad K_2(y_1,y_2,\xi')=a_{\neq}^{-1}(y_1,y_2,\xi')\widetilde{b}_0^{-1}(y_1,\xi'),$$
$$\widetilde{G}_1(y_2,\xi')=\widetilde{g}_1(y_2,\xi')\widetilde{a}_0^{-1}(y_2,\xi'),\quad \widetilde{G}_2(y_1,\xi')=\widetilde{g}_2(y_1,\xi')\widetilde{b}_0^{-1}(y_1,\xi'),$$
$$\widetilde{a}_0(y_2,\xi')=\int_{-\infty}^{\infty}a_{\neq}^{-1}(y_1,y_2,\xi')dy_1,\quad \widetilde{b}_0(y_1,\xi')=\int_{-\infty}^{\infty}a_{\neq}^{-1}(y_1,y_2,\xi')dy_2.$$

Theorem 2.1. *We assume that the symbol $A(\xi)$ the operator A admits the wave factorization with respect to $-W_+^a$ with index æ such that $æ-s=1+\delta$, $|\delta|<1/2$. Then the Dirichlet problem* (2.1),(1.4) *is equivalent to the system of linear integral equations* (2.3) *with the unknown functions $\widetilde{C}_0,\widetilde{D}_0\in H^{s-æ+\frac{1}{2}}(\mathbb{R}^{m-1})$ under the condition $\widetilde{a}_0(y_2,\xi')\neq 0$, $\widetilde{b}_0(y_1,\xi')\neq 0$.*

2.1. Limit case $a \to \infty$. We consider the system of integral equations (2.3) and determine its structure as $a \to \infty$; this corresponds to the case where W_+^a is degenerated to a half-space. Since $\widetilde{a}_0(y_2, \xi')$ depends on a, assume that the limit

$$\lim_{a\to\infty} a_{\neq}^{-1}(y_1, y_2, \xi') = \mathfrak{A}(y_1, y_2, \xi') \tag{2.4}$$

exists and one can pass to the limit under the integral sign.

Remark 2.1. In fact, we have

$$a_{\neq}^{-1}(y_1, y_2, \xi') = A_{\neq}^{-1}\Big(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2a}, \xi'\Big),$$

so that our assumption means that the function $\mathfrak{A}(y_1, y_2, \xi')$ is symmetric with respect to y_1, y_2; it depends on $y_1 + y_2$.

We begin with the functions $\widetilde{a}_0(y_2, \xi'), \widetilde{b}_0(y_1, \xi')$ as $a \to \infty$:

$$\widetilde{a}_0(y_2, \xi') = \int_{-\infty}^{\infty} a_{\neq}^{-1}(y_1, y_2, \xi')dy_1 \longrightarrow \int_{-\infty}^{\infty} \mathfrak{A}(y_1, y_2, \xi')dy_1 \equiv \widetilde{A_0}(y_2, \xi'),$$

$$\widetilde{b}_0(y_1, \xi') = \int_{-\infty}^{\infty} a_{\neq}^{-1}(y_1, y_2, \xi')dy_2 \longrightarrow \int_{-\infty}^{\infty} \mathfrak{A}(y_1, y_2, \xi')dy_2 \equiv \widetilde{B_0}(y_1, \xi').$$

Taking into account the above remarks, it is easy to see that $\widetilde{A}_0(y, \xi') = \widetilde{B}_0(y, \xi')$. Similarly, we check the characteristics $K_1(y_1, y_2, \xi')$ and $K_2(y_1, y_2, \xi')$:

$$K_1(y_1, y_2, \xi') = a_{\neq}^{-1}(y_1, y_2, \xi')\widetilde{a_0}^{-1}(y_2, \xi') \longrightarrow \mathfrak{A}(y_1, y_2, \xi')\widetilde{A_0}^{-1}(y_2, \xi'), \quad a \to \infty,$$

$$K_2(y_1, y_2, \xi') = a_{\neq}^{-1}(y_1, y_2, \xi')\widetilde{b_0}^{-1}(y_2, \xi') \longrightarrow \mathfrak{A}(y_1, y_2, \xi')\widetilde{B_0}^{-1}(y_1, \xi'), \quad a \to \infty.$$

The kernels K_1 and K_2 converge to a common symmetric kernel, say, $K(y_1, y_2, \xi')$.

Finally, the last limit is connected with the boundary function; namely,

$$\widetilde{G}_1(y_2, \xi') = \widetilde{g_1}(y_2, \xi')\widetilde{a}_0^{-1}(y_2, \xi') \longrightarrow \widetilde{g}_1(y_2, \xi')\widetilde{A}_0^{-1}(y_2, \xi') \equiv \widetilde{\psi_1}(y_2, \xi'), \quad a \to \infty,$$

$$\widetilde{G}_2(y_1, \xi') = \widetilde{g}_2(y_1, \xi')\widetilde{b}_0^{-1}(y_1, \xi') \longrightarrow \widetilde{g}_2(y_1, \xi')\widetilde{B}_0^{-1}(y_1, \xi') \equiv \widetilde{\psi_2}(y_1, \xi'), \quad a \to \infty$$

Then the system (2.3) take the following form as $a \to \infty$:

$$\int_{-\infty}^{\infty} K(y_1, y_2, \xi')\widetilde{C}_0(y_1, \xi')dy_1 + \widetilde{D}_0(y_2, \xi')$$

$$= \widetilde{\psi}_1(y_2, \xi')\widetilde{C}_0(y_1, \xi') + \int_{-\infty}^{\infty} K(y_1, y_2, \xi')\widetilde{D}_0(y_2, \xi')dy_2 = \widetilde{\psi}_2(y_1, \xi'). \tag{2.5}$$

Exchanging the variables y_1 and y_2 in the second equation and taking into account that the kernel is symmetric, we can write

$$\widetilde{C}_0(y_2, \xi') + \int_{-\infty}^{\infty} K(y_1, y_2, \xi')\widetilde{D}_0(y_1, \xi')dy_1 = \widetilde{\psi}_2(y_2, \xi').$$

Further, adding with the first equation, we obtain the single equation

$$(\widetilde{C}_0(y_2,\xi^{'})+\widetilde{D}_0(y_2,\xi^{'}))+\int\limits_{-\infty}^{\infty}K(y_1,y_2,\xi^{'})(\widetilde{C}_0(y_1,\xi^{'})+\widetilde{D}_0(y_1,\xi^{'}))dy_1=\widetilde{\psi}_1(y_2,\xi^{'})+\widetilde{\psi}_2(y_2,\xi^{'}).$$

However, $\widetilde{C}_0(y_2,\xi^{'})+\widetilde{D}_0(y_2,\xi^{'})=\widetilde{c}_0(y_2,\xi^{'})$ because $P+Q=I$, and, finally, we can represent the equation as follows:

$$\widetilde{c}_0(y,\xi^{'})+\int\limits_{-\infty}^{\infty}K(y,\tau,\xi^{'})\widetilde{c}_0(\tau,\xi^{'})d\tau_1=\widetilde{\Psi}(y,\xi^{'}),\quad \widetilde{\Psi}(y,\xi^{'})=\widetilde{\psi}_1(y,\xi^{'})+\widetilde{\psi}_2(y,\xi^{'}) \tag{2.6}$$

Theorem 2.2. *We assume that the symbol $A(\xi)$ of the operator A admits the wave factorization with respect to $-W_+^a$ with index æ such that $æ-s=1+\delta$, $|\delta|<1/2$, for all sufficiently large a and satisfies* (2.4)*, $g_1,g_2\in H^{s-\frac{1}{2}}(\mathbb{R}_+^{m-1})$. Then the boundary value problem* (2.1)*,* (1.4) *is equivalent to the linear integral equation* (2.6) *with unknown function $\widetilde{c}_0\in\widetilde{H}^{s-æ+\frac{1}{2}}(\mathbb{R}^{m-1})$.*

2.2. Dirichlet condition. Here, we consider the Dirichlet problem in the domain $\mathbb{R}^m$ with cut along D_0 of the form $D_0=\{x\in\mathbb{R}^m|x=(x_1,x_2,...,x_m),x_1=0,x_2>0\}$. We consider the equation with the Dirichlet boundary condition

$$(Au)(x)=0,\quad x\in\mathbb{R}^m\setminus D_0,\quad u|_{D_0}=\varphi, \tag{2.7}$$

with a given function $\varphi\in H^{s-\frac{1}{2}}(\mathbb{R}_+^{m-1})$.

We introduce into the consideration the wedge W_+^a for sufficiently large a, then we take an extension Φ of φ on $H^s(W_+^a)$ and study Equation (2.1) in W_+^a with the Dirichlet data

$$u|_{x_1-ax_2=0}=\Phi(ax_1+x_2,x'),u|_{x_1+ax_2=0}=\Phi(ax_1-x_2,x'),\quad x'\in\mathbb{R}^{m-2}$$

where it is assumed that $a\to\infty$.

Using the above constructions, we obtain instead of (2.6) the following equation:

$$\widetilde{c}_0(y,\xi^{'})+\int\limits_{-\infty}^{\infty}K(y,\tau,\xi^{'})\widetilde{c}_0(\tau,\xi^{'})dy_1=\frac{2\widetilde{\varphi}(y,\xi^{'})}{\widetilde{A}(y,\xi^{'})}, \tag{2.8}$$

where

$$K(y,\tau,\xi^{'})=\frac{\mathfrak{A}(y,\tau,\xi^{'})}{\widetilde{A}(y,\xi^{'})},\quad \widetilde{A}(y,\xi^{'})=\int\limits_{-\infty}^{\infty}\mathfrak{A}(y,\tau,\xi^{'})d\tau.$$

Corollary 2.1. *Assume that the symbol $A(\xi)$ admits the wave factorization with respect to $-W_+^a$ with index æ such that $æ-s=1+\delta$, $|\delta|<1/2$, for all sufficiently large a* (2.4) *holds, $\varphi(x)\in H^{s-\frac{1}{2}}(D_0)$. Then the unique solvability of the boundary value problem* (2.7) *is equivalent to the unique solvability of the linear integral equation* (2.8) *with unknown function $\widetilde{c}_0\in\widetilde{H}^{s-æ+\frac{1}{2}}(\mathbb{R}^{m-1})$.*

3 Inhomogeneous Equation

In this section, we consider the inhomogeneous case of a nonzero right-hand side of $g \in H^{s-\alpha}(\mathbb{R}^m \setminus \overline{W}_+^a)$, In other words, we consider the equation

$$(Au)(x) = g(x), \quad x \in \mathbb{R}^m \setminus \overline{W}_+^a \tag{3.1}$$

According to (1.3),

$$A_{\neq}^{-1}((\xi)(Q(\xi)(I - G_{m-2}^{'})Q^{-1}(\xi)(A_{=}^{-1}\widetilde{l}g)(\xi) = F(\xi).$$

Then we can write (1.3) as

$$\begin{aligned}\widetilde{u}(\xi) &= F(\xi) + A_{\neq}^{-1}(\xi)\Big(\frac{\widetilde{c_0}(\xi_1 + a\xi_2, \xi^{'}) + \widetilde{c}_0(\xi_1 - a\xi_2, \xi^{'})}{2}\Big) \\ &+ A_{\neq}^{-1}(\xi)\left(\text{v. p.}\,\frac{i}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\widetilde{c}_0(\eta,\xi)}{\xi_1 + a\xi_2 - \eta}d\eta - \text{v. p.}\,\frac{i}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\widetilde{c_0}(\eta,\xi)}{\xi_1 - a\xi_2 - \eta}d\eta\right).\end{aligned} \tag{3.2}$$

Using the above change of variables y_1, y_2 and operators P, Q, we can write

$$\widetilde{U}(y_1, y_2, \xi^{'}) = \widetilde{f}(y_1, y_2, \xi^{'}) + \frac{\widetilde{C}_0(y_1, \xi^{'}) + \widetilde{D}_0(y_2, \xi^{'})}{a_{\neq}(y_1, y_2, \xi^{'})} \tag{3.3}$$

where

$$\widetilde{f}(y_1, y_2, \xi^{'}) = F\Big(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2a}, \xi^{'}\Big).$$

Here, we used the notation introduced above.

Now, we first integrate (3.3) with respect to y_1 and then with respect to y_2. Using (1.4) in the Fourier images, we find the system of linear integral equations

$$\begin{aligned}&\int\limits_{-\infty}^{\infty} K_1^*(y_1, y_2, \xi^{'})dy_1 + \int\limits_{-\infty}^{\infty} K_1(y_1, y_2, \xi^{'})\widetilde{C}_0(y_1, \xi^{'})dy_1 + \widetilde{D}_0(y_2, \xi^{'}) \\ &= \widetilde{G}_1(y_2, \xi^{'})\widetilde{C}_0(y_1, \xi^{'}) + \int\limits_{-\infty}^{\infty} K_2^*(y_1, y_2, \xi^{'})dy_2 + \int\limits_{-\infty}^{\infty} K_2(y_1, y_2, \xi^{'})\widetilde{D}_0(y_2, \xi^{'})dy_2 = \widetilde{G}_2(y_1, \xi^{'}),\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}&K_1(y_1, y_2, \xi^{'}) = a_{\neq}^{-1}(y_1, y_2, \xi^{'})\widetilde{a}_0^{-1}(y_2, \xi^{'}), \quad K_2(y_1, y_2, \xi^{'}) = a_{\neq}^{-1}(y_1, y_2, \xi^{'})\widetilde{b}_0^{-1}(y_1, \xi^{'}), \\ &K_1^*(y_1, y_2, \xi^{'}) = \widetilde{a}_0^{-1}(y_2, \xi^{'})\widetilde{f}(y_1, y_2, \xi^{'}), \quad K_2^*(y_1, y_2, \xi^{'}) = \widetilde{b}_0^{-1}(y_2, \xi^{'})\widetilde{f}(y_1, y_2, \xi^{'}), \\ &\widetilde{G}_1(y_2, \xi^{'}) = \widetilde{g}_1(y_2, \xi^{'})\widetilde{a}_0^{-1}(y_2, \xi^{'}), \quad \widetilde{G}_2(y_1, \xi^{'}) = \widetilde{g}_2(y_1, \xi^{'})\widetilde{b}_0^{-1}(y_1, \xi^{'}).\end{aligned}$$

Theorem 3.1. *Let the symbol $A(\xi)$ admit the wave factorization with respect to $-W_+^a$ with index æ such that $æ - s = 1 + \delta$, $|\delta| < \frac{1}{2}$, If $\widetilde{a}_0(y_2, \xi^{'}) \neq 0$, $\widetilde{b}_0(y_1, \xi^{'}) \neq 0$, then the Dirichlet problem* (3.1), (1.4) *is equivalent to the system of linear integral equations* (3.4) *with unknown functions $\widetilde{C}_0, \widetilde{D}_0 \in H^{s-æ+\frac{1}{2}}(\mathbb{R}^{m-1})$.*

Remark 3.1. Each of these equations can be written with a single unknown function, but then we obtain double integrals.

3.1. Limit case $a \to \infty$. In this subsection, we modify the system of linear integral equations (3.4) in the case of essentially large values a as $a \to \infty$. This corresponds to the case where W_+^a is degenerated to an $m-1$-dimensional half-space.

In accordance with the aforesaid, we need to study an additional term connected with the right-hand side. Since there is the operator G'_{m-2}, we consider this term. Making the change of variables

$$\begin{aligned} t_1 &= \eta_1 + a\eta_2, \qquad & y_1 &= \xi_1 + a\xi_2, \\ t_2 &= \eta_1 - a\eta_2, \qquad & y_2 &= \xi_1 - a\xi_2, \end{aligned}$$

we find

$$\begin{aligned} G'_{m-2}\widetilde{u}(\xi) &= \frac{a}{2\pi^2}\lim_{\tau\to 0}\int\limits_{\mathbb{R}^2} \frac{\widetilde{u}(\eta_1,\eta_2,\xi')d\eta_1 d\eta_2}{(\xi_1-\eta_1)^2 - a^2(\xi_2-\eta_2+i\tau)^2} \\ &= \lim_{\tau\to 0}\frac{a}{2\pi^2}\int\limits_{\mathbb{R}^2}\frac{1}{2a}\frac{\widetilde{u}(\frac{t_1+t_2}{2},\frac{t_1-t_2}{2a},\xi')dt_1dt_2}{(y_1-t_1-ai\tau)(y_2-t_2+i\tau)} \\ &= \lim_{\tau\to 0}\frac{1}{4\pi^2}\int\limits_{\mathbb{R}^2}\frac{\widetilde{u}(\frac{t_1+t_2}{2},\frac{t_1+t_2}{2a},\xi')dt_1dt_2}{(t_1-y_1+ai\tau)(t_2-y_2-i\tau)} = \frac{1}{(2i\pi)^2}\int\limits_{\mathbb{R}^2}\frac{\widetilde{u}(\frac{t_1+t_2}{2},\frac{t_1+t_2}{2a},\xi')dt_1dt_2}{(t_1-z_1)(t_2-z_2)} \\ &= \frac{1}{4}\Big(\widetilde{U}(y_1,y_2,\xi') - S_1\widetilde{U}(y_1,y_2,\xi') + S_2\widetilde{U}(y_1,y_2,\xi') + S_1S_2\widetilde{U}(y_1,y_2,\xi')\Big), \end{aligned}$$

where $z_1 = y_1 - ai\tau,\ z_2 = y_2 + ai\tau,\ \tau \to 0$,

$$\widetilde{U}(t_1,t_2,\xi') = \widetilde{u}\Big(\frac{t_1+t_2}{2},\frac{t_1+t_2}{2a},\xi'\Big)$$

and S_1, S_2 are two one-dimensional singular integral operators with respect to the variables t_1, t_2 defined by

$$S_1\widetilde{U}(y_1,y_2,\xi') = \text{v. p.}\,\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{\widetilde{U}(t_1,y_2,\xi')}{t_1-y_1}dt_1,$$

$$S_2\widetilde{U}(y_1,y_2,\xi') = \text{v. p.}\,\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{\widetilde{U}(y_1,t_2,\xi')}{t_2-y_2}dt_2.$$

Here, we used formulas from [6] for the limit values of the bisingular integrals.

Further, we have

$$\begin{aligned} &\lim_{a\to\infty}(G'_{m-2}\widetilde{u})\Big(\frac{y_1+y_2}{2},\frac{y_1-y_2}{2a},\xi'\Big) = \lim_{a\to\infty}\frac{1}{4}(\widetilde{U}(y_1,y_2,\xi') - S_1\widetilde{U}(y_1,y_2,\xi') \\ &+ S_2\widetilde{U}(y_1,y_2,\xi') + S_1S_2\widetilde{U}(y_1,y_2,\xi'))\lim_{a\to\infty}\frac{1}{4}\Big(\widetilde{u}\Big(\frac{y_1+y_2}{2},\frac{y_1-y_2}{2a},\xi'\Big) \\ &- S_1\widetilde{u}\Big(\frac{y_1+y_2}{2},\frac{y_1-y_2}{2a},\xi'\Big) + S_2\widetilde{u}\Big(\frac{y_1+y_2}{2},\frac{y_1-y_2}{2a},\xi'\Big) + S_1S_2\widetilde{u}\Big(\frac{y_1+y_2}{2},\frac{y_1-y_2}{2a},\xi'\Big)\Big) \end{aligned}$$

$$= \frac{1}{4}\Big(\widetilde{u}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big) - S_1\widetilde{u}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big) + S_2\widetilde{u}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big) + S_1S_2\widetilde{u}\Big(\frac{y_1+y_2}{2},0,\xi^{'})\Big) \tag{3.5}$$

It is easy to see that

$$S_1\widetilde{u}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big) = S_2\widetilde{u}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big)$$

and

$$S_1S_2\widetilde{u}\left(\frac{y_1+y_2}{2},0,\xi^{'}\right) = \frac{1}{\pi i}\,\text{v. p.}\int\limits_{-\infty}^{\infty}\left(\frac{1}{\pi i}\,\text{v. p.}\int\limits_{-\infty}^{\infty}\frac{1}{t_1-y_1}\frac{\widetilde{u}(\frac{t_1+t_2}{2},0,\xi^{'})}{t_2-y_2}dt_2\right)dt_1$$

$$= S_1^2\widetilde{u}\Big(\frac{t_1+t_2}{2},0,\xi^{'}\Big) = \widetilde{u}\Big(\frac{t_1+t_2}{2},0,\xi^{'}\Big).$$

Consequently,

$$\lim_{a\to\infty}(G^{'}_{m-2}\widetilde{u})\Big(\frac{y_1+y_2}{2},\frac{y_1-y_2}{2a},\xi^{'}\Big) = \frac{1}{4}\Big(\widetilde{u}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big) + \widetilde{u}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big)\Big)$$

$$= \frac{1}{2}\widetilde{u}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big).$$

According to the aforesaid, we need to correct the limit values K_1^* and K_2^* in (3.4). We recall that these are two terms in (3.4)

$$\int\limits_{-\infty}^{+\infty}\widetilde{a}_0^{-1}(y_2,\xi^{'})\widetilde{f}(y_1,y_2,\xi^{'})dy_1, \quad \int\limits_{-\infty}^{+\infty}\widetilde{b}_0^{-1}(y_2,\xi^{'})\widetilde{f}(y_1,y_2,\xi^{'})dy_2.$$

Taking into account the calculations with $G^{'}_{m-2}$, we need to determine the limit of $\widetilde{f}(y_1,y_2,\xi')$, under the assumption that the limit passage under the integral sign is allowed. We recall

$$\widetilde{f}(y_1,y_2,\xi^{'}) = F\left(\frac{y_1+y_2}{2},\frac{y_1-y_2}{2a},\xi^{'}\right), \quad F(\xi) = A^{-1}_{\neq}((\xi)(Q(\xi)(I-G^{'}_{m-2})Q^{-1}(\xi)(A^{-1}_{=}\widetilde{\ell g})(\xi).$$

As above, we denote

$$a^{-1}_{=}(y_1,y_2,\xi^{'}) = A^{-1}_{=}\left(\frac{y_1+y_2}{2},\frac{y_1-y_2}{2a},\xi'\right)$$

and assume that the limit

$$\lim_{a\to\infty} a^{-1}_{=}(y_1,y_2,\xi^{'}) = \mathfrak{B}(y_1,y_2,\xi') \tag{3.6}$$

exists, and it is possible to pass to the limit under the integral sign and the function $\mathfrak{B}(y_1,y_2,\xi')$ is symmetric with respect to the variables y_1,y_2, it depends on y_1+y_2.

We simplify $F(\xi)$ as follows:

$$F(\xi) = A^{-1}(\xi)\widetilde{\ell g}(\xi) - A^{-1}_{\neq}((\xi)(Q(\xi)G^{'}_{m-2}Q^{-1}(\xi)(A^{-1}_{=}\widetilde{\ell g})(\xi).$$

We write in the variable y_1,y_2:

$$\lim_{a\to\infty}\widetilde{f}(y_1,y_2.\xi') = A^{-1}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big)\widetilde{g}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big)$$

$$-\frac{1}{2}\mathfrak{A}(y_1,y_2,\xi')\mathfrak{B}(y_1,y_2,\xi')\widetilde{g}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big)$$

at least for g in the Schwarz space $S(\mathbb{R}^m)$. However,

$$\begin{aligned}a^{-1}(y_1,y_2,\xi') &= \lim_{a\to\infty} A_{=}^{-1}\Big(\frac{y_1+y_2}{2},\frac{y_1-y_2}{2a},\xi'\Big)\\ &= A^{-1}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big)=\mathfrak{A}(y_1,y_2,\xi')\mathfrak{B}(y_1,y_2,\xi'),\end{aligned}$$

so that we have

$$\lim_{a\to\infty}\widetilde{f}(y_1,y_2.\xi')=\frac{1}{2}A^{-1}\left(\frac{y_1+y_2}{2},0,\xi^{'}\right)\widetilde{g}\left(\frac{y_1+y_2}{2},0,\xi^{'}\right),$$

and, in this case,

$$\begin{aligned}K_1^*(y_1,y_2,\xi') &= \frac{1}{2}\widetilde{A}_0^{-1}(y_2,\xi')A^{-1}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big)\widetilde{g}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big),\\ K_2^*(y_1,y_2,\xi') &= \frac{1}{2}\widetilde{B}_0^{-1}(y_1,\xi')A^{-1}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big)\widetilde{g}\Big(\frac{y_1+y_2}{2},0,\xi^{'}\Big),\end{aligned}$$

We denote by $K^*(y_1,y_2,\xi')$ their common value.

Then the system (3.4) takes the following form as $a\to\infty$:

$$\begin{aligned}&\int_{-\infty}^{\infty}K^*(y_1,y_2,\xi^{'})dy_1+\int_{-\infty}^{\infty}K(y_1,y_2,\xi^{'})\widetilde{C_0}(y_1,\xi^{'})dy_1+\widetilde{D_0}(y_2,\xi^{'})=\widetilde{\psi}_1(y_2,\xi^{'}),\\ &\widetilde{C_0}(y_1,\xi^{'})+\int_{-\infty}^{\infty}K^*(y_1,y_2,\xi^{'})dy_2+\int_{-\infty}^{\infty}K(y_1,y_2,\xi^{'})\widetilde{D_0}(y_2,\xi^{'})dy_2=\widetilde{\psi}_2(y_1,\xi^{'}).\end{aligned}\tag{3.7}$$

In the second equation, we exchange the variables and take into account the symmetry of kernels. Then

$$\widetilde{C_0}(y_2,\xi^{'})+\int_{-\infty}^{\infty}K^*(y_1,y_2,\xi^{'})dy_1+\int_{-\infty}^{\infty}K(y_1,y_2,\xi^{'})\widetilde{D_0}(y_1,\xi^{'})dy_1=\widetilde{\psi}_2(y_2,\xi^{'}).$$

Adding the last equation with the first one, we obtain the equation

$$\begin{aligned}&(\widetilde{C_0}(y_2,\xi^{'})+\widetilde{D_0}(y_2,\xi^{'}))+2\int_{-\infty}^{\infty}K^*(y_1,y_2,\xi^{'})dy_1+\int_{-\infty}^{\infty}K(y_1,y_2,\xi^{'})(\widetilde{C_0}(y_1,\xi^{'})+\widetilde{D_0}(y_1,\xi^{'}))dy_1\\ &=\widetilde{\psi}_1(y_2,\xi^{'})+\widetilde{\psi}_2(y_2,\xi^{'})\end{aligned}$$

However, $\widetilde{C_0}(y_2,\xi^{'})+\widetilde{D_0}(y_2,\xi^{'})=\widetilde{c_0}(y_2,\xi^{'})$. Finally, the last equation can be written as

$$\widetilde{c_0}(y,\xi^{'})+2\int_{-\infty}^{\infty}K^*(y,\tau,\xi^{'})d\tau+\int_{-\infty}^{\infty}K(y,\tau,\xi^{'})\widetilde{c_0}(\tau,\xi^{'})d\tau=\widetilde{\Psi}(y,\xi^{'})\tag{3.8}$$

Theorem 3.2. *We assume that the symbol $A(\xi)$ admits the wave factorization with respect to $-W_+^a$ with index æ such that $æ - s = 1 + \delta$, $|\delta| < \frac{1}{2}$ for all sufficiently large a the conditions* (2.4) *and* (3.6) *hold, $g_1, g_2 \in H^{s-\frac{1}{2}}(\mathbb{R}_+^{m-1})$. Then the boundary value problem* (3.1),(1.4) *is equivalent to the integral Equation* (3.8) *with unknown $\widetilde{c}_0 \in H^{s-æ+\frac{1}{2}}(\mathbb{R}^{m-1})$.*

3.2. Dirichlet condition on cut. We consider the Dirichlet problem in $\mathbb{R}^m$ with cut $D_0D_0 = \{x \in \mathbb{R}^m | x = (x_1, x_2, ..., x_m), x_1 = 0, x_2 > 0\}$, which can be written as follows:

$$(Au)(x) = f(x), \quad x \in \mathbb{R}^m \setminus D_0, \quad u|_{D_0} = \theta, \tag{3.9}$$

where $\theta \in H^{s-\frac{1}{2}}(\mathbb{R}_+^{m-1})$. The function Θ is chosen as an extension of θ to W_+^a in such a way that $\Theta \in H^s(W_+^a)$ for sufficiently large a. The study of the solvability of Equation (3.1) with boundary condition

$$u|_{x_1 - ax_2 = 0} = \Theta(ax_1 + x_2, x^{'}), \quad u|_{x_1 + ax_2 = 0} = \Theta(ax_1 - x_2, x^{'}), \quad x^{'} \in \mathbb{R}^{m-2}, \tag{3.10}$$

leads to the conclusion that we obtain the equation below instead of Equation (3.8)

$$\widetilde{c_0}(y, \xi^{'}) + 2 \int\limits_{-\infty}^{\infty} K^*(\tau, y, \xi^{'}) d\tau + \int\limits_{-\infty}^{\infty} K(\tau, y, \xi^{'}) \widetilde{c_0}(\tau, \xi^{'}) d\tau = \frac{2\widetilde{\theta}(y, \xi^{'})}{\widetilde{A_0}(y, \xi^{'})}, \tag{3.11}$$

where $K(\tau, y, \xi^{'}), K^*(\tau, y, \xi^{'})$ are as above.

Corollary 3.1. *If the assumptions of Theorem* 3.2 *hold and $\theta \in H^{s-\frac{1}{2}}(D_0)$, then the unique solvability the problem* (3.9) *is equivalent to the unique solvability of the linear integral equation* (3.11) *in the space $H^{s-æ+\frac{1}{2}}(\mathbb{R}^{m-1})$.*

Declarations

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflict of Interests: The authors declare that there is no conflict of interest.

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Submitted on March 23, 2025