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SOLVING THE CAUCHY PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION WITH AN INTEGER POWER OF THE BESSEL OPERATOR USING TRANSMUTATION OPERATORS

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Abstract
This work investigates the Cauchy problem for a high-order ordinary differential equation involving the Bessel operator and a spectral parameter. Due to the lack of appropriate tools, this problem has not been previously studied. The primary aim of the paper is to introduce one such tool, namely the generalized Erdélyi–Kober fractional operator, which possesses the property of a transmutation operator. By applying this operator, the considered problem is reduced to an equation without degeneration and without a lower-order term. An explicit formula for the solution of the problem is constructed. Another aim is to demonstrate the effectiveness of the proposed method, which allows for obtaining an exact solution to the formulated problem. Despite the advancement of modern computational tools, constructing exact solutions for boundary value problems for ordinary differential equations remains an important and relevant task. Such solutions provide deeper insight into the qualitative properties of the processes and phenomena being described, the properties of mathematical models, and may also serve as benchmark examples for asymptotic, approximate, and numerical methods.

Keywords High-order ordinary differential equation · Cauchy problem · Generalized Erdélyi–Kober fractional operator

Mathematics Subject Classification (2010) 34A05 · 34A12 · 34B05

Introduction

In this article, we consider the following ordinary differential equation

$$A_{2\alpha,\lambda}^{m} y(x) \equiv \left(\frac{d^2}{dx^2} + \frac{2\alpha}{x}\frac{d}{dx} + \lambda^2\right)^{m} y(x) = f(x), \ \ \alpha > 0, \ \ m \in N, \tag{1}$$

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with an integer power of the Bessel operator

$$A_{\gamma,\lambda} \equiv B_\gamma + \lambda^2, \gamma, \lambda \in R, \quad \gamma = 2\alpha, \tag{2}$$

where $B_\gamma = \dfrac{d^2}{dx^2} + \dfrac{\gamma}{x}\dfrac{d}{dx}$ is the Bessel operator, $\lambda \in R$, $m \in N$, $f(x)$ is a given function, $A^m_{\gamma,\lambda} = A^{m-1}_{\gamma,\lambda} A_{\gamma,\lambda}$ is the power of the operator in (2).

We now outline some related problems and topics addressed in previous studies. Differential equations with Bessel operators were studied by many authors, most extensively and completely by I.A. Kipriyanov and his school in Voronezh, cf. [1–3]. These studies are closely connected and often based on transmutation theory, cf. [4, 6, 7]. Also, methods and results from fractional calculus are very useful, cf. [3, 5–8], including such important in applications to differential equations special types of fractional operators as, namely, Erdélyi-Kober (see [2, 5, 7–9]), Buschman-Erdélyi (see [3, 10]) ones and fractional type operators with Gauss and Legendre kernels (see [11]). Some problems for products of Bessel–type operators are considered in [12], for products of Sturm–Lioville operators in [13].

We highlight in particular the remarkable paper [14], which presents many original and valuable findings. Among them, problems for 3 types of integer powers of the next differential operators were explicitly solved in terms of integral operators with hypergeometric function kernels

$$D_1 = \left(\frac{1}{x}\frac{d}{dx}\right)^m, D_2 = \left(\frac{d^2}{dx^2} + \frac{\gamma}{x}\frac{d}{dx}\right)^k, D_3 = \left(\frac{1}{x}\frac{d}{dx}\right)^m \left(\frac{d^2}{dx^2} + \frac{\gamma}{x}\frac{d}{dx}\right)^k$$

with additional conditions at $x = 1$ on the solution and its derivatives. Choice of $x = 1$ avoids problems at $x = 0$. It should be noted that we consider a more complex case with additional conditions imposed precisely at $x = 0$. For differential equations for Bessel-type operators with conditions on lines of singularities as in (1) special conditions are needed, namely Kipriyanov's evenness conditions, cf. [12], p.235, [3], p.33. Regarding the difficulties with conditions on singularity lines for Bessel-type equations, see [15].

Another valuable part of the paper [14] introduced and outlined explicit constructions for fractional powers of the Bessel operator (2). Further developments of these results can be found in [3, 6, 16, 17].

It should also be noted that for operators of type (1) with a spectral parameter, a special class of transmutations was introduced by S.M. Sitnik. They were named Vekua-Erdélyi-Lowndes (VEL) transmutations after the authors of the first known special cases, for any operator A and a constant λ these VEL transmutations T are defined by a property

$$T(A + \lambda) = AT,$$

see [18, 19]. In this paper, we use such VEL transmutations introduced by Lowndes [20]. Also, our main method is to use as transmutations the generalized Erdélyi–Kober fractional operator [5, 8, 9].

The Cauchy problem

In the domain $R^+ = \{x \in R : x > 0\}$ we seek the solution $y(x) \in C^{2m}\left(R^+\right)$ of the equation (1) satisfying the following initial conditions

$$y^{(j)}(0) = 0, \quad j = 0, 1, \ldots, 2m - 1. \tag{3}$$

It should also be noted that the Cauchy problem for the equation (1) on the interval [0,1] with initial conditions $y^{(j)}(1) = 0, \quad j = 0, 1, \ldots, 2m - 1$ was studied in [14] as mentioned before.

We now turn to the main properties of the generalized Erdélyi–Kober fractional operator.

Generalized Erdélyi–Kober fractional operator with Bessel functions in the kernel

In the works of Erdélyi and Kober, the following modification of fractional integration was introduced:

$$I_{\eta,\alpha}\varphi(x) = \frac{2x^{-2(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^x \left(x^2 - t^2\right)^{\alpha-1} t^{2\eta+1} \varphi(t)\, dt \tag{4}$$

$$K_{\eta,\alpha}\varphi(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^{+\infty} \left(x^2 - t^2\right)^{\alpha-1} t^{1-2(\eta+\alpha)} \varphi(t) dt, \tag{5}$$

where $\alpha > 0, \quad \eta > -1/2, \quad \varphi(x) \in L\left(R^{+}\right)$.

Operators (4), (5) and their generalizations are called Erdélyi–Kober operators [5].

The results of Erdélyi and Kober are generalized in the works of J.S. Lowndes [20], who introduced and studied generalized Erdélyi–Kober operators of the form:

$$J_\lambda(\eta, \alpha) f(x) = 2^\alpha \lambda^{1-\alpha} x^{-2\alpha-2\eta}$$

$$\times \int_0^x t^{2\eta+1} (x^2 - t^2)^{(\alpha-1)/2} J_{\alpha-1}\left(\lambda\sqrt{x^2 - t^2}\right) f(t) dt \tag{6}$$

$$R_\lambda(\eta, \alpha) f(x) = 2^\alpha \lambda^{1-\alpha} x^{2\eta}$$

$$\times \int_x^{+\infty} t^{1-2\alpha-2\eta} (t^2 - x^2)^{(\alpha-1)/2} J_{\alpha-1}\left(\lambda\sqrt{t^2 - x^2}\right) f(t) dt, \tag{7}$$

where $\eta, \alpha, \lambda \in R$, and $\alpha > 0, \quad \eta \geq -1/2$, $J_\nu(z)$ is the Bessel function of the first kind.

It is clear that when $\lambda \to 0$ operators (6) and (7) coincide with the classical operators (4) and (5):

$$J_0(\eta, \alpha) = I_{\eta,\alpha}, \; R_0(\eta, \alpha) = K_{\eta,\alpha}.$$

Later, we will need the following representation of the operator (6):

$$J_\lambda(\eta, \alpha) f(x) = \frac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^x (x^2 - t^2)^{\alpha-1} \bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2 - t^2}\right) t^{2\eta+1} f(t)) dt, \tag{8}$$

where $\bar{J}_\nu(z)$ is the normalized Bessel function [6, p. 530] defined by

$$\bar{J}_\nu(z) = \Gamma(\nu+1)(z/2)^{-\nu} J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{(\nu+1)_k k!}. \tag{9}$$

The function $\bar{J}_\nu(z)$ is even and infinitely differentiable. Moreover, $\left|\bar{J}_\nu(z)\right| \leq 1$ for $\nu > -1/2$, and it satisfies the following equation:

$$B_\gamma^x \bar{J}_\gamma(\lambda x) + \lambda^2 \bar{J}_\gamma(\lambda x) = 0$$

with the initial conditions $\bar{J}_\nu(0) = 1, \; (d/dx)\bar{J}_\nu(\lambda x)\big|_{x=0} = 0$.

The inverse operator to (8) is given by [5, p. 536]:

$$J_\lambda^{-1}(\eta, \alpha) f(x) = \frac{2x^{-2\eta}}{\Gamma(p-\alpha)} \left(\frac{1}{2x}\frac{d}{dx}\right)^p \int_0^x \frac{\bar{I}_{p-\alpha-1}\left(\lambda\sqrt{x^2 - s^2}\right)}{(x^2 - s^2)^{\alpha-p+1}} s^{2(\eta+\alpha)+1} f(s) ds, \tag{10}$$

where $p = [\alpha] + 1$, $\bar{I}_\nu(z) = \bar{J}_\nu(iz) = \Gamma(\nu+1)(z/2)^{-\nu} I_\nu(z)$ is the Bessel function with imaginary argument.

From [21, 22] we cite the following theorem:

Theorem 1 ([21, 22]) *Let* $\alpha > 0, \quad \gamma \geq 0, \quad f(x) \in C^{2m}(0, b), \quad b > 0$ *and the function* $x^\gamma A_{\gamma,0}^{k+1} f(x)$, *be integrable at zero and* $\lim_{x\to 0} x^\gamma A_{\gamma,0}^{k+1} f(x) = 0, \; k = 0, 1, \ldots, m-1$

Then, the following equality holds true:

$$A_{\gamma+2\alpha,\lambda}^m J_\lambda\left(\frac{\gamma-1}{2}, \alpha\right) f(x) = J_\lambda\left(\frac{\gamma-1}{2}, \alpha\right) A_{\gamma,0}^m f(x).$$

In particular, if $\lambda = 0$, *then*

$$\left[B_{\gamma+2\alpha}\right]^m I_{\frac{\gamma-1}{2},\alpha} f(x) = I_{\frac{\gamma-1}{2},\alpha} \left[B_\gamma\right]^m f(x).$$

Moreover, for $\gamma = 0$ we have:

$$A_{2\alpha,\lambda}^m J_\lambda\left(-\frac{1}{2}, \alpha\right) f(x) = J_\lambda\left(-\frac{1}{2}, \alpha\right) A_{0,0}^m f(x)$$

The last equality can be written in the following expanded form

$$\left(\frac{d^2}{dx^2}+\frac{2\alpha}{x}\frac{d}{dx}+\lambda^2\right)^m J_\lambda\left(-\frac{1}{2},\alpha\right)f(x)=J_\lambda\left(-\frac{1}{2},\alpha\right)f^{(2m)}(x).$$

Application of the generalized Erdélyi-Kober operator to the solution of the Cauchy problem

Assuming that the solution to problem (1), (3) exists, we seek it in the form

$$y(x)=J_\lambda(-1/2,\alpha)z(x)=\frac{2x^{1-2\alpha}}{\Gamma(\alpha)}\int\limits_0^x\left(x^2-t^2\right)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)z(t)\,dt, \tag{11}$$

where $z(x)$ is an unknown function, and we assume that $z(x)$ is sufficiently smooth.

Substituting (11) into the initial conditions (3) and equation (1), and using Theorem 1, we obtain the following equation for the function $z(x)\in C^{2m}\left(R^+\right)$ to

$$z^{(2m)}(x)=F(x),\quad x>0, \tag{12}$$

with the initial conditions

$$z^{(j)}(0)=0,\quad j=0,1,\ldots,2m-1, \tag{13}$$

where

$$F(x)=J_\lambda^{-1}\left(-\frac{1}{2},\alpha\right)f(x). \tag{14}$$

It is easy to verify that the general solution of equation (12) has the form

$$z(x)=\sum_{k=0}^{2m-1}C_kx^k+\frac{1}{(2m-1)!}\int\limits_0^x(x-t)^{2m-1}F(t)\,dt, \tag{15}$$

where C_k, $k=0,1,\ldots,2m-1$ are arbitrary constants.

Substituting (15) into the initial conditions (13), we obtain the solution to the problem (12)–(13) in the following form

$$z(x)=\frac{1}{(2m-1)!}\int\limits_0^x(x-t)^{2m-1}F(t)\,dt$$

. Hence, taking into account the formula $\Gamma(n)=(n-1)!$, we have

$$z(x)=\frac{1}{\Gamma(2m)}\int\limits_0^x(x-t)^{2m-1}F(t)\,dt=I_{0+}^{2m}F(x), \tag{16}$$

where $I_{0+}^{\alpha}\varphi(x)=\frac{1}{\Gamma(\alpha)}\int\limits_0^x(x-t)^{\alpha-1}\varphi(t)\,dt$ is the left-sided Riemann–Liouville integral fractional operator.

Thus, considering (14), we obtain

$$z(x)=I_{0+}^{2m}F(x)=I_{0+}^{2m}J_\lambda^{-1}(-1/2,\alpha)f(x) \tag{17}$$

To simplify equation (17), we substitute (10) for $p=1$, $\eta=-1/2$ the inverse operator into (17) and get:

$$z(x)=\frac{1}{\Gamma(2m)\Gamma(1-\alpha)}\int\limits_0^x(x-t)^{2m-1}F'(t)dt, \tag{18}$$

where

$$F(t)=\int\limits_0^t\left(t^2-s^2\right)^{-\alpha}s^{2\alpha}\bar{I}_{-\alpha}\left(\lambda\sqrt{t^2-s^2}\right)f(s)\,ds.$$

Hence, applying the rule of integration by parts and taking into account that $F(0) = 0$, we get

$$z(x) = \frac{2m-1}{\Gamma(2m)\Gamma(1-\alpha)} \int\limits_0^x (x-t)^{2m-1} F(t) dt. \tag{19}$$

Substituting (18) into the last equality and changing the order of integration, we obtain

$$z(x) = \frac{1}{\Gamma(2m-1)\Gamma(1-\alpha)} \int\limits_0^x s^{2\alpha} f(s)\, g(x,s)\, ds, \tag{20}$$

where

$$g(x,s) = \int\limits_s^x (x-t)^{2m-2} \left(t^2 - s^2\right)^{-\alpha} \bar{I}_{-\alpha}\left(\lambda\sqrt{x^2-t^2}\right) dt.$$

We compute the last integral. Using the series expansion of the normalized Bessel function with imaginary argument and considering the uniform convergence of this series for any values of the argument, by changing the order of integration and summation, we get

$$g(x,s) = \sum_{k=0}^{\infty} \frac{\left[\lambda^2/4\right]^k}{(1-\alpha)_k k!} \int\limits_s^x (x-t)^{2m-2} \left(t^2 - s^2\right)^{k-\alpha} dt. \tag{21}$$

Making a substitution and using the integral representation of the Gauss hypergeometric function, we rewrite (21) as follows

$$g(x,s) = \frac{\Gamma(2m-1)\Gamma(1-\alpha)}{\Gamma(2m-\alpha)} (x-s)^{2m-1} \left(x^2-s^2\right)^{-\alpha}$$
$$\times \sum_{k=0}^{\infty} \frac{\left[\lambda^2(x^2-s^2)/4\right]^k}{(1-\alpha)_k k!} F(2m-1,\ \alpha-k;\ 2m-\alpha+k;\ \tau), \tag{22}$$

where $\tau = \dfrac{x-s}{x+s}$.

Applying the formula to (22) ([23])

$$F(a,\ b;\ a-b+1;\ z) = (1-z)^{-a} F\left[\frac{a}{2}, -b + \frac{a+1}{2}; a-b+1; -4z(1-z)^{-2}\right],$$

we transform equation (22), and then substituting it with the obtained expression of (21) into (20), we have

$$z(x) = \frac{(2x)^{1-2m}}{\Gamma(2m-\alpha)} \int\limits_0^x s^{2\alpha} f(s) \left(x^2-s^2\right)^{2m-\alpha-1} \times$$
$$\times \sum_{k=0}^{\infty} \frac{\sigma^k}{(2m-\alpha)_k k!} F(m,\ m-\frac{1}{2}; 2m-\alpha+k; \omega) ds,$$

where $\omega = 1 - \dfrac{s^2}{x^2}$, $\sigma = \dfrac{\lambda^2}{4}(x^2-s^2)$.

Considering the formula [5]

$$\Xi_2(a,b,c;x,y) = \sum_{n=0}^{\infty} \frac{y^n}{(c)_n n!} F(a,b;c+n;x), \tag{23}$$

we get

$$z(x) = \frac{(2x)^{1-2m}}{\Gamma(2m-\alpha)}$$
$$\times \int\limits_0^x s^{2\alpha} f(s) \left(x^2-s^2\right)^{2m-\alpha-1} \Xi_2\left(m, m-\frac{1}{2}; 2m-\alpha; \omega, \sigma\right) ds, \tag{24}$$

where $\Xi_2\,(a, b, c; x, y)$ is the confluent hypergeometric function of Humbert [24].

If the Gaussian hypergeometric function in expression (22)

$$F(a,\ b; a-b+1; \omega) = \Gamma\,(a-b+1)\,(-\omega)^{\frac{b-a}{2}}\,(1-\omega)^{-b}\,P^{b-a}_{-b}\left(\frac{1+\omega}{1-\omega}\right),\ [-\infty<\omega<0]$$

is expressed in terms of the associated Legendre function [24, p.386, 60], then expression (24) takes the following form:

$$z\,(x) =$$

$$\frac{1}{2^{\alpha}}\int\limits_0^x\left(\sqrt{s^2-x^2}\right)^{2m-1-\alpha}\sum_{k=0}^{\infty}\frac{(2m-\alpha)_k\left[-\frac{\lambda^2}{2}s\sqrt{s^2-x^2}\right]^k}{(1-\alpha)_k k!}P^{\alpha-k-2m+1}_{k-\alpha}\left(\frac{x}{s}\right)s^{\alpha}f\,(s)\,ds.$$

Substituting (24) into equation (11) and changing the order of integration, we obtain

$$y\,(x) = \frac{2^{2-2m}x^{1-2\alpha}}{\Gamma\,(\alpha)\,\Gamma(2m-\alpha)}\int\limits_0^x s^{2\alpha}f\,(s)\,h\,(x,s)\,ds, \tag{25}$$

where

$$h(x,s) = \int\limits_s^x (x^2-t^2)^{\alpha-1}(t^2-s^2)^{2m-\alpha-1}$$

$$\times\,\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)\Xi_2\left(m, m-\frac{1}{2}; 2m-\alpha; \omega_0, \sigma_0\right)t^{1-2m}dt. \tag{26}$$

Here $\omega_0 = 1-\dfrac{s^2}{t^2}$, $\sigma_0 = \dfrac{\lambda^2}{4}(t^2-s^2)$.

To compute integral (26), we make a change of variable according to the formula $t^2 = x^2-\left(x^2-s^2\right)\tau$

$$h(x,s) = \frac{1}{2}x^{-2m}(x^2-s^2)^{2m-1}\int\limits_0^1\tau^{\alpha-1}(1-\tau)^{2m-\alpha-1}(1-\omega\tau)^{-m}$$

$$\times\,\bar{J}_{\alpha-1}\left(2\sqrt{\sigma\tau}\right)\Xi_2\left(m, m-\frac{1}{2}; 2m-\alpha; \frac{\omega(1-\tau)}{1-\omega\tau}, \sigma(1-\tau)\right)d\tau,$$

where $\omega = 1-\dfrac{s^2}{x^2}$, $\sigma = \dfrac{\lambda^2}{4}(x^2-s^2)$.

Using the series expansion of the normalized Bessel function and the uniform convergence of the series, changing the order of integration and summation, we come

$$h(x,s) = \frac{1}{2}x^{-2m}(x^2-s^2)^{2m-1}\sum_{k=0}^{\infty}\frac{(-\sigma)^k}{(\alpha)_k k!}\int\limits_0^1\tau^{k+\alpha-1}(1-\tau)^{2m-\alpha-1}$$

$$\times\,(1-\omega\tau)^{-m}\,\Xi_2\left(m, m-\frac{1}{2}; 2m-\alpha; \frac{\omega(1-\tau)}{1-\omega\tau}, \sigma(1-\tau)\right)d\tau. \tag{27}$$

The following lemma is valid:

Lemma 1 *If* $\operatorname{Re} c > \operatorname{Re}\gamma > 0$, $|\arg(x-1)| < \pi$, *then the following formula holds true*

$$\Xi_2\,(a, b, c; x, y) = \frac{\Gamma(c)}{\Gamma(\gamma)\Gamma(c-\gamma)}\int\limits_0^1\tau^{\gamma-1}(1-\tau)^{c-\gamma-1}(1-x\tau)^{-\delta}$$

$$\times\,F(a-\delta,\ b; \gamma; x\tau)\,\Xi_2\left(\delta, b-\gamma; c-\gamma; \frac{x(1-\tau)}{1-x\tau}, y(1-\tau)\right)d\tau. \tag{28}$$

Proof Substituting the following formula [24] into (23)

$$F(a,b,c;\omega,\sigma)=\frac{\Gamma(c)}{\Gamma(\gamma)\Gamma(c-\gamma)}$$

$$\times\int_0^1 \tau^{\gamma-1}(1-\tau)^{c-\gamma-1}(1-\omega\tau)^{-\delta}F(a-\delta,\ b;\gamma;\omega\tau)$$

$$\times F\left(\delta,b-\gamma;c-\gamma;\frac{\omega(1-\tau)}{1-\omega\tau},\sigma(1-\tau)\right)d\tau$$

and then changing the order of integration and summation, we obtain the desired formula (28).

Substituting into equation (28) $\delta=a$ and taking into account $F(0,\ b;\gamma;\omega)=1$, we get the following equality:

$$\Xi_2(a,b,c;x,y)=\frac{\Gamma(c)}{\Gamma(\gamma)\Gamma(c-\gamma)}$$

$$\times\int_0^1 \tau^{\gamma-1}(1-\tau)^{c-\gamma-1}(1-x\tau)^{-a}\Xi_2\left(a,b-\gamma;c-\gamma;\frac{x(1-\tau)}{1-x\tau},y(1-\tau)\right)d\tau. \tag{29}$$

Applying formula (29) to equation (25), we get

$$h(x,s)=\frac{\Gamma(\alpha)\Gamma(2m-\alpha)}{2\Gamma(2m)}x^{-2m}(x^2-s^2)^{2m-1}$$

$$\times\sum_{k=0}^{\infty}\frac{(-\sigma)^k}{(2m)_k k!}\Xi_2\left(m,m+\alpha-\frac{1}{2}+k;2m+k;\omega,\sigma\right), \tag{30}$$

where $\omega=1-\frac{s^2}{x^2},\quad \sigma=\frac{\lambda^2}{4}(x^2-s^2)$.

Substituting equation (30) into (25), we have

$$y(x)=\frac{1}{\Gamma(2m)}\int_0^x\left(\frac{s}{x}\right)^{2\alpha}\left(\frac{x^2-s^2}{2x}\right)^{2m-1}$$

$$\times\sum_{k=0}^{\infty}\frac{(-\sigma)^k}{(2m)_k k!}\Xi_2\left(m,m+\alpha-\frac{1}{2}+k;2m+k;\omega,\sigma\right)f(s)\,ds. \tag{31}$$

As the result, the following theorem is proved in the article: □

Theorem 2 *If $f(x)\in C[0,+\infty)$, then the solution of the Cauchy problem can be represented in the form (31).*

Remark 1 When $\lambda=0$, equation (1) $A^m_{\gamma,\lambda}y(x)=f(x)$ turns into the equation $A^m_{\gamma,0}y(x)\equiv B^m_\gamma y(x)=f(x)$, and the solution, by the equalities $\Xi_2(a,b,c;x,0)=F(a,b;c;x)$, $\gamma=2\alpha$, corresponds to the result obtained in [14]

$$y(x)=\frac{1}{\Gamma(2m)}\int_0^x\left(\frac{s}{x}\right)^{2\alpha}\left(\frac{x^2-s^2}{2x}\right)^{2m-1}F\left(m+\frac{\gamma-1}{2},m;2m;1-\frac{s^2}{x^2}\right)f(s)\,ds.$$

Remark 2 Equation (31) can be interpreted even for non-integer m. Since the inverse to the differentiation operator is the integration operator, we can write

$$A^{-\beta}_{\gamma,\lambda}f(x)=\frac{1}{\Gamma(2\beta)}\int_0^x\left(\frac{s}{x}\right)^{\gamma}\left(\frac{x^2-s^2}{2x}\right)^{2\beta-1}\sum_{k=0}^{\infty}\frac{(-\sigma)^k}{(2m)_k k!}$$

$$\times \Xi_2 \left(\beta, \beta + \frac{\gamma - 1}{2} + k; 2\beta + k; \omega, \sigma \right) f(s)\, ds$$

where β is a fractional number.

The last equality can be regarded as a fractional power of operator (2), or a left-sided fractional Bessel integral generalizing the Bessel fractional integral.

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Declarations

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