DOI 10.1007/s10958-025-07881-3
Journal of Mathematical Sciences, Vol. 292, No. 1, July, 2025

UNIMODALITY OF THE PROBABILITY DISTRIBUTION OF THE EXTENSIVE FUNCTIONAL OF SAMPLES OF A RANDOM SEQUENCE

Yu. P. Virchenko and A. M. Tevolde UDC 519.213

Abstract. We establish a criterion for the unimodality of the probability distribution of a functional that is represented by the sum of a set of independent identically distributed random nonnegative variables $\tilde{x}_k$ with a random number of terms distributed according to Poisson. The general distribution of terms $\tilde{x}_k$ is concentrated on the interval $[0, 1]$ and is such that $\Pr\{\tilde{x}_k = 0\} \neq 0$. Its absolutely continuous part is asymptotically close to a uniform distribution. We introduce the concept of smoothing functions and establish an explicit form of the distribution of any fixed number of terms uniformly distributed on $[0, 1]$.

Keywords: sum of independent identically distributed random variables, unimodality of probability distribution, smoothing function, single-peak function.

1. Introduction

The study of unimodality of probability distributions determined by some natural conditions, from the point of view of setting the problems of probability theory, seems to be very important, since its presence, from the point of view of mathematical modeling of natural processes, reflects the fact of the absence of any causes that regularly influence their course. Among the first publications devoted to this direction, we name the work [6]. Without giving any significant historical overview of this direction in the present article, we will point out the works [4, 7, 8, 16], which were devoted to establishing the unimodality of distributions associated with sums of independent identically distributed random variables — a traditional object of research in probability theory. Another direction of research on the unimodality of distributions began to take shape in connection with the study of the qualitative properties of distributions for functionals of samples of random processes, in particular, for extensive functionals [1, 9–11], in the terminology of the present work, and for sample maxima [12, 13].

The problem of this work is connected with the study of the mathematical model of electrical breakdown of heterogeneously disordered condensed matter as an object of statistical mathematical physics [2] (see also [14, 15]). From the mathematical point of view, it consists in finding the conditions under which the absolutely continuous part of the probability distribution of the sum of independent, identically distributed, nonnegative random variables is unimodal.

Let us describe the general statement of the problem. Consider a random sequence $\langle \tilde{x}_n; n \in \mathbb{N} \rangle$, whose components are elements of the space Ω. The probability space of the sequence is constructed as $\langle \Omega^{\mathbb{N}}, \mathfrak{B}^{\mathbb{N}}, \mathsf{P}^{\mathbb{N}} \rangle$ where $\mathfrak{B}$ is the σ-algebra on Ω, P is the probability measure defined on $\mathfrak{B}$, $\mathsf{P}(A) = \Pr\{\tilde{x} \in A\}$, $\tilde{x} \in \Omega$, $A \in \mathfrak{B}$.

Let a measurable functional with values in $\mathbb{R}$, $\mathsf{V} : \Omega \mapsto \mathbb{R}$ be defined on the set Ω. Each such functional generates functionals $\mathsf{S}^{(N)}[\cdot; \mathsf{H}]$, $N \in \mathbb{N}$ on the spaces Ω^N of finite subsequences of length N. These functionals, which we call *extensive*, are defined by the formula

$$\mathsf{S}^{(N)}[\langle \tilde{x}_n; n \in I_N \rangle; \mathsf{H}] = \sum_{n=1}^{N} \mathsf{H}(\tilde{x}_n), \quad I_n = \{1 \div N\}.$$

Translated from Sovremennaya Matematika. Fundamental'nye Napravleniya (Contemporary Mathematics. Fundamental Directions), Vol. 70, No. 4, 2024.

They are obviously measurable. Since the probability distribution on Ω^N is an N-fold power of the measure P, the values of each functional $\mathsf{S}^{(N)}[\cdot]$ are represented by sums of independent identically distributed quantities. The probability distribution function $G^{(N)}(x)$ of each of them, determined by the measure P, is defined by the formula (see, for example, [3])

$$G^{(N)}(x) = \int\limits_{\mathbb{R}^N} \theta\Big(x - \sum_{n=1}^{N} x_n\Big) \prod_{n=1}^{N} d\mathrm{Pr}\{\mathsf{H}(\tilde{x}_n) \le x_n\}, \tag{1.1}$$

where $\theta(x) = \{1, x \ge 0; 0, x < 0\}$ is the Heaviside function. We restrict ourselves to the case where the general distribution function $V(x) = \mathrm{Pr}\{\tilde{x} \le x\}$, $x \in \mathbb{R}$ of random variables $\tilde{x}_k$, $k \in \mathbb{N}$ does not contain a singular component. In this case, the distribution function $G^{(N)}(x)$ does not contain a singular component either, i.e., it has a distribution density $g^{(N)}(x)$ in terms of generalized functions over the underlying space of locally continuous functions on $\mathbb{R}$.

There is a rich literature devoted to the study of probability distributions of sums of independent random variables, since this mathematical object is related to the foundations of probability theory. It should be noted, however, that these studies mainly solve problems related to the limiting at $N \to \infty$ probability distributions for centered sums of random independent variables with their appropriate normalization. These results are very important for processing statistical data under conditions of little information about the distribution of P, but at the same time there is the possibility of freely operating with experimental data regarding each individual term of the sum. In the opposite case, where such a possibility is absent, namely, neither the number of terms of the sum nor each of the terms is determined experimentally, and only the result — the random value of the sum is registered, then statistical estimates of the parameters of the limit distributions become impossible. The consequence of this is that it is necessary to study the probability distributions $G^{(N)}$ with a specific value N, which can be either a small value or a very large one. In such a situation, due to the significant dependence of the analytical form of the probability distributions $G^{(N)}$ on the probability distribution P, it is natural to be interested, first of all, not in this form, but in its qualitative properties and to classify the distributions according to these properties.

Following the described ideological setup, in the present work we investigate the possibility of the occurrence of unimodality of distributions $G^{(N)}$ of probabilities for sums of independent nonnegative random variables in the case where the distribution of each of the terms is concentrated on a finite segment. The object of our study is to establish the signs of unimodality of distributions $G^{(N)}$ in the case of the Poisson limit as $N \to \infty$. In this case, the probability distribution of each of the terms is assumed to be asymptotically close to a uniform one. Such statement of the problem arises naturally in the analysis of the electrical strength of a thin polymer film with respect to electrical breakdown under conditions of randomly distributed heterogeneous low-density foreign inclusions with random geometric characteristics [2]. The above-described formulation of the problem arises due to the limited possibility of experimentally studying the probability distribution of the geometric characteristics of individual inclusions, as well as due to the randomness of their number in the material.

In the next section, we give a concrete particular formulation of the problem to which the result obtained in the paper pertains. In Sec. 3, we introduce the concept of a class of nonnegative functions with a smoothing property. In Sec. 4, we calculate an explicit analytical form of the distribution densities of sums of independent uniformly distributed random variables. Finally, in Sec. 5, we establish necessary and sufficient conditions for the unimodality of the distribution of $G^{(N)}$ in the Poisson limit $N \to \infty$ in the case where $\tilde{x}_k$ are uniformly distributed on $[0, 1]$.

2. Statement of the Problem

Let us assume that the components of the random sequence $\langle \tilde{x}_n; n \in \mathbb{N}\rangle$ take values in $\Omega = \mathbb{R}_+ = [0, \infty)$ and $\mathfrak{B}$ is a σ-algebra of Borel subsets in $\mathbb{R}_+$. The probability measure P on Ω is represented by the distribution function V on $\mathbb{R}_+$. We assume that it consists of an absolutely continuous component with density $pv(x) = dV(x)/dx > 0$ for $x > 0$, where $1 > p > 0$, $\int_0^\infty v(x)dx = 1$, and a discrete component represented by the distribution $(1-p)\delta(x)$ with the Dirac δ-function. We assume that piecewise continuous on $[0,\infty)$ nonnegative density v is right-continuous.

Due to the continuity from the right of the θ-function, according to its definition, the limit as $x \to +0$ of the derivative with respect to x of the integral $\int_0^\infty \theta(x-y)u(y)dy$ with any continuous in the neighborhood of $x = 0$ and piecewise continuous on $[0,\infty)$ function should be understood as $\int_{-0}^\infty \delta(x-y)u(y)dy = u(0)$.

Thus, a random sequence $\langle \tilde{x}_n; n \in \mathbb{N}\rangle$ can be viewed as a generalization of a homogeneous *sequence of independent trials* with the probability of occurrence of "success" p, $0 < p < 1$, and with the space $\Omega = [0,\infty)$ of possible states for each of the trials.

The operation of convolution of two piecewise continuous distribution densities $v_1(x)$ and $v_2(x)$ concentrated on $[0,\infty)$, is defined by the formula

$$(v_1 * v_2)(x) = \int_{-0}^{\infty} v_1(y)v_2(x-y)dy. \tag{2.1}$$

This binary operation can be viewed as a commutative multiplication on the set of all piecewise continuous functions on $[0,\infty)$. The set of such functions, equipped with the convolution operation, is a commutative semigroup. With this definition of multiplication, the corresponding mth power $(v_*^m)(x)$, $m \in \mathbb{N}$ of any piecewise continuous function $v(x)$ is defined by the recurrence relation

$$(v_*^m)(x) = \int_0^x v(x-y)(v_*^{m-1})(y)dy, \quad N \geq 2.$$

Since the convolution of a pair of piecewise continuous functions on the half-axis is a piecewise continuous function, then, representing the differential of the distribution function V of the type described above in the form $dV(x) = [(1-p)\delta(x) + pv(x)]dx$, we find that the distribution density $g^{(N)}(x) = dG^{(N)}(x)/dx$ of the probabilities of the values of the functional $\mathsf{S}^{(N)}[\cdot]$ takes the form

$$g^{(N)}(x) = (1-p)^N\delta(x) + \sum_{m=1}^{N} \mathrm{C}_N^m p^m (1-p)^{N-m} v_*^N(x).$$

Thus, the probability distribution of the values of the sum $\mathsf{S}^{(N)}[\cdot]$ with density $[(1-p)\delta + pv]_*^N$, obviously consists of a discrete component with one growth point $x = 0$ and an absolutely continuous component with density

$$g_N(x) = \sum_{m=1}^{N} \mathrm{C}_N^m p^m (1-p)^{N-m} v_*^N(x). \tag{2.2}$$

In accordance with what was said in Sec. 1, we are interested in the unimodality conditions of the absolutely continuous part of g of the distribution density obtained from g_N in the so-called *Poisson limit* $p = \lambda/N \to 0$ for $N \to \infty$, $\lambda > 0$

$$g(x) = e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{m!} v_*^m(x). \tag{2.3}$$

The result we obtained concerns the case where the density v depends on some parameter $\eta > 0$ such that $v(x) \equiv w(x;\eta) = w(x) + o(1)$ by $\eta \to +0$ is uniformly in $x \in [0,1]$ where $w(x) = \theta(x)\theta(1-x)$.

3. Smoothing of Nonnegative Piecewise Continuous Functions

Let us clarify the concepts in terms of which the study will be conducted further. In doing so, we do not strive for formulations in the most general form, but limit ourselves to their reasonable sufficiency for solving the problem.

We consider piecewise continuous functions u on $[0,\infty)$. This means that at each point $x \in [0,\infty)$ the function u has a one-sided limit on the right and its domain $[0,\infty)$ can be represented as an at most countable disjunctive union $[0,\infty) = \bigcup_j [a_j, a_{j+1})$, where $a_1 = 0$, $a_j < a_{j+1}$, so that in all intervals (a_j, a_{j+1}), $j \in \mathbb{N}$ the function u is continuous, and at the points a_j, $j = 2, 3, 4, \ldots$ it can have discontinuities of the first kind.

For any point x of the maximum of a piecewise continuous function u on $[0,\infty)$ there exists a segment $[a,b]$ such that $x \in [a,b]$ and $u(y) = \text{const}$ for $y \in (a,b)$, where $\lim\limits_{y\to a+0} u(y) = \lim\limits_{y\to b-0} u(y)$, for which, for a sufficiently small $\varepsilon > 0$, we have $u(x) < u(a)$, if $x \in (a-\varepsilon, a)$, and $u(x) < u(b)$, if $x \in (b, b+\varepsilon)$, $x \in (a,b)$. Moreover, if $x = a$ or $x = b$, then it is possible that $u(x) \neq \text{const}$. The segment $[a,b]$ will be called the *segment of maximum* of the function u.

We call a maximum point a *local maximum* of a piecewise continuous function u, if a sufficiently small neighborhood of it does not contain maximal points different from it and $a = b$. A local maximum can either be achieved by the function u, if the point a is a point of its continuity from the left/right, or, otherwise, not be achieved.

For any point x of minimum of a piecewise continuous function u on $[0,\infty)$ there exists a segment $[a,b]$, such that $x \in [a,b]$ and $u(y) = \text{const}$ for $y \in (a,b)$, where $\lim\limits_{y\to a+0} u(y) = \lim\limits_{y\to b-0} u(y)$ for which, for sufficiently small $\varepsilon > 0$, we have $u(y) > u(a)$, if $y \in (a-\varepsilon, a)$, and $u(y) > u(b)$, if $y \in (b, b+\varepsilon)$. If $x = a$ or $x = b$, then it is possible that $u(x) \neq \text{const}$. The segment $[a,b]$ will be called the *minimality segment* of the function u.

We call a minimum point a *local minimum* of a piecewise continuous function u, if it is isolated, $a = b$, when a sufficiently small neighborhood of it does not contain minimum points distinct from it. Similarly to the concept of a local maximum, a local minimum can be achieved by u, if the point a is a point of its left/right continuity, or not achieved otherwise.

Remark 3.1. From these definitions it follows that the point $x = 0$, by definition, is always either a maximum point or a minimum point.

Definition 3.1. A nonnegative piecewise continuous function u on $[0,\infty)$ that has a unique maximality segment is called *single-peaked* (*unimodal*).

The introduction of this term is related to the concept of a single-peaked (unimodal) probability distribution on $\mathbb{R}$, when the function u represents its density. The following rather obvious statement is true.

Theorem 3.1. *For any piecewise continuous function u on $[0,\infty)$ there exists at most countable disjunctive partition of the domain $[0,\infty) = \bigcup_j [b_j, b_{j+1})$, $b_1 = 0$, $b_j < b_{j+1}$, where each of the segments $[b_j, b_{j+1}]$ consists only of the maximal points of the function u or only of its minimal points, or in the segment included in this partition the function u is monotonically nonincreasing/nondecreasing.*

Proof. It suffices to consider the intervals of variation of the function u that do not intersect any interval consisting of maximum/minimum points. Let $[a,b]$ be such a segment. Without loss of generality, such a segment can be considered as having no intersection with any interval in which there are maximum/minimum points. For it, there exist points $a' < a$ and $b' > b$, such that u is constant on the intervals (a', a), (b, b'). Otherwise, we can extend the segment $[a,b]$ by setting the point a equal to the largest of all points for which the interval (a', a) has the specified property, or $a' = -\infty$, and the point b' can be chosen to be the smallest of all possible points, or $b' = \infty$.

Let x and y be a pair of arbitrary points from the interval (a, b), $x < y$, such that $u(x) \neq u(y)$. Assume that u is not monotone on (x, y). Let, for definiteness, $u(x) \leq u(y)$. Then there exists a point z from (x, y) such that $u(z) < u(x)$, $u(z) < u(y)$. Consequently, inside the interval $(x, y) \subset (a, b)$ the piecewise continuous function u has a minimum point. If on the interval (x, y) there exists a point z at which $u(z) > u(x)$, $u(z) > u(y)$, then the function u on the interval $(x, y) \subset (a, b)$ has a maximum point. □

Corollary 3.1. *On each segment* $[0, L]$, $L > 0$, *a piecewise continuous function* u *can have at most a finite set of maximal and minimal segments.*

Corollary 3.2. *The maximal and minimal segments of a piecewise continuous function* u *alternate on the semiaxis* $[0, \infty)$.

A piecewise continuous function u is called *piecewise smooth* if its domain $[0, \infty)$ can be represented as an at most countable disjunctive union $[0, \infty) = \bigcup_j [a_j, a_{j+1})$, where $a_1 = 0$, $a_j < a_{j+1}$, and in all intervals (a_j, a_{j+1}), $j \in \mathbb{N}$ the function u is continuously differentiable. At each of the points a_j, $j = 2, 3, 4, \ldots$ its derivative can have only discontinuities of the first kind.

Definition 3.2. A nonnegative piecewise smooth function v on $[0, \infty)$ is called *smoothing* if for any nonnegative piecewise continuous function u with a finite set of maximal and minimal segments, its image under the action of the convolution operation $v * u$ is a function for which each of the numbers of maximal and minimal segments does not exceed, respectively, the numbers of maximal and minimal segments of the function u.

Remark 3.2. The smoothing property of a function is a broader concept than the concept of a strictly unimodal distribution [4, 5]. If a function v is smoothing and is the density of an absolutely continuous distribution, then this distribution is necessarily strictly unimodal.

Let v on $[0, \infty)$ have the support $[a, b]$, $a \geq 0$. We define the function $v^{(a)}$ by the formula $v(x + a) = v^{(a)}(x)$, $x \in [0, \infty)$. This function has the support $[0, b - a]$.

Theorem 3.2. *In order for the function* v *to be smoothing, it is necessary and sufficient that the function* $v^{(a)}$ *is smoothing.*

Proof. Let $\operatorname{supp} v = [a, b]$. Consider its convolution with an arbitrary piecewise continuous function u:

$$(v * u)(x) = \int_0^x v(y)u(x - y)dy = \int_a^x v(y)u(x - y)dy = \int_0^{x-a} v(y + a)u(x - a - y)dy$$

$$= \int_0^{x-a} v^{(a)}(y)u(x - a - y)dy.$$

Since the function on the right-hand side of the equality is continuous, all its maximal/minimal points on the segment $[0, L]$ coincide with the maximal/minimal points of the function $v * u$ on the segment $[0, L + a]$. Due to the arbitrariness of the number $L > 0$, we are convinced of the validity of the statement of the theorem. □

Corollary 3.3. *The set of all maximal/minimal points of the function* $v * u$ *on the segment* $[0, L]$ *is obtained by transferring by* $-a$ *the set of maximal/minimal points of the function* $v^{(a)} * u$ *on* $[a, L]$.

The following fairly obvious statements are true.

Theorem 3.3. *Let* $C > 0$ *be an arbitrary constant. In order for the function* v *to be smoothing, it is necessary and sufficient that the function* Cv *is smoothing.*

Proof. For any function u its maximum/minimum points coincide with the maximum/minimum points of the function Cu with an arbitrarily chosen constant $C > 0$. □

Theorem 3.4. *Let $\mu > 0$ be an arbitrary constant. In order for the function $v(\cdot)$ to be smoothing, it is necessary and sufficient that the function $v(\mu(\cdot))$ is smoothing.*

Proof. Let Σ be the set of maxima/minima of the function $u(\cdot)$. Then for any $\mu > 0$ the set $\mu\Sigma$ represents the maxima/minima of the function $u(\mu^{-1}(\cdot))$. The validity of the statement follows

$$\int_0^x v(\mu(x-y))u(y)dy = \mu^{-1}\int_0^{\mu x} v(\mu(\mu x - y)u(\mu^{-1}y)dy. \tag{3.1}$$

□

Let us define the function $w(x; a, b, C)$, $a < b$ on $[0, \infty)$, $C > 0$ by the formula

$$w(x; a, b, C) = \begin{cases} C, & x \in [a, b]; \\ 0, & x \notin [a, b]. \end{cases}$$

We calculate the convolution of $(w(\cdot; a, b, C) * u)(x)$ with an arbitrary piecewise continuous function u whose support contains $[a, b]$:

$$\begin{aligned}(w(\cdot; a, b, C) * u)(x) &= \int_0^x w(y; a, b, C)u(x-y)dy \\ &= C\theta(x-a)\Big[\theta(b-x)\int_a^x u(x-y)dy + \theta(x-b)\int_a^b u(x-y)dy\Big] \\ &= C\theta(x-a)\Big[\theta(b-x)\int_0^{x-a} u(y)dy + \theta(x-b)\int_{x-b}^{x-a} u(y)dy\Big].\end{aligned}$$

The values of its derivatives for $x > a$ are represented by the formula

$$\frac{d}{dx}(w(x; a, b, C) * u)(x) = C\begin{cases} \theta(x-a)u(x-0), & x < b; \\ \theta(x-a)\big[u(x-a+0) - u(x-b-0)\big], & x > b. \end{cases} \tag{3.2}$$

Let $w(x) = w(x; 0, 1, 1)$ denote the density of the uniform distribution on $[0, 1]$.

Theorem 3.5. *If the function u has a unique interval of maximum or minimum on $[0, \infty)$, then the function $w * u$ also has a unique interval of maximum, respectively, minimum. If the maximum/minimum point of the nonnegative function u on $[0, \infty)$ is unique and the function u has no intervals of constancy, then the maximum/minimum point of the function $w * u$ is also unique and this function has no intervals of constancy.*

Proof. From formula (3.2) for $a = 0$, $b = C = 1$ it follows that $(w(\cdot) * u)(x)$ increases for $x < 1$, and therefore its stationarity points are possible only for $x \geq 1$. Let us first consider the case where the function u is continuous. In this case, the function

$$(w * u)(x) = \int_{\max\{0, x-1\}}^x u(y)dy$$

is continuously differentiable, and therefore each of its extreme points x_*, determined by the vanishing of the derivative, can only be found at $x > 1$, and it must satisfy the equation

$$\frac{d}{dx}(w * u)(x) = u(x) - u(x-1) = 0. \tag{3.3}$$

Let us consider the case where u has a unique maximum segment on $[0, \infty)$. The case where u has a unique minimum segment is considered similarly.

First, let us assume that u has a unique maximum point, and suppose the contrary, that the function $w * u$ has a pair of points x_j, $j \in \{1, 2\}$ of maximum on $(0, \infty)$ where $x_1 < x_2$. They are solutions of Eq. (3.3). Since the maximum point x_* of the function is unique, then $x_j \geq x_*$, $j \in \{1, 2\}$. Equality is impossible here, since otherwise the point x_* is not unique. It also turns out that $x_j - 1 < x_*$, $j \in \{1, 2\}$. If in this case $u(x_2) = u(x_1)$, then the function u has an interval of constancy (x_1, x_2), which is excluded by the condition of the theorem. Thus, there is only one possibility $u(x_2) < u(x_1)$. Reasoning in the same way, we find that $u(x_2 - 1) > u(x_1 - 1)$. Then, subtracting the equalities $u(x_j) = u(x_j - 1)$, from each other, we obtain a contradiction, since $u(x_2) - u(x_1) < 0$ and simultaneously $u(x_2) - u(x_1) = u(x_2 - 1) - u(x_1 - 1) > 0$.

Now let u be piecewise continuous. There is a sequence $\langle u_\varepsilon; \varepsilon > 0\rangle$ of single-peaked functions u_ε that converges to the function u as $\varepsilon \to +0$ at its points of continuity. This sequence is constructed as follows. Let u have a discontinuity of the first kind at some point x_*. For u_ε, this discontinuity point is eliminated by connecting the points $\langle x_* - \varepsilon, u(x_* - \varepsilon)\rangle$ and $\langle x_* + \varepsilon, u(x_* + \varepsilon)\rangle$ on the graph of u by a straight line segment, where ε is chosen small enough that on the interval $(x_* - \varepsilon, x_* + \varepsilon)$ there are no discontinuity points of u different from x_*. This can be done since u has only a finite set of discontinuity points. Since the point x_* is chosen arbitrarily, then, carrying out the described procedure for eliminating discontinuity points at each of these points of the function u, we construct the function u_ε with a fixed value ε. Since the value $\varepsilon > 0$ is chosen arbitrarily under the condition that it is sufficiently small, then, having chosen a sequence $\langle \varepsilon_n > 0; n \in \mathbb{N}\rangle$ monotonically tending to zero and having carried out for each of the values of ε_n the described procedure for eliminating discontinuity points, we construct the above sequence of functions $\langle u_\varepsilon; \varepsilon > 0\rangle$. Obviously, all functions u_ε in this sequence are single-peaked and the sequence as $\varepsilon \to +0$ converges to the function u pointwise at each of the continuity points of the function u.

All functions u_ε do not have intervals of constancy if the function u does not have such intervals. Then for each of the functions $w * u_\varepsilon$ due to the continuity of all functions u_ε, the statement of the theorem is true, as was established above. Passing to the limit in the sequence of single-peaked functions $w * u_\varepsilon$, we obtain that the limit function $w * u$ is also single-peaked. It does not have an interval of constancy in the case where the function u does not have intervals of constancy.

Let us now consider, without loss of generality, the case where the function u has a maximal segment $[a, b]$ of nonzero length. Assume that the function $w*u$ has a maximal segment $[a', b']$. Moreover, $a' \geq b$. Assume that there is at least one more segment of extremality of the function $w * u$. This segment is necessarily a minimal segment, and it is located to the right of $[a', b']$ without intersecting it. Then there is a point $y \in (a', b')$ of continuity of the function u at which $u(y) = u(y-1)$, and there is a point $z > y$ at which the function u is continuous. Therefore, the function $w*u$ is continuously differentiable and its derivative is positive, i.e., $u(z) > u(z - 1)$. In this case, $z - 1 < b$, $u(z - 1) \leq u(b)$. Then $y - 1 = z - 1$ and therefore $u(y - 1) \leq u(z - 1)$. From these inequalities it follows that $u(y) < u(z)$. But this inequality contradicts the fact that the points y and z are on the interval of nonincreasing function u, which indicates the absence of any minimal point. □

Corollary 3.4. *Let, if the statement of the theorem is satisfied regarding the function u, its only maximum/minimum is realized at the point of discontinuity of the first kind. In the case of realization of the maximum, the points x_ε of maxima of functions $w * u_\varepsilon$, each of which is a solution of the equation $u_\varepsilon(x_\varepsilon) = u_\varepsilon(x_\varepsilon - 1)$, can lead at $\varepsilon \to 0$ to the maximum point x_* of function u — the solution of equation $u(x_* + 0) = u(x_* - 1)$, if function u has a discontinuity at point x_* with a negative* *jump. Conversely, they can tend to the solution of equation $u(x_*) = u(x_* - 1 - 0)$ if function u has a discontinuity at point $x_* - 1$ with a positive jump.*

Similarly, if the minimum of the function u is realized at a discontinuity point, then the points x_ε of the minima of the functions $w * u_\varepsilon$, each of which is a solution of the equation $u_\varepsilon(x_\varepsilon) = u_\varepsilon(x_\varepsilon - 1)$, can tend as $\varepsilon \to 0$ to the point x_* of the minimum of the function $w * u$ — the solution of the equation $u(x_*) = u(x_* - 1 - 0)$, if the function u has a discontinuity at the point $x_* - 1$ with a negative jump. Conversely, they can tend to the solution of the equation $u(x_* + 0) = u(x_* - 1)$ if the function u has a discontinuity at the point x_* with a positive jump.

Remark 3.3. In connection with the proved statement, the function w is the density of the so-called *strictly unimodal* distribution. The convolution of such a distribution with any unimodal distribution is the density of the unimodal distribution. However, in the formulation of the theorem not only the unimodal nature of the convolution is asserted, but also the type of the vertex is indicated.

Theorem 3.6. *The function $w(x) \equiv w(x; 0, 1; 1)$ s smoothing on $[0, \infty)$.*

Proof. The function $(w(\cdot) * u)(x)$ increases for $x < 1$, and therefore its stationarity/constancy points are possible only for $x \geq 1$. As in the proof of Theorem 3.5, we first consider the case where the function u is continuous. If the function u has a unique maximum/minimum, then the statement of the theorem is true by virtue of the indicated Theorem 3.5.

Without loss of generality, we assume that all extremal points of u are isolated. In this case, the concept of smoothness is defined for functions u that have a finite set of maximum and minimum points. Then we can assume that all points $x_1, x_2, \ldots, x_N$ of extremality of u are on the interval $(0, 1)$, i.e., the function u either does not monotonically decrease or does not monotonically increase for $x > 1$. Such a situation can always be achieved by choosing, according to Theorem 3.4, a suitable scaling factor $\mu > 0$ and by passing to a function equivalent in the sense of the smoothing property.

Thus, the interval $(0, 1)$ consists of $N + 1$ intervals (x_j, x_{j+1}) of monotone variation of the function u. They are separated by the boundary points x_j, $j = 0 \div N$, $x_0 = 0$, $x_{N+1} = 1$. Let us assume for definiteness that the last extreme point x_N of the function u is a maximum. Then the function u does not increase for $x > 1$.

The points x of extremality of the function $w*u$ necessarily satisfy the equation $u(x) = u(x-1)$ and, in the case under consideration, are on the interval $(1, 2)$ due to the monotonic change of the function u for $x > 1$. They are the points at which the graphs of the monotonic function u and the function $u_- \equiv u(x - 1)$, which has intervals $(x_j - 1, x_{j+1} - 1)$ of monotonic change in $(1, 2)$ separated by the boundary points $x_j - 1$, $j = 0 \div N$, coincide. Due to the monotonic nonincreasing of the function u on $(1, 2)$ its graph can have no more than one intersection point with the graph of the function u_- on each interval from those indicated where it does not decrease. Each such intersection point is a maximum point of the function $w * u$ since $(w * u)(x - \delta) = u(x - \delta) - u(x - 1 - \delta) > 0$ and $(w * u)(x + \delta) = u(x + \delta) - u(x - 1 + \delta) < 0$ for sufficiently small $\delta > 0$. On the same intervals $(x_j - 1, x_{j+1} - 1)$, where the function u_- is nonincreasing, its graph can have several intersection points with the graph of the function u. The intersection points in such intervals can only be minimum points of the function $w * u$ since for sufficiently small $\delta > 0$ the following must hold: $(w * u)(x - \delta) = u(x - \delta) - u(x - 1 - \delta) < 0$ and $(w * u)(x + \delta) = u(x + \delta) - u(x - 1 + \delta) > 0$. And since the points of maximum and minimum must alternate, only one of the intersection points on each of the intervals of nonincreasing function u_- can be a minimum point. Thus, the function $w * u$ can have no more than $N + 1$ points of extremality, since the interval $(1, 2)$ is divided by the points $x_j + 1$, $j = 0 \div N$ into $N + 1$ intervals.

Finally, we note that on the first interval $(1, x_1 + 1)$ there is no intersection point of the graphs of the functions $u_-(x)$ and $u(x)$, since such a point would correspond to the extreme point of the function $w * u$ on the interval $(0, x_1)$ which is absent due to the monotonic nondecreasing of this function on it. Thus, on the interval $(1, 2)$ there are no more than N intersection points of the graphs of the functions u_- and u. □

A generalization of the proven theorem is the following theorem.

Theorem 3.7. *For any ordered pair* $\{a,b\} \subset [0,\infty)$, $a < b$ *and any constant* $C > 0$ *the function* $w(x;a,b,C)$ *on* $[0,\infty)$ *is smoothing.*

Proof. By virtue of Statement 3.3, the set of all maximal/minimal points of the function $w(\cdot;a,b,C)*u$ on the segment $[0,L]$ is obtained by transferring by $(-a)$ the set of maximal/minimal points of the function $w(\cdot;0,b-a,C)*u$ on $[a,L+a]$. Therefore, the number of maximal/minimal segments of the function $w(\cdot;0,b-a,C)*u$ does not exceed the number of such segments of the function $w(\cdot;a,b,C)*u$. Thus, it suffices to prove that the function $w(\cdot;0,b-a,C)$ is smoothing. By virtue of Theorem 3.3, the smoothing property does not depend on the value of the constant C, and by virtue of Theorem 3.4, the smoothing property of the function $w(\cdot;0,b-a,C)$ does not depend on the size $(b-a)$ of its support.

Since the function $w(x)$ is smoothing on $[0,\infty)$ by virtue of Theorem 3.5, the function $w(x;a,b,C)$ is also smoothing. □

Remark 3.4. The smoothing function $w(x;a,b,(b-a)^{-1})$ is the density of the strictly unimodal distribution according to Ibragimov. It is obvious that Ibragimov's theorem [4, 5] is valid for it, since, setting

$$u(x;\eta) = \begin{cases} w(x;a,b,(b-a)^{-1}), & x \in [a,b]; \\ (b-a)^{-1}\exp(-\eta(x-b)^2), & x \geq b; \\ (b-a)^{-1}\exp(-\eta(x-a)^2), & x \leq a, \end{cases}$$

we obtain a logarithmically concave density on $\mathbb{R}$. Passing to the limit $\eta \to \infty$, we obtain that the probability distribution with density $w(x;a,b,(b-a)^{-1})$ is the limit of distributions, each of which has a logarithmically concave density.

Theorem 3.8. *The convolution of any two smoothing functions* v_1 *and* v_2 *on* $[0,\infty)$ *is a smoothing function on* $[0,\infty)$.

Proof. Let u be an arbitrary piecewise continuous function on $[0,\infty)$ with N extremal points. Then, according to the definition of the smoothing function, $v_1 * u$ has at most N extremal points. Applying the contraction operation to this function with the smoothing function v_2, we find that the function $v_2 * (v_1 * u) = (v_2 * v_1) * u$ also has at most N extremal points. □

Theorem 3.9. *The limit of a sequence* $\langle v_n;\ n \in \mathbb{N}\rangle$ *of smoothing functions is a smoothing function on* $[0,\infty)$.

Proof. For an arbitrary piecewise continuous function u with N extreme points, consider the sequence $\langle v_n * u; n \in \mathbb{N}\rangle$. Due to the smoothing property of functions v_n, each function in this sequence has at most N extreme points. Since this sequence, by assumption, has a limit as $N \to \infty$, then the sequence $\langle M_n; n \in \mathbb{N}\rangle$ has some limit M where M_n is the number of extreme points of the function $v_n * u$, $M_n \leq N$. Passing to the limit in the inequality $M_n \leq N$ we obtain that $M \leq N$. □

A consequence of Theorems 3.7–3.9 is the following theorem.

Theorem 3.10. *For any at most countable set* $A \subset \mathbb{N}$, *the convolution of functions* $w(\cdot;a_j,b_j,C_j)$, *defined by the set of triples of numbers* $\{\langle a_j,b_j,C_j\rangle; j \in A\}$, *is a smoothing function*

$$v(x) = \Big(\bigotimes_{j\in A} w(\cdot;a_j,b_j,C_j)\Big)(x), \quad x \in [0,\infty).$$

Proof. The proof is obtained by applying Statement 3.9 to a sequence of smoothing functions, by virtue of Theorem 3.8

$$v_n(x) = \Big(\bigotimes_{j\in A_N} w(\cdot;a_j,b_j,C_j)\Big)(x), \quad x \in [0,\infty),$$

where $A_N = A \cap I_N$. □

4. Properties of Functions w_*^N

Nonnegative piecewise continuous functions u on $[0,\infty)$ form a semigroup with respect to the binary operation of convolution of a pair of such functions. Smoothing functions in this semigroup form a subsemigroup. Each of the functions $w(\cdot;a,b,C)$, $0<a<b$, $C>0$ is its generator. Let us study the qualitative behavior of the powers $w_*^m(x)$, $m\in\mathbb{N}$ of the distribution density $w(x)=\theta(x)\theta(1-x)$. First of all, let us prove the following formula.

Theorem 4.1. *For densities $w_*^m(x)$, $m\in\mathbb{N}$ the following formula is valid:*

$$w_*^m(x)=w_*^m(m-x). \tag{4.1}$$

Proof. Let us denote

$$\bar u_j(k)=\int_{-\infty}^{\infty}u_j(x)e^{ikx}dx,\quad j\in\{1,2\}.$$

For Fourier images of densities $u_1(x)$ and $u_2(x)$, according to (2.1), one has

$$(\overline{u_1*u_2})(k)=\int_{-\infty}^{\infty}e^{ikx}(u_1*u_2)(x)dx=\int_{-\infty}^{\infty}e^{ikx}\Big(\int_{-\infty}^{\infty}u_1(y)u_2(x-y)dy\Big)dx$$

$$=\int_{-\infty}^{\infty}e^{iky}u_1(y)\Big(\int_{-\infty}^{\infty}e^{ik(x-y)}u_2(x-y)dx\Big)dy=\bar u_1(k)\bar u_2(k).$$

Therefore, the Fourier transform of the mth power of the density $u(x)$

$$(\overline{u_*^m})(k)=\int_{-\infty}^{\infty}(u_*^m)(x)e^{ikx}dx,\quad m=1,2,3,\dots$$

satisfies the relation $(\overline{u_*^m})(k)=(\overline{u_*^{m-1}})(k)\cdot\bar u(k)$, and therefore

$$(\overline{u_*^m})(k)=\bar u^m(k).$$

The density $w(x)$ has the property $w(x)=w(1-x)$. Therefore, for its Fourier transform the following equalities hold:

$$\bar w(k)=\int_{-\infty}^{\infty}e^{ikx}w(x)dx=\int_{-\infty}^{\infty}e^{ikx}w(1-x)dx=e^{ik}\int_{-\infty}^{\infty}e^{ik(x-1)}w(1-x)dx=e^{ik}\bar w(-k),$$

i.e.,

$$\bar w(k)=e^{ik}\bar w(-k).$$

It follows that $\bar w^m(k)=e^{ikm}\bar w^m(-k)$. Applying the inverse Fourier transform to both sides of this equality, we find

$$w_*^m(x)=\frac{1}{2\pi}\int e^{-ikx}\bar w^m(k)dk=\frac{1}{2\pi}\int e^{-ik(x-m)}\bar w^m(-k)dk=w_*^m(m-x).$$

□

Corollary 4.1. *The maximum of the function $w_*^m(x)$ is at the point $x=m/2$.*

Proof. The statement follows from (4.1). □

The following statement is a refinement of the well-known Ibragimov theorem (see [4]) on the so-called *strictly unimodal* distributions when applied to the density $w(x)$.

Corollary 4.2. *The functions $w_*^m(x)$ for $m\geq 2$ have a single maximum point.*

Proof. The function $w_*^2(x)$ has an explicitly unique maximum point at $x = 1$. The validity of the general statement for any number $m \geq 2$ is obtained by induction on $m > 2$ using the smoothing property of the function w. □

The successive calculation of the degrees of the density convolution operation $w(x)$ is carried out on the basis of the formula

$$w_*^{m+1}(x) = \int_0^x w(x-y)w_*^m(y)dy = \int_0^x \theta(x-y)\theta(1-x+y)w_*^m(y)dy. \tag{4.2}$$

Obviously, $w_*^m(x) = \theta(x)w_*^m(x)$. Moreover, based on formula (4.2), the following theorem is proved by induction on m.

Theorem 4.2. *Each density $w_*^m(x)$ is concentrated on $[0, m]$, $m = 1, 2, 3, \ldots$, i.e., the formula $w_*^m(x) = \theta(m-x)w_*^m(x)$ holds.*

Proof. We replace the density $w_*^m(y)$ in the integrand by $w_*^m(y)\theta(m-y)$ according to the induction hypothesis. Since for $x > m+1$ and $y < m$, $1+y > x$ we must have $1+m > 1+y > x > m+1$, which is impossible, we have $\theta(m-y)\theta(1-x+y)\theta(x-m-1) = 0$. Consequently, the integral in (4.2) is proportional to $\theta(m+1-x)$. □

According to (4.2), given that the function $w_*^{m+1}(x)$ is concentrated on $[0, m+1]$, we write its expression on this segment as

$$w_*^{m+1}(x) = \int_0^x \theta(x-y)\theta(1-x+y)w_*^m(y)dy = \int_0^x w_*^m(y)dy + \theta(x-1)\int_{x-1}^x w_*^m(y)dy. \tag{4.3}$$

By induction on m using (4.2) we prove that the functions $w_*^m(x)$ are continuous starting from $m = 2$, and for $m > 2$ they are s times continuously differentiable, where $s < m-2$.

Based on formula (4.3) we prove the following statement.

Theorem 4.3. *For functions $w_*^m(x)$ there is a representation*

$$w_*^m(x) = \sum_{k=0}^{m-1} \theta(x-k)\theta(k+1-x)P_{m,k}(x), \tag{4.4}$$

where for polynomials $P_{m,k}(x)$, $k = 0, 1, \ldots, m-1$, $m = 1, 2, 3, \ldots$ the following recurrence relations hold:

$$P_{m+1,0}(x) = \int_0^x P_{m,0}(y)dy, \quad x \in [0, 1); \tag{4.5}$$

$$P_{m+1,m}(x) = \int_{x-1}^m P_{m,m-1}(y)dy, \quad x \in [m, m+1); \tag{4.6}$$

$$P_{m+1,k}(x) = \int_{x-1}^k P_{m,k-1}(y)dy + \int_k^x P_{m,k}(y)dy, \quad x \in [k, k+1), k = 1 \div m-1. \tag{4.7}$$

Proof. The representation (4.4) holds for $m = 1$ with $P_{1,0}(x) = 1$. Let us construct an induction step from m to $m+1$. Substituting expansion (4.4) into the right-hand side of formula (4.7) for $x \in [0, m+1]$ leads to the equality

$$w_*^{m+1}(x) = \theta(1-x)\int_0^x P_{m,0}(y) + \theta(x-1)\sum_{k=0}^{m-1}\int_{x-1}^x \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy, \tag{4.8}$$

where we take into account that only the term with the polynomial $P_{m,0}(y)$ makes a nonzero contribution to the first integral. We write the last integral for $k = 0, 1, 2, \dots$ in the form

$$\int\limits_{x-1}^{x} \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy = \theta(x-k)\theta(k+2-x)\int\limits_{x-1}^{x} \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy,$$

since for $x - 1 > k + 1$ and for $x < k$ it is equal to zero.

For $k < m$, if $k + 1 < x < k + 2$, then $k < x - 1 < k + 1$, and in this case

$$\int\limits_{x-1}^{x} \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy = \int\limits_{x-1}^{k+1} P_{m,k}(y)dy;$$

if $k < x < k + 1$, then $x - 1 < k$, and in this case

$$\int\limits_{x-1}^{x} \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy = \int\limits_{k}^{x} P_{m,k}(y)dy.$$

Therefore,

$$\int\limits_{x-1}^{x} \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy = \theta(x-k-1)\theta(k+2-x)\int\limits_{x-1}^{k+1} P_{m,k}(y)dy$$

$$+ \theta_k\theta(x-k)\theta(k+1-x)\int\limits_{k}^{x} P_{m,k}(y)dy,$$

where $\theta_k = 1 - \delta_{k,0}$. Substituting the obtained representations for the integrals into (4.8), we find that

$$w_*^{m+1}(x) = \theta(x)\theta(1-x)\int\limits_{0}^{x} P_{m,0}(y)$$

$$+ \sum_{k=0}^{m-1}\Big[\theta_k\theta(x-k)\theta(k+1-x)\int\limits_{k}^{x} P_{m,k}(y)dy + \theta(x-k-1)\theta(k+2-x)\int\limits_{x-1}^{k+1} P_{m,k}(y)dy\Big]$$

$$= \sum_{k=1}^{m-1} \theta(x-k)\theta(k+1-x)\Big[\int\limits_{k}^{x} P_{m,k}(y)dy + \int\limits_{x-1}^{k} P_{m,k-1}(y)dy\Big]$$

$$+ \theta(x)\theta(1-x)\int\limits_{0}^{x} P_{m,0}(y)dy + \theta(x-m)\theta(m+1-x)\int\limits_{x-1}^{m} P_{m,m-1}(y)dy.$$

Defining the functions $P_{m+1,0}(x)$, $P_{m+1,m}(x)$, $P_{m+1,k}$, $k = 1 \div m-1$, according to (4.5)–(4.7), we obtain the desired representation for the density $w_*^{m+1}(x)$:

$$w_*^{m+1}(x) = \sum_{k=0}^{m} \theta(x-k)\theta(k+1-x)P_{m+1,k}(x).$$

□

Corollary 4.3. *Polynomials $P_{m+1,k}(x)$, $k = 0, 1, \dots, m-1$ satisfy the identities*

$$P_{m,k}(x) = P_{m,m-1-k}(m-x). \tag{4.9}$$

Proof. Substituting into identity (4.1) the expansions for the functions $w_*^m(x)$ and $w_*^m(m-x)$ according to formula (4.4), we find that the equality

$$\sum_{k=0}^{m-1} \theta(x-k)\theta(k+1-x)P_{m,k}(x) = \sum_{k=0}^{m-1} \theta(m-x-k)\theta(k+1-m+x)P_{m,k}(m-x).$$

Replacing the summation variable $m-1-k$ with k in the sum on the right-hand side of the equality leads to the identity

$$\sum_{k=0}^{m-1} \theta(x-k)\theta(k+1-x)P_{m,k}(x) = \sum_{k=0}^{m-1} \theta(x-k)\theta(k+1-x)P_{m,-1-k}(m-x),$$

from which it follows that (4.9) holds for $x \in [k, k+1]$. □

Remark 4.1. From the condition of continuity of densities w_*^m, $m \geq 2$ and continuity of their derivatives $(w_*^m)^{(s)}$ of order $s \leq m-2$, $m > 2$ at points $x = 1 \div m$, we conclude that the polynomials $P_{m,k}$ must have the property $P_{m,k}^{(s)}(k+1) = P_{m,k+1}^{(s)}(k+1)$ for $k = 0 \div m-2$ and the specified values of s, and $P_{m,m-1}(m) = 0$ must also hold.

The explicit form of the polynomials $P_{m,k}$ is found on the basis of formulas (4.3)–(4.5), $k = 1 \div m$, $m \in \mathbb{N}$. It is determined by them uniquely, if we take into account that $P_{1,0}(x) = 1$ due to the definition of the function $w(x) = \theta(x)\theta(1-x)$, $x \in \mathbb{R}$ and expansion (3.1). This fact is easily established by induction for $m \in \mathbb{N}$ for a given function $P_{1,0}(x)$ first for the polynomials $P_{m,0}$ and $P_{m,m-1}$, and then for each fixed value $k < m$. We precede the proof of the statement establishing the form of the polynomials $P_{m,k}$ with the following combinatorial lemmas.

Lemma 4.1. *For any $m \in \mathbb{N}$, $s \in \mathbb{N}_+$, $s < m$ the following identity holds:*

$$\sum_{l=s}^{m} \frac{(-1)^l}{(l-s)!(m-l)!} = 0.$$

Proof. The following identity transformations are valid:

$$\sum_{l=s}^{m} \frac{(-1)^l}{(l-s)!(m-l)!} = \sum_{l=s}^{m} (-1)^l \frac{l!}{((l-s)!} \mathrm{C}_m^l = \left[\frac{d^s}{d\xi^s} \sum_{l=0}^{m} (-\xi)^l \mathrm{C}_m^l\right]_{\xi=1} = \left[\frac{d^s}{d\xi^s}(1-\xi)^m\right]_{\xi=1} = 0$$

due to $x < m$. □

Corollary 4.4. *For any $m \in \mathbb{N}$ and $s \in \{0, 1, \ldots, m-1\}$ the following identity holds:*

$$\sum_{l=1}^{m} (-1)^l l^s \mathrm{C}_m^l = 0. \tag{4.10}$$

Proof. For $s = 0$ we have a well-known combinatorial identity. Suppose that (4.10) holds for all $s \in \{0, 1, \ldots, t\}$, $t < m-1$. Then for any polynomial Π of degree not higher than t we have a similar identity

$$\sum_{l=1}^{m-1} (-1)^l \Pi(l) \mathrm{C}_m^l = 0.$$

Since $l(l-1)\ldots(l-t+2) = l^{t+1} + \Pi(l)$, where $\deg \Pi \leq t$, then using the identity of Lemma 4.2 for $s = t+1$, we obtain equality (4.10) for $s = t+1$. □

Lemma 4.2. *For any $m \in \mathbb{N}$, $n \in \mathbb{N}_+$, $n < m$ at $z \in \mathbb{C}$ there holds the identity*

$$(-1)^{m-1}(z-m)^n = \sum_{k=0}^{m-1} (-1)^k (z-k)^n \mathrm{C}_m^k. \tag{4.11}$$

Proof. Setting $s = n - k$, $k \in \{0, 1, \ldots, n\}$, we write (4.11) in the form

$$(-1)^{m-1} m^{n-k} = \sum_{l=1}^{m-1} (-1)^l l^{n-k} \mathrm{C}_m^l.$$

Summing these identities over $k \in \{0, 1, \ldots, n\}$ using the Newton binomial formula, having first multiplied them by $(-1)^{n-k} z^k \mathrm{C}_n^k$, we obtain (4.11). □

Theorem 4.4. *For polynomials $P_{m,k}$ the following formula is valid:*

$$P_{m,k}(x) \equiv \frac{S_{m,k}(x)}{(m-1)!}, \quad S_{m,k}(x) = \sum_{l=0}^{k} (-1)^l (x-l)^{m-1} \mathrm{C}_m^l \tag{4.12}$$

for $x \in [k, k+1]$, $k = 0 \div m - 1 \in \mathbb{N}$, $m = 2, 3, \ldots$

Proof. By induction on $m \in \mathbb{N}$ using formula (4.5) and the condition $P_{1,0}(x) = 1$ for $m = 1$, we establish the formula $P_{m,0}(x) = [(m-1)!] x^{m-1}$, $x \in [0, 1]$. Likewise, by inductive reasoning for $m \in \mathbb{N}$, using formula (4.6) under the same condition, if $m = 1$, we find that $P_{m,m-1}(x) = (-1)^{m-1}[(m-1)!](x-m)^{m-1}$, $x \in [m-1, m]$.

For fixed functions $P_{m,0}$, $m \in \mathbb{N}$, formula (4.7) can be considered as a system of inhomogeneous integral equations with respect to a family of piecewise continuous functions $\{P_{m,k}(x);\, k = 1 \div m - 1, m \in \mathbb{N} \setminus \{1\}\}$, which are defined, respectively, on $[k, k+1]$. This system of equations defines the specified set of polynomials in a unique way. It is necessary to prove that all functions of this family have form (4.12) provided that $P_{m,0}(x) = [(m-1)!] x^{m-1}$, $x \in [0, 1]$ and $P_{m,m-1}(x) = (-1)^{m-1}[(m-1)!](x-m)^{m-1}$, $x \in [m-1, m]$.

From this system of equations, by differentiating with respect to x, we obtain a system of differential equations

$$\dot{P}_{m+1,k}(x) = P_{m,k}(x) - P_{m,k-1}(x-1), \quad k = 1 \div m - 1, m \in \mathbb{N}, \tag{4.13}$$

where each equation is satisfied on the interval $[k, k+1]$ for fixed values of k and m. This system of differential equations is uniquely solvable under additional conditions at the points $x = k$, $k \in \{1, 2, \ldots, m-1\}$. These are the continuity conditions $P_{m,k-1}(k) = P_{m,k}(k)$, where $P_{m,1}(1) = P_{m,0}(1) = [(m-1)!]^{-1}$ and $P_{m,m-2}(m-1) = P_{m,m-1}(m-1) = [(m-1)!]^{-1}$, according to the already calculated functions $P_{m,0}$, $P_{m,m-1}$.

The proof now consists of verifying that the set of polynomials (4.12) satisfies the system of Eqs. (4.13) and the continuity conditions. For the given values of k and m we have $\dot{S}_{m+1,k}(x) = m S_{m+1,k}(x)$. In addition, for the same values, we have the equality

$$S_{m,k}(x) - S_{m,k-1}(x-1) = \left[x^{m-1} + \sum_{l=1}^{k} (-1)^l (x-l)^{m-1} \left(\mathrm{C}_m^{l-1} + \mathrm{C}_m^l \right) \right]$$

$$= \sum_{l=0}^{k} (-1)^l (x-l)^{m-1} \mathrm{C}_{m+1}^l = S_{m+1,k}(x), \quad k = 1 \div m - 1.$$

where the binomial coefficients C_m^l are introduced and the identity $\mathrm{C}_m^{l-1} + \mathrm{C}_m^l = \mathrm{C}_{m+1}^l$ s used. Then the polynomials $S_{m,k}(x)/(m-1)!$ satisfy the system of differential Eqs. (4.13). The validity of the continuity conditions for these polynomials at the points $2, \ldots, m-2$ follows from the fact that the last term at $l = k+1$ in the sum $S_{m,k+1}(k+1)$ is equal to zero and therefore it coincides with the sum $S_{m,k}(k+1)$. At the point $x = 1$ we have $S_{m,1}(1) = S_{m,0}(1) = 1$. Finally, the validity of the continuity condition at the point $x = m - 1$ follows from formula (4.13) at $x = m$. □

5. Unimodality of the Function $g(x)$

In this section we will find conditions for the unimodality of the function g — the absolutely continuous part of the probability density $g^{(N)}$ of the extensive functional of the random sequence $\langle \tilde{x}_k, k = 1 \div N \rangle$ in the Poisson limit for $0 < p < 1$ in the case where the components of the sequence are distributed with a common density $w(x)$.

Then formula (2.2), taking into account expansion (4.4), takes the form of expansion over integer intervals

$$g_N(x) = \sum_{m=1}^{N} \mathrm{C}_N^m p^m (1-p)^{N-m} w_*^m(x) = \sum_{m=1}^{N} \mathrm{C}_N^m p^m (1-p)^{N-m} \sum_{k=0}^{m-1} \theta(x-k)\theta(k+1-x) P_{m,k}(x)$$

$$= \sum_{k=1}^{N} \theta(x-k+1)\theta(k-x) R_k^{(N)}(x,p)$$

with coefficients

$$R_k^{(N)}(x,p) = (1-p)^N \sum_{m=k}^{N} \mathrm{C}_N^m \frac{\nu^m}{(m-1)!} \sum_{l=0}^{k-1} (-1)^l (x-l)^{m-1} \mathrm{C}_m^l,$$

$\nu = p/(1-p)$, according to (4.12). In the Poisson limit $N \to \infty$, $p = \lambda/N$, function (2.3) $g(x) = \lim\limits_{N\to\infty} g^{(N)}(x)$ is represented by the expansion

$$g(x) = \sum_{k=0}^{\infty} \theta(x-k)\theta(k+1-x) R_{k+1}(x,\lambda) \tag{5.1}$$

with coefficients $R_k(x,\lambda) = \lim\limits_{N\to\infty} R_k^{(N)}(x,p)$,

$$R_k(x) = e^{-\lambda} \sum_{m=k}^{\infty} \frac{\lambda^m}{m!} P_{m,k-1}(x) = e^{-\lambda} \sum_{m=k}^{\infty} \frac{\lambda^m}{(m-1)!} \sum_{l=0}^{k-1} \frac{(-1)^l}{l!(m-l)!} (x-l)^{m-1}. \tag{5.2}$$

The following statement describes the differential properties of the function g.

Theorem 5.1. *For any $p \in (0,1)$, the function g is piecewise smooth on $\mathbb{R}_+$. It has a discontinuity of the 1st kind at $x_* = 1$, with $g(1-0) > g(1+0)$. The point $x_* = 1$ is its maximum point. At $x = 2$ the function g is continuous, but has a discontinuity of the 1st kind of the derivative at $x = 2$ and it is continuously differentiable on $\mathbb{R}_+ \setminus \{1,2\}$.*

Proof. The presence of discontinuities in g at $x = 1$ and in $\dot{g}$ at $x = 2$, as well as the continuous differentiability of g on $\mathbb{R}_+ \setminus \{1,2\}$ follow from representation (5.1), in which the functions R_k, $k \in \mathbb{N}$ according to (5.2) are continuously differentiable on $\mathbb{R}_+$, and it also follows from (5.2) that $R_1(1-0) > R_2(1+0)$, $R_k(k-0) = R_{k+1}(k+0)$, $k \geq 2$ and $\dot{R}_2(2-0) \neq R_3(2+0)$, $\dot{R}_k(k-0) = R_{k+1}(k+0)$, $k \geq 3$. Indeed,

$$g(1-0) = R_1(1-0) = e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{m!(m-1)!},$$

$$g(1+0) = R_2(1+0) = e^{-\lambda} = R_1(1-0) - \lambda.$$

In the same way, based on (5.2), we find

$$\dot{g}(2-0) = \dot{R}_2(2-0) = e^{-\lambda} \sum_{m=2}^{\infty} \frac{\lambda^m}{m!(m-2)!} [2^{m-2} - m], \tag{5.3}$$

$$\dot{g}(2+0) = \dot{R}_3(2+0) = R_2(2-0) - \frac{\lambda^2}{2}. \tag{5.4}$$

Taking into account (5.3), (5.4) and the increase of the function R_1 on the segment $[0,1]$, we see that at the point x_* the vertex of the function g with the value $R_1(1)$ is realized. $\square$

Finally, we prove the statement representing the conditions for the unimodality of the function g.

Theorem 5.2. *If the parameter $\lambda > 0$ satisfies the condition*

$$1 > \sum_{m=2}^{\infty} \frac{\lambda^{m-1}}{m!(m-1)!}, \tag{5.5}$$

then the function g has a unique maximum at the point $x_* = 1$. If, on the contrary, g has a unique maximum at x_*, then the parameter λ necessarily satisfies the condition $\sum_{m=3}^{\infty} \frac{\lambda^{m-2}}{m!(m-2)!} < 1/2$.

Proof. Let us introduce the functions

$$w_k^{(N)}(x) = e^{-\lambda} \sum_{l=1}^{N} \frac{\lambda^m}{(m+k)!} w_*^m, \qquad k \in \mathbb{N}_+,\ N \geq 2. \tag{5.6}$$

Sufficiency. We prove it by induction on N. For $N = 2$ all functions $w_k^{(2)} = \lambda w/(1+k)! + \lambda^2 w_*^2/(2+k)!$, $k \in \mathbb{N}$ are piecewise continuous with a discontinuity of the first kind at x_*, are single-peaked with a vertex at x_* and have no intervals of constancy, since the function $w_*^2(x) = x\theta(x)\theta(1-x) + (2-x)\theta(x-1)\theta(2-x)$ is continuous and has no intervals of constancy. It has a unique vertex at x_*, and the function $w(x) = \{1, x \in [0,1]; 0, x \in (0,\infty)\}$.

Let us construct an induction step from the value N to the value $N+1$. Let us assume that all functions $w_k^{(N)}$ are piecewise continuous with a jump at x_*, are piecewise continuous with a jump at x_*. Then each function $w * w_k^{(N)}$, $k \in \mathbb{N}_+$ is continuous, since all functions w_*^m, $m \geq 2$ are continuous. Each of them has no intervals of constancy and has a unique vertex z_k, $k \in \mathbb{N}_+$ due to the smoothing property of the function w. In this case, the points z_k satisfy the inequality $z \geq x_*$ due to the increase of all functions w_*^m on $[0,1]$, $m \geq 2$. By the induction hypothesis, the functions $w_k^{(N)}$ decrease by (x_*, ∞). From the restriction on the parameter λ in the conditions of the theorem it follows that

$$w_k^{(N)}(0) = e^{-\lambda}\frac{\lambda}{k!} > e^{-\lambda}\frac{\lambda}{k!}\sum_{m=2}^{N}\frac{\lambda^m}{m!(m-1)!} > e^{-\lambda}\frac{\lambda}{k!}\sum_{m=2}^{N}\frac{\lambda^m}{(m+k)!(m-1)!} = w_k^{(N)}(1+0),$$

where the inequality $(m+k)! > k!m!$ is used. Thus, the jump $w_k^{(N)}(x_*-0) - w_k^{(N)}(x_*+0) = \lambda^1/(k+1)!$ for each of the functions $w_k^{(N)}$ at the point x_* exceeds the difference $w_k^{(N)}(0) - w_k^{(N)}(1+0)$. Then, according to Statement 3.4, each of the functions $(w * w_k^{(N)})(x)$, $k \in \mathbb{N}_+$ has a vertex $z_k \leq 1$, and therefore $z_k = x_*$.

Let us now consider the functions $\lambda(w/(k+1)! + w * w_k^{(N)})$ for $k \in \mathbb{N}$. Since $w(x) = \{1, x \in [0,1]; 0, x \in (0,\infty)\}$ they are piecewise continuous with one discontinuity of the first kind at x_*, have no intervals of constancy and are unimodal with vertex at x_*. Moreover,

$$\lambda\big[(w/(k+1)! + w * w_k^{(N)}\big] = \sum_{m=1}^{N} \frac{\lambda^{m+1} w_*^{m+1}}{(k+m)!} = w_{k-1}^{N+1}, \quad k \in \mathbb{N},$$

which completes the construction of the induction step. From the unimodal property of all functions $w_k^{(N)}$ with vertex at x_* it follows, in particular, that such is the function with $k = 0$. Passing to the limit as $N \to \infty$, we obtain that the limit function $g = \lim_{N\to\infty} w_0^{(N)}$ is single-peaked with the vertex at the point $x_* = 1$, since the limit of single-peaked functions is a single-peaked function.

Necessity. If g is a single-peaked function with vertex at x_*, then $\dot{g}(x_*+0) < 0$. Hence, $\dot{R}_2(1+0) < 0$. From formula (5.2), having calculated the derivative at $x = 1$, we find $\sum_{m=2}^{\infty} \frac{\lambda^m}{m!(m-2)!} < \lambda^2$. □

6. Conclusion

In this paper, we study the possibility of realizing unimodality of the absolutely continuous part of the probability distribution of the sum of nonnegative independent identically distributed random variables. The solved problem should be considered as a special case of the general problem of determining qualitatively different probability distributions for values of extensive functionals of trajectories of stationary random processes and, in particular, determining the situations when these distributions are unimodal. Research in this direction seems important from the point of view of applications in theoretical physics, since the loss of the unimodality property of a probability distribution (and more generally, exact unimodality [1, 10, 11]) indicates the manifestation of some physical mechanism leading to such a situation. It should be noted that even within the framework of the relatively simple mathematical model that was investigated in this paper, the problem has not been solved exhaustively. First, apparently, with an increase in the parameter λ the unimodality property should be lost. In this case, the question of determining the number of distribution vertices arises. Secondly, it is important to study the functions g_N for finite values of N from the point of view of determining the range of the parameter p in which they are unimodal. Thirdly, it is very important to extend the results of this paper to probability distributions of a more general form for random variables $\tilde{x}_k$, for example, to the class of all strictly unimodal distributions.

Conflict-of-interest. The authors declare no conflicts of interest.

Acknowledgments and funding. The authors declare no financial support.

REFERENCES

1. N. K. Bakirov, "Extrema of distributions of quadratic forms from Gaussian variables," *Teor. Ver. i Ee Primen.*, **34**, No. 2, 241–250 (1989).
2. R. P. Braginskii, B. V. Gnedenko, G. M. Zaytseva, and S. A. Molchanov, "Theoretical and statistical study of defect set in enamel-varnish electrical insulating coatings," *Dokl. AN SSSR*, **303**, No. 2, 270–274 (1988).
3. B. V. Gnedenko, *Course in Probability Theory* [in Russian], Librokom, Moscow (2011).
4. I. A. Ibragimov, "On the composition of single-peak distributions," *Teor. Ver. i Ee Primen.*, **1**, No. 2, 283–288 (1956).
5. E. Lukach, *Characteristic Functions* [in Russian], Nauka, Moscow (1979).
6. A. Ya. Khinchin, "On unimodal distributions," *Izv. NII Tomsk. Mat. In-ta*, **2**, No. 2, 1–7 (1938).
7. P. Medgyssy, "On the interconnection between the representation theorems of characteristic functions of unimodal distributions and of convex characteristic functions," *Maguar Tud. Acad. Mat. Kutató Intér. Közl.*, **8**, 425–430 (1963).
8. P. Medgyssy, "On a new class of infinitely divisible functions and related topics," *Studia Sci. Math. Hungar.*, **2**, 441–446 (1967).
9. Yu. P. Virchenko and A. S. Mazmanishvili, "Unimodality of a class of distributions connected with complex Ornstein–Uhlenbeck process," *Dokl. Akad. Nauk Ukr. SSR*, No. 1, 55–87 (1988).
10. Yu. P. Virchenko and A. S. Mazmanishvili, "Essential unimodality of probability distributions of random quadratic functionals," *Dokl. Akad. Nauk Ukr. SSR*, No. 12, 3-4 (1990).
11. Yu. P. Virchenko and A. S. Mazmanishvili, "Essential unimodality of probabilistic distributions for random quadratic functionals," *Cybernet. Systems Anal.*, **285**, No. 2, 312–315 (1992).

12. Yu. P. Virchenko and A. D. Novoseltsev, “Unimodality of probability distributions for maxima of samples of independent Erlang random variables,” *Prikl. Mat. i Fiz.*, **51**, No. 3, 366–373 (2019).
13. Yu. P. Virchenko and A. D. Novoseltsev, “Probability distributions unimodality of finite sample extremes of independent Erlang random variables,” *J. Phys. Conf. Ser.*, **1479**, 012104 (2020), DOI: 10.1088/1742-6596/1479/1/012104.
14. Yu. P. Virchenko and A. D. Novoseltsev, “Bifurcation of distribution function of electric breakdown voltages of polymer enamel–lacquer coatings,” *J. Phys. Conf. Ser.*, **1902**, 012091 (2021), DOI: 10.1088/1742-6596/1902/1/012091.
15. Yu. P. Virchenko and A. D. Novoseltsev, “Bifurcation of the distribution of electrical breakdown voltages of enamel-varnish polymer coatings,” In: *Current Issues in Applied Mathematics, Computer Science and Mechanics*, Nauch.-Issl. Publ., Voronezh, pp. 1530–1534 (2021).
16. D. F. Vysochanskii and Yu. I. Petunin, “Justification of the 3σ rule for unimodal distributions,” *Teor. Ver. i Mat. Stat.*, **21**, 25–36 (1980).

Yu. P. Virchenko
Belgorod State Technological University named after V. G. Shukhov, Belgorod, Russia
E-mail: `virch@bsu.edu.ru`
A. M. Tevolde
Belgorod State University, Belgorod, Russia
E-mail: `amanuelmt1@gmail.com`