ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2025, Vol. 46, No. 7, pp. 3466–3476. © Pleiades Publishing, Ltd., 2025.

Strongly Unimodal Distributions of Measure on $\mathbb{R}_+$ with Smoothness Condition of Densities

Yu. P. Virchenko[1,2*] and A. M. Tewolde[2]

(Submitted by I. B. Petrov)

[1]*Belgorod State Technological University, Belgorod, 308012 Russia*

[2]*Belgorod State University, Belgorod, 308015 Russia*

Received March 17, 2025; revised April 27, 2025; accepted April 30, 2025

Abstract—The problem of describing the class of non-negative densities v of finite measure distributions on $\mathbb{R}_+$, which have the property of strong unimodality according to I.A. Ibragimov, is solved. It is based on the study of the properties of integral transformation corresponding to the convolution of distributions. The approach to unimodality investigation based on the Volterra integral equation of first kind with a difference kernel is proposed. It allows to describe the class of all such densities. For this, it is used the smooth parametrization by means of $\lambda \in (0, \infty)$ connected with the set $\mathfrak{D}(\mathbb{R}_+)$ of nonnegative unimodal densities f on $\mathbb{R}_+$ satisfying the equation. This makes it possible to select the entire class of all unimodal densities those that have the property of strong unimodality. It is based on the requirement that solutions to the equation have no some bifurcations in the parameter λ.

2010 Mathematics Subject Classification: 60E05

DOI: 10.1134/S1995080225609373

Keywords and phrases: *bifurcation, densities convolution, unimodal distributions, strong unimodality, Wolterra's equation of first kind*

1. INTRODUCTION

We will study *unimodal* distribution functions F of measures on $\mathbb{R}_+$. This means that the functions F are nonnegative, do not decrease monotonously when the argument $x \in \mathbb{R}$ is changed, they tend to 1 when $x \to \infty$ and to 0 when $x \to -\infty$. Without limiting generality, in the future, we will assume that the functions F are continuous on the right. In applications, as a rule, absolutely continuous distributions are used, which have densities f corresponding to each of them. In this case, those of them are of particular importance, in which the density has a single maximum point, which we will later call a *vertex* one. The presence of more than one maximum point means, from the point of view of applications, that along with the "natural" stochastic mechanism that forms the probability distribution of corresponding random variable, there is some additional "physical" mechanism that leads to the appearance of several maximum points of the distribution density. Such a situation is realized, for example, in models of statistical mechanics when describing so-called phase transitions [1].

Distributions with a single vertex point are called the *unimodal* ones. Generally speaking, from the viewpoint of applications, the presence of additional regular mechanisms which form the probability distribution is indicated not only by its loss of the unimodality, but also by the distortion of its shape in the presence of a single vertex point, for example, by the appearance of additional inflection points in the density. In the absence of such distortions, namely, in the presence of only two such inflection points, they say about the property of so-called *exact unimodality* [2–4]. However, in this paper we will be interested only in the phenomenon of preserving the distribution F unimodality. The existence of such a position turns out to be important in various problems of statistical mathematical physics (see, for example, [5–7]).

*E-mail: virch@bsu.edu.ru

Establishing the unimodality presence in a fixed distribution function F, which is represented by an explicit analytical formula, of course, reduces to the standard mathematical analysis problem. Of interest, however, are the problems of establishing unimodality for classes of distribution functions such as those determined by the needs of applications in mathematical modeling. To such problems relate those of which are connected with sums of independent random variables or with extremes of samples of such random variables. In this paper, we study the first of these types of problems in the case when the measure F is concentrated on $\mathbb{R}_+$ and the entire class of such measures is formed by means of a Wolterra integral transformation with defference kernel. In the next section, we give the general concept of unimodality of a one-dimensional distribution, as well as the concept of strong unimodality introduced by I.A. Ibragimov. In the following sections, we give some examples of problems related to establishing the unimodality property for classes of distributions and develop a method for investigating the presence of the strong unimodality property of distributions with continuously differentiable densities.

2. UNIMODAL DISTRIBUTIONS

Since the maximum distribution density can be achieved not at a solitary vertex point, but at each point of a certain interval of its constancy, when studying unimodality of distributions, a more general definition of unimodality is used, allowing for such a possibility. Moreover, such a definition is suitable for the broadest interpretation of distribution function, which is not based on the assumption that it has a density f, but is given in terms of the distribution function F (see, for example, [8–10, 13]).

Definition 1. *A distribution function F is called the unimodal one, if there exists a point $a \in \mathbb{R}$ for which it is convex at $x \in (-\infty, a)$ and concave at $x \in [a, \infty)$.*

Since the set of points a satisfying the condition pointed out in the given definition may consist of more than one point, we will further operate with the concept of the set Σ of all such vertex points having the specified property. It is obvious that the set Σ of vertex points of a unimodal distribution consists either of a single point or is the segment $\Sigma \subset \mathbb{R}$ of finite length. It is also clear that at any vertex point $a \in \Sigma$ of a unimodal distribution F, the last is continuous in the intervals (a, ∞) and $(-\infty, a)$. In the case when the vertex point is not unique, then the function F is continuous. Moreover, the unimodal distribution function F has the right $D_+F(x)$ derivative and the left $D_-F(x)$ one in each point of intervals (a, ∞) and $(-\infty, a)$. They are finite with the exception may be of a single point a which is the unique vertex point of the function F in this case. For Dini's derivatives the inequalities $D_+F(x) \geq D_-F(x)$ for $x < a$ and $D_+F(x) \leq D_-F(x)$ for $x > a$ are valid.

If the set Σ of vertex points of the unimodal distribution F is one-point, $\Sigma = \{a\}$, then the distribution F may have a gap of first kind at the point a. In this case the right and left derivatives may be finite and infinite at this point.

Described properties of unimodal distributions allow to state that they are absolutely continuous except for, may be, the vertex point a, corresponding to each of them, in the case when such a point is unique and there is a gap of the function in it. At the same time the absolutely continuous component F_c of the distribution F has the derivative at all points in $\mathbb{R}$ except for no more than a countable set of theirs where the right and left Dini derivatives do not coincide, $D_+F(x) \neq D_-F(x)$.

The following concept and the statement connected with it are very important from theoretical point of view.

Definition 2. *The sequence $\langle F^{(n)}; n \in \mathbb{N}\rangle$ of distribution functions is weakly converges when $n \to \infty$ to the distribution F, if it pointwise converges to this distribution at all continuity points of this function.*

From the above-pointed out properties of unimodal distribution and from the Definition 2 it follows that the sequence $\langle F^{(n)}; n \in \mathbb{N}\rangle$ of unimodal distribution functions weakly converges to the distribution F, if it pointwise converges to the function F at all points except for, may be, the vertex point a of the distribution and, in the case, it is unique.

Thus, the following statement takes place (see, for example, [10]).

Theorem 1. *Let $\langle F^{(n)}; n \in \mathbb{N}\rangle$ be a sequence of unimodal distributions, which weakly converges to the distribution function F. Then, F is the unimodal distribution. At the same time, the sequence $\langle F^{(n)}; n \in \mathbb{N}\rangle$ point wise converges to F at all points except for, may be, the vertex point a of the distribution F, if it is unique and there is a gap of the function F in it.*

Corollary 1. *Let $\langle \Sigma_n; n \in \mathbb{N}\rangle$ be the sequence of segments of vertex points and each of them corresponds to the unimodal distribution function, which is the component of the sequence $\langle F^{(n)}; n \in \mathbb{N}\rangle$ and this sequence weakly converges to the distribution function F, when $n \to \infty$. Then, the set Σ_∞ of limit points of all sequences $\langle a^{(n)}; n \in \mathbb{N}\rangle$ of vertex points $a^{(n)} \in \Sigma_n$, $n \in \mathbb{N}$ of the distribution functions $F^{(n)}$ coincides with the set Σ of vertex points of the distribution F.*

The following property of unimodal distributions is very important.

Theorem 2. *If $F(x)$, $x \in \mathbb{R}$ is a unimodal distribution, then $F(\lambda x - c)$, $x \in \mathbb{R}$ is also the unimodal distribution for any $c \in \mathbb{R}$, $\lambda \in (0, \infty)$.*

Proof. If a is a vertex point of the distribution $F(x)$, $x \in \mathbb{R}$, then $(a + c)/\lambda$ is the vertex point of the distribution $F(\lambda x - c)$, since $F(\lambda x - c)$ is convex for $x < (a + c)/\lambda$ and is concave for $x > (a + c)/\lambda$. □

3. STRONGLY UNIMODAL DISTRIBUTION

In this paper, we are interested in conservation of the unimodality property of a distribution F after its convolution with other unimodal distribution V. From this viewpoint, it is very important the concept of the *strong unimodality* of distributions introduced by I.A. Ibragimov.

Definition 3. *A distribution function V is named the strongly unimodal one, if it is unimodal and its convolution*

$$G(x) = (V * F)(x) \equiv \int\limits_{\mathbb{R}} V(x - y)dF(y) \tag{1}$$

with any unimodal distribution F is also unimodal distribution function where the integral is understood by the Stieltjes sense.

The normal distribution V with the density

$$v(x) = \frac{\nu}{\sqrt{2\pi}} \exp\Big(- \frac{\nu^2(x - x_0)^2}{2}\Big)$$

is the example of strongly unimodal distribution for any $x_0 \in \mathbb{R}$, $\nu > 0$ [13].

It is correct the following statement which is analogous to Theorem 2.

Theorem 3. *If V is a strongly unimodal distribution, then $V(\lambda x - c)$ is also the strongly unimodal one for any $c \in \mathbb{R}$, $\lambda \in (0, \infty)$.*

Proof. Since $V(x)$ is unimodal, then $V(\lambda x - c)$ is also unimodal for $x \in \mathbb{R}$. Therefore, for any unimodal distribution $F(y)$ we may regard the distribution $F((y + c)/\lambda)$ which is unimodal, according to Theorem 2. The distribution

$$G(x) = \int\limits_{\mathbb{R}} V(x - y)dF((y + c)\lambda) = \int\limits_{\mathbb{R}} V(x - c - \lambda y)dF(y)$$

is unimodal for it. Therefore, according to the same theorem, the distribution

$$G(\lambda x) = \int\limits_{\mathbb{R}} V(\lambda(x - y) - c)dF(y)$$

is also unimodal. Since F is an arbitrary unimodal distribution, then it follows the statement of theorem. □

In the paper of Ibragimov [13], by means of rather subtle construction, it is found the necessary and sufficient conditions of strong unimodality of distribution functions on $\mathbb{R}$ being very convenient in applications.

Theorem 4. *For strong unimodality of a unimodal distribution function V, it is necessary and sufficient that V should be absulutely continuous, and the function $\ln V'(x)$ be concave on the set $\{x \in \mathbb{R} : (D_+V(x)) \cdot (D_-V(x)) \neq 0\}$.*

It is obvious that convolution of a pair of strongly unimodal distributions is strongly unimodal distribution and, based on Definition 3 and Theorem 1, the following statement is valid.

Theorem 5. *The limit distribution function V, in the sense of weak convergence, of the sequence $\langle V^{(n)}; n \in \mathbb{N}\rangle$ of strongly unimodal distribution functions is the strongly unimodal distribution.*

Proof. Since the components $V^{(n)}$, $n \in \mathbb{N}$ of the sequence $\langle V^{(n)}; n \in \mathbb{N}\rangle$ are strongly unimodal distribution functions, then, for any unimodal distribution F, the sequence $\langle V^{(n)} * F; n \in \mathbb{N}\rangle$ consists of unimodal distribution functions. Further, since the sequence $\langle V^{(n)}; n \in \mathbb{N}\rangle$ converges in the weak sense to the distribution V, then the sequence $\langle V^{(n)} * F; n \in \mathbb{N}\rangle$ also converges in weak sense to unimodal distribution function $V * F$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} F(x-y)dV^{(n)}(y) = \int_{\mathbb{R}} F(x-y)dV(y)$$

where the transition to the limit under the integral sign has been justified by second Helly theorem (see, for example, [10]). □

It is obvious, if distributions V, F and G have corresponding densities v, f, and g on $\mathbb{R}$, then the formula (1) is written down in the form of the integral transformation

$$g(x) = \int_{\mathbb{R}} v(x-y)f(y)dy, \tag{2}$$

which one may consider as an integral Fredholm equation of first kind for the function f at given densities g and v. Its solution is formally obtained on the base of the Fourier transformation. It may be fulfilled due to g and v belong to the space $\mathbb{L}_1(\mathbb{R})$. But the production of those Fourier transformations is not necessarily a positively defined function, that is, it is not necessarily density. Due to this reason, the specific class of problems arises (it is so-called the decomposition problems, see, for example, [10]), connected with the description of classes of densities g and v, for which such positively defined solutions exist.

We introduce the designation $\mathfrak{D}(\mathbb{R})$ for the set of unimodal distribution on $\mathbb{R}$, and also $\mathfrak{S}(\mathbb{R})$ for the set of strongly unimodal distributions on $\mathbb{R}$.

4. STRONGLY UNIMODAL DISTRIBUTIONS ON $\mathbb{R}_+$

This paper is dedicated to the study of measures distributions on $\mathbb{R}_+$, that is, to distribution functions F, which is equal identically to zero when $x \in (-\infty, 0)$. Therefore, according to analogy with Definition 1, we give the following

Definition 4. *A distribution function V on $\mathbb{R}$, which is equal identically to zero for $x < 0$, is named strongly unimodal one on $\mathbb{R}_+ = [0,\infty)$, if it is unimodal and its convolution*

$$G(x) = (V * F)(x) = \int_{-0}^{x} V(x-y)dF(y) \tag{3}$$

with any unimodal distribution function F on $\mathbb{R}_+$, which is equal identically to zero for $x < 0$, is also the unimodal distribution function on $\mathbb{R}_+$.

In the pointed out formula the integration is understood according to Stieltjes, the low integration limit implies that the distribution F may have the gap in the point $y = 0$ and its contribution should be taken into account when integration is done. In the case when the distribution F has such a jump, its vertex point is equal to zero and it is represented by a concave function on $(0,\infty)$.

In the case when the distributions V, F, and G are absolutely continuous, the formula (3) is represented in the form

$$g(x) = \int_{0}^{x} v(x-y)f(y)dy, \tag{4}$$

which one may consider as an integral Volterra equation of first kind for an integrable function f at fixed densities g and v. However, as above, its solution, which is formally obtained on the base of the

Laplace transformation being fulfilled, since g and v belong to the space $\mathbb{L}_1(\mathbb{R}_+)$, is not necessarily is a positively defined function, that is, it is not necessarily the density. Consequently, it arises the question about description of classes of densities g and v, for which such positively defined solutions exist. Further, everywhere, if unless otherwise specified, we will consider that densities f are continuous and continuously differentiable on intervals, which are internal parts of their supports $\mathrm{Int\,supp}\, f$. In the same case, we will say simply that they are continuously differentiable.

Lemma 1. *Let f_j be densities of unimodal distributions F_j on $\mathbb{R}_+$, which are continuously differentiable. Let, further, Σ_j, $j \in \{1,2\}$ be the segments of their vertex points. Then, the vertex points b of the density g of the distribution $G = F_1 * F_2$ satisfy the condition $b > \max\{c : c \in \Sigma_1 \cup \Sigma_2\}$.*

Proof. We write down the expression for the density g,

$$g(x) = \int_0^x f_1(x-y)f_2(y)dy = \int_0^x f_1(y)f_2(x-y)dy.$$

Due to continuous differentiability of densities f_j, $j \in \{1,2\}$, we have

$$g'(x) = \int_0^x f_1'(x-y)f_2(y)dy + f_1(0)f_2(x),$$

$$g'(x) = \int_0^x f_1(y)f_2'(x-y)dy + f_1(x)f_2(0),$$

where the infinite value for $f_1(0)$ is possible in the first formula and, in this case, the vertex point of the density f_1 is located in $x = 0$, and in the second formula the infinite value for $f_2(0)$ is possible when $x = 0$ is the vertex point of the density f_2.

By replacing the integration variable $y \Rightarrow x - y$ in the given formulas,

$$g'(x) = \int_0^x f_1'(y)f_2(x-y)dy + f_1(0)f_2(x),$$

$$g'(x) = \int_0^x f_1(x-y)f_2'(y)dy + f_1(x)f_2(0),$$

we find that the integral in the first formula is positive for $x \le \max \Sigma_1$, and in the second formula, the integral is positive for $x < \max \Sigma_2$. Therefore, $g'(x) > 0$, due to non-negativity of second summands in both formulas. □

The simplest and, along with that, very important from the theoretical point of view example of strongly unimodal distribution, concentrated on the semiaxis $[0, \infty)$, is the exponential distribution.

Theorem 6. *The distribution functions, which are defined be the formula $V(x) = 1 - \exp(-\rho x)$ for $x \ge 0$, $\rho > 0$, are strongly unimodal.*

Proof. Let the distribution F have a unimodal density f with a vertex point a and this density is continuously differentiable. Then, using the density $v(x) = \rho \exp(-\rho x)$, $x \ge 0$ of exponential distribution, the density g of its convolution with the distribution F, on the basis of (3), is written down in the form

$$g(x) = \int_0^x v(x-y)f(y)dy = \rho e^{-\rho x} \int_0^x e^{\rho y} f(y)dy. \quad (5)$$

The density g is continuously differentiable, due to continuity of the function v' and the density f. Differentiating on x both sides of the equality, we find

$$g'(x) + \rho g(x) = \rho f(x). \tag{6}$$

Since $g'(b) = 0$ should be take place at the vertex point b of the distribution G and, due to continuous differentiability of the densities f and g, according to (6), there exists the second derivative g'' of the density g. Then, using the equality, which is obtained after second differentiation, and also the fact that, according to Lemma 1, the equation $g'(x) = 0$ may have the solution only at points x from the interval (a, ∞) in which the density f is decreased, $f'(b) < 0$, we find that in such a point it is valid $g''(b) = \rho f'(b) < 0$. Therefore, the density g may have only maximum points as its extreme points. Consequently, such a maximum vertex point is unique.

In order to go to general case, we notice that the functions

$$F_\gamma(x) = \gamma \int_0^x e^{-\gamma(x-y)} F(y)dy, \quad \rho > 0 \tag{7}$$

for any distribution function F on $\mathbb{R}_+$ are absolutely continuous distributions, since $F_\gamma(x) \geq 0$, $F_\gamma(0) = 0$, and the limit when $x \to \infty$, according to the l'Hôpital rule, is equal

$$\lim_{x\to\infty} F_\gamma(x) = \gamma \lim_{x\to\infty} \frac{\int_0^x e^{\gamma y} F(y)dy}{e^{\gamma x}} = \lim_{x\to\infty} F(x) = 1.$$

Differentiating $F_\gamma(x)$,

$$F'_\gamma(x) = \gamma \left(F(x) - \gamma \int_0^x e^{-\gamma(x-y)} F(y)dy \right),$$

and after that using the theorem about the average value of integral for the estimate of second summand, due to positivity of the weight function $e^{-\gamma(x-y)}$, we find

$$F'_\gamma(x) = \gamma\Big(F(x) - (1 - e^{-\gamma x})F(y(x))\Big),$$

where $y(x)$ is a midpoint in the segment $[0, x]$. Due to monotonous non-decreasing of the distribution F, it is valid $F(x) \geq F(y(x))$. From where it follows that $F_\gamma(x)$ monotonously increases.

Having applied again the transformation (7) to the distributions F_γ, we obtain the collection of distributions $\langle \gamma \int_0^x e^{-\gamma(x-y)} F_\gamma(y)dy; \gamma > 0 \rangle$, linearly ordered according to increasing of the parameter γ, in which all distribution functions have continuously differentiable densities. Consequently, application the transformation (5) to their densities gives the unimodal distribution densities, according to above-proved fact. It remains now to go to the limit $\gamma \to \infty$ and after that to use Theorem 1. □

Corollary 2. *The distributions V_n, $n \in \mathbb{N}$ with densities $v_n(x) = dV_n/dx = \rho^{n+1} x^n \exp(-\rho x)$, $\rho > 0$ are strongly unimodal.*

Proof. The proof follows by means of application of the statement about that the convolution of a pair of strongly unimodal distributions is the strongly unimodal distribution, and after that one may to apply the induction on $n \in \mathbb{N}$, in which the distribution $V_1 = V$ is strongly unimodal. □

5. THE CONDITION OF STRONG UNIMODALITY

We introduce the designations $\mathfrak{D}(\mathbb{R}_+)$ for the class of unimodal distributions and $\mathfrak{S}(\mathbb{R}_+)$ for the class of strongly unimodal distributions on $\mathbb{R}_+$. According to the definition of classes $\mathfrak{D}(\mathbb{R}_+)$ and $\mathfrak{D}(\mathbb{R})$, the inclusion $\mathfrak{D}(\mathbb{R}_+) \subset \mathfrak{D}(\mathbb{R})$ takes place.

Let $V \in \mathfrak{S}(\mathbb{R})$ and this distribution is equal identically to zero for $x < 0$. Then, according to the definition of distributions belonging to $\mathfrak{S}(\mathbb{R})$, a convolution V with any distribution from $\mathfrak{D}(\mathbb{R}_+)$ is the unimodal distribution. Consequently, $V \in \mathfrak{S}(\mathbb{R}_+)$. Otherwise, if $V \in \mathfrak{S}(\mathbb{R}_+)$, then the statement that $V \in \mathfrak{S}(\mathbb{R})$ not necessarily should be correct, since for validity of such an inclusion, it is necessary to do verification of conservation of the unimodality property of convolutions $V * F$ for more wider class $\mathfrak{D}(\mathbb{R})$ of distributions F. In this case it may be realized the situation, when a necessary condition of affiliation of the distribution V to the class $\mathfrak{S}(\mathbb{R}_+)$ is weaker than the statement in Theorem 4. Certainly, the sufficient condition of affiliation of the distribution V, which is equal to zero identically for $x < 0$, to the set $\mathfrak{S}(\mathbb{R})$ is sufficient for its affiliation to the set $\mathfrak{S}(\mathbb{R}_+)$. Due to this reason it occurs the problem of description of the functional class $\mathfrak{S}(\mathbb{R}_+)$ of distributions on $\mathbb{R}_+$. In this paper we solve this problem in the case, when the distribution V has the density v, which is twice continuously differentiable on $\mathbb{R}_+$.

Let the distributions $V \in \mathfrak{S}(\mathbb{R}_+)$ and $F \in \mathfrak{D}(\mathbb{R}_+)$ have the unimodal densities v and f, correspondingly. At that, the set of vertex-points of the density v consists of one point x_0. Consider the density g, obtained by means of the convolution (4). A vertex-point (one of points at the segment of its vertex-points) of the density f we denote by a. Further, we are needed the following fact for the analysis of the convolution (4).

Lemma 2. *Let a density f be finite and has the support* $\operatorname{supp} f$*, which is contained in* $[0, A]$*, $A > 0$, and the density v is absolutely continuous and unimodal. Then, the density g defined by (4) is continually differentiable and its stationary points contained in $[0, A + x_0)$ where x_0 is the right extreme point of the segment Σ of vertex-points of the density v.*

Proof. It is necessary to prove the fact of location of stationary points in $[0, A + x_0)$, which are the solutions of the equation $g'(x) = 0$ connected with the density g. Since $\operatorname{supp} f \subset [0, A]$, then the derivative of the density g for $x > A$, according to (4), is defined by the formula

$$g'(x) = \int_0^x v'(x - y) f(y) dy.$$

Consider the solution of the equation

$$\int_0^x v'(x - y) f(y) dy = 0.$$

Since $v'(x - y) \geq 0$ for $x - y \leq x_0$, then the integration segment $[0, A]$ should be such that the set $\{y \in [0, A] : x - y > x_0\}$ is non-empty. It is possible only in the case when $x - x_0 \in [0, A]$, that is, $x < A + x_0$. □

Further, we analyze the case when the density v is twice continuously differentiable. We will prove the following statement which is the main result of our paper.

Theorem 7. *Let a distribution V on $\mathbb{R}_+$ have the twice continuously differentiable on $(0, \infty)$ monotonously non-increasing density v. Further, let $\lim_{x \to +0} v(x) = \lim_{x \to +0} v'(x) = 0$ and in those points which are contained in $\operatorname{Int} \operatorname{supp} v$ do not simultaneously vanish the derivatives v' and v''. Then, if $v \in \mathfrak{D}(\mathbb{R}_+)$,* it is fulfilled $v \in \mathfrak{S}(\mathbb{R}_+)$.

Proof. We divide our proof into several items.

I. We introduce the parameter $\lambda > 0$. Let us associate a density f_λ with each unimodal density f, the values of which are determined by the formula $f_\lambda(x) = \lambda f(\lambda x)$ at each point $x \in (0, \infty)$. According to Theorem 3, it is unimodal. Let a be the vertex point of the density f. Then a/λ is the vertex point of the density f_λ. To establish a strong unimodality of the density v, it is necessary to prove the unimodality of any density g, obtained according to the formula (4). Analysis of unimodality of this density is equivalent to unimodality analysis of anything of densities g_λ, $\lambda > 0$, where each of them is defined by the formula

$$g_\lambda(x) = \int_0^x v(x - y) f_\lambda(y) dy. \tag{8}$$

Therefore, it is sufficient to establish the unimodality of the density g_λ at least for a value $\lambda > 0$. For this, by replacing the integration variable $\lambda y \Rightarrow y$ in the integral (8), we find that it is necessary and sufficient to establish those conditions for the density v, for which the density defined by the transformation

$$g_\lambda(x) = \int_0^{\lambda x} v(x - y/\lambda) f(y) dy, \quad x \in (0, \infty) \tag{9}$$

is unimodal. It means that for such conditions, it is necessary and sufficient to prove the existence of the unique point $b \in (0, \infty)$, where the following conditions

$$g'_\lambda(b) = 0, \quad g_\lambda^{(2k)}(b) < 0 \tag{10}$$

take place for a value $k \in \mathbb{N}$ such that $g^{(j)}(b) = 0$, $j = 1 \div 2k - 1$.

We notice that under suppositions relative to the density v in the formulation of theorem, $g_\lambda(0) = 0$, that is, the point $x = 0$ does not the vertex one.

Differentiating on x the density (9) with account the condition $v(0) = v'(0) = 0$ and the continuity of the density f, we find

$$g'_\lambda(x) = \int_0^{\lambda x} v'(x - y/\lambda) f(y) dy, \quad x \in (0, \infty). \tag{11}$$

II. Let the density f be continuously differentiable. In this case, due to twice continuous differentiability of the density v, continuous second derivative of the density g_λ,

$$g''_\lambda(x) = \int_0^{\lambda x} v''(x - y/\lambda) f(y) dy, \quad x \in (0, \infty) \tag{12}$$

is defined.

The solution b of the equation (10) is an implicit function $b(\lambda)$ on the parameter $\lambda \geq 0$. This function is defined by the equation (10) as the continuously differentiable one for sufficiently large values. On the basis of the theorem of implicit function $b(\lambda)$ existence, its derivative $db(\lambda)/d\lambda$ is defined by the equation $g'_\lambda(b(\lambda)) = 0$, after calculation of full derivative on λ of its right side.

Using the expression (11), we find

$$g''_\infty(b(\lambda)) \frac{db(\lambda)}{d\lambda} + \Big(\frac{\partial g'_\lambda(x)}{\partial s}\Big)_{x=b(\lambda)} = 0. \tag{13}$$

Since as in the case of the existence of unique function $b(\lambda)$, it should be defined at a sufficiently large value λ such the point $b(\lambda) = b$ is a vertex point for the density g_λ, then, according to theorem of implicit function, it is necessary and sufficient for this that the conditions $g''_\lambda(b) \neq 0$ and

$$\Big(\frac{\partial g'_\lambda(x)}{\partial \lambda}\Big)_{x=b} \neq 0 \tag{14}$$

should be fulfilled in this point. On he basis of (11) we find that

$$\frac{\partial g'_\lambda(x)}{\partial \lambda} = \frac{1}{\lambda^2} \int_0^{\lambda x} v''(x - y/\lambda) f(y) dy, \quad x \in (0, \infty). \tag{15}$$

Going over the limit $\lambda \to \infty$ in (11), (12), and (15), we find that the following equalities

$$g'_\infty(x) = v'(x), \quad g''_\infty(x) = v''(x) \tag{16}$$

and asymptotic formula

$$\frac{\partial g'_\lambda(x)}{\partial \lambda} = M v''(x)(1 + o(1)), \quad \lambda \to \infty, \tag{17}$$

with $M = \int_0^\infty yf(y)dy$ take place for any x in bounded segment on $[0, \infty)$.

III. We put that the differentiable function f is finite and it has the support $\mathrm{supp}\, f \subset [0, A]$. Then, according to Lemma 2, for the application of formulas (15) and (16) for the density g_λ, it is sufficient to choose values x in the segment $[0, A + x_0]$ where x_0 is the unique vertex point of the density v. Consequently, at first, from the equality in (10) it follows that $g'_\infty(b(\infty)) = v'(b(\infty)) = 0$ and, therefore, $b(\infty) = x_0$, and from the inequality at $k = 1$ we conclude that $g''_\infty(b(\infty)) = v''(b(\infty)) < 0$ due to the condition of theorem. From here it follows that the points $b(\lambda)$ exist with needed properties for all sufficiently large values λ. In these points, the condition (13) is fulfilled and the point $b(\lambda)$ is unique for each such a value λ. In this case, we have $\lim_{\lambda\to\infty} b(\lambda) = x_0$. Thus, the statement of theorem has been proved in the case of finiteness and continuous differentiability of the density f.

It remains to extend this statement to arbitrary unimodal distributions F on $\mathbb{R}_+$, that is, for such of them which have not necessarily finite and continuously differentiable densities f.

IV. Let us fix a small value $\delta > 0$. Further, let f be a continuously differentiable unimodal density on $\mathbb{R}_+$. Define an unboundedly increasing sequence $\langle A_n \in (0, \infty); n \in \mathbb{N}\rangle$ such that the density f does not increase for $x \in (A_1, \infty)$. On the base of the sequence $\langle A_n \in (0, \infty); n \in \mathbb{N}\rangle$ and the density f, we define the sequence $\langle f_n; n \in \mathbb{N}\rangle$ of finite, continuously differentiable densities. Let $f_n(x) = Z_n f(x)$ for $x \in [0, A_n]$, and we put that $f_n(x) = \alpha_n x^2 + \beta_n x + \gamma_n$ for $x \in [A_n, A_n + \delta]$, where coefficients $\alpha_n, \beta_n, \gamma_n$ are chosen such that the densities f_n were continuous at points A_n and $A_n + \delta$ and also they should have continuous derivatives at A_n, $n \in \mathbb{N}$. Namely, $f_n(A_n) = \alpha_n A_n^2 + \beta_n A_n + \gamma_n$, $f'_n(A_n) = 2\alpha_n A_n + \beta_n$, $\alpha_n(A_n + \delta)^2 + \beta_n(A_n + \delta) + \gamma_n = 0$. It is possible, since, in this case,

$$\alpha_n = -\delta^{-2} Z_n(f(A_n) + \delta f'(A_n)), \quad \beta_n = \beta_n = Z_n f'(A_n) - 2\alpha_n A_n,$$
$$\gamma_n = Z_n f(A_n) - \alpha_n A_n^2 - \beta_n A_n,$$

where Z_n are defined on the basis of the condition $Z_n \int_{\mathbb{R}_+} f_n(x)dx = 1$. It is obvious that the densities f_n defined by this way are non-negative and unimodal. Since $f(A_n) \to 0$ when $n \to \infty$, then $Z_n \to 1$ at such limit transition. Therefore, it is fulfilled $f_n(x) \to f(x)$ at each point $x \in \mathbb{R}_+$.

For each of the constructed densities f_n, it is valid the statement proved in the previous item, that is, the densities g_n defined by convolutions $g_n(x) = \int_0^x v(x - y)f_n(y)dy$ are strongly unimodal under the same conditions of theorem relative to the density v. Since the density v is continuous and bounded, and the integrals $\int_{A_n}^\infty f_n(y)dy$ tend to zero when $n \to \infty$, then we may go to the limit under the sign of integral. Therefore, it is valid

$$g_n(x) = \int_0^\infty v(x - y)f_n(y)dy \to g(x) = \int_0^\infty v(x - y)f(y)dy,$$

and we obtain, according to Theorem 1, that g is the density of unimodal distribution.

V. Let F be an arbitrary unimodal distribution on $[0, \infty]$. Since the distribution V_ρ with the density $v(x, \rho) = \rho^2 x e^{-\rho x}$ is strongly unimodal on $\mathbb{R}_+$, then the function $f(x, \rho) = \int_{\mathbb{R}_+} v(x - y, \rho)dF(y)$ is the density of unimodal distribution $V_\rho * F$. It is continuous and continuously differentiable with the derivative $f'(x, \rho) = \int_{\mathbb{R}_+} v'(x - y, \rho)dF(y)$. For the distribution with such a density, according items **I**–**IV**, it is correct the statement of theorem. Then, for any strongly unimodal distribution with the density v, satisfying conditions of theorem, the distribution $V * V_\rho * F = V_\rho * V * F$ is unimodal. But each of distributions $V_\rho * V$ is strongly unimodal, since, according to the definition of strong unimodality, convolution of strongly unimodal distributions is the strongly unimodal one. It remains to notice that $\lim_{\rho\to\infty} \int_0^x v(x - y, \rho)v(y)dy$ tends to $v(x)$ when $\rho \to \infty$, and to apply Theorem 5. □

6. CONCLUSIONS

The paper proposes an approach to the study of the integral transformation, which relates the probability distributions of non-negative random variables in the case when one of them is the sum of two statistically independent ones. In this case, such a transformation generates a Volterra integral equation of first kind. This approach is applied to study the properties of strongly unimodal distributions on $\mathbb{R}_+$, that is, they retain the unimodal property when they are composed with other unimodal distributions. From the point of view of application to probability theory, the study of strongly unimodal distributions has fundamental interest. It is connected with the fact that, on the one hand, unimodal distributions describe random variables such that the formation of their probability distribution, as a rule, is not influenced by extraneous non-random factors, and, on the other hand, if a random variable is the sum of random variables with unimodal distributions, then one can expect that its distribution is also unimodal. In this regard, a general mathematical problem arises of studying those situations where the above considerations are correct. In particular, it is of interest to answer the simple question of what analytical properties a unimodal distribution must have in order for its convolution with itself to have the unimodality property. In this case, the resulting distribution describes the sum of two equivalent, statistically independent random variables. This and similar problems turn out to be quite difficult problems of mathematical analysis. In this connection, we can mention the problem of unimodality of the so-called stable distributions and the related more general problem of unimodality of the self-decomposable distributions introduced by A.N. Kolmogorov (see, for example, [11, 12]). The relevance of solving such problems is also related to the fact that similar problems arise in problems of statistical radiophysics (see, for example, [14]), when distributions of measure are determined by characteristic functions.

Since integral transformations (2) (or transformation (3)) contain a kernel that depends on the difference of arguments, it seems natural to approach to their study using of the Fourier transform (the Laplace transform), that is, to solve problems associated with this transformation in terms of characteristic functions [10]. However, this approach faces serious analytical difficulties due to the complex relationship between the characteristic functions and the corresponding distributions. Therefore, establishing the property of strong unimodality of distribution based on the study of the analytical properties of the integral transformation used in [13] seems more promising to us.

As for the specific result obtained in the work, it is, of course, still insufficient from the point of view of a complete answer to the question posed in the work about the description of the class of strongly unimodal distributions on $\mathbb{R}_+$. Further progress towards a complete solution to the problem involves extending the description of this class to distributions of V that do not have twice continuously differentiable densities.

FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. M. Kac, "Mathematical mechanisms of phase transitions," in *Statistical Physics Phase Transitions and Superfluidity* (Gordon and Beach Science, New York, 1968).
2. N. K. Bakirov, "Extrema of distributions of quadratic forms in gaussian variables," Teor. Veroyatn. Primen. **34**, 241–250 (1989).
3. Yu. P. Virchenko and A. S. Mazmanishvili, "Essential unimodality of probability distributions of random quadratic functionals," Dokl. Akad. Nauk Ukr. SSR, No. 12, 3–4 (1990).
4. Yu. P. Virchenko and A. S. Mazmanishvili, "Essential unimodality of probabilistic distributions for random quadratic functionals," Cybern. Syst. Anal. **285**, 312–315 (1992).
5. Yu. P. Virchenko and A. S. Mazmanishvili, "Unimodality of a class of distributions connected with complex Ornstein–Uhlenbeck process," Dokl. Akad. Nauk Ukr. SSR, Ser. A, No. 1, 55–87 (1988).
6. Yu. P. Virchenko and A. D. Novoseltsev, "Unimodality of probability distributions for sample maxima of independent Erlang random variables," Nauch. Vedom. Belgor. Univ., Mat. Fiz. **51**, 366–373 (2019).
7. Yu. P. Virchenko and A. D. Novoseltsev, "Probability distributions unimodality of finite sample extremes of independent Erlang random variables," J. Phys.: Conf. Ser. **1479**, 012104 (2020).
8. A. Ya. Khinchin, "On unimodal distributions," Izv. Tomsk. Mat. Inst. **2** (2), 1–7 (1938).

9. P. Medgyssy, “On the interconnection between the representation theorems of characteristic functions of unimodal distributions and of convex characte-ristic functions,” Maguar Tud. Acad. Mat. Kutató Intér. Közl. **8**, 425–430 (1963).
10. E. Lukacs, *Characteristic Functions* (Hafner, New York, 1970).
11. M. Yamazato, “Unimodality of infinitely divisible distribution functions of class L,” Ann. Probab. **6**, 523–531 (1978).
12. V. M. Zolotarev, *One-Dimensional Stable Distributions* (Am. Math. Soc., Provindence, RI, 1986).
13. I. A. Ibragimov, “On the composition of unimodal distributions,” Teor. Veroyatn. Primen. **1**, 283–288 (1956).
14. Yu. P. Virchenko and A. S. Mazmanishvili, “Characteristic function of quadratic convolution functional on the space of trajectories of gaussian processes,” Lobachevskii J. Math. **45**, 5328–5340 (2024).

Publisher’s Note. Pleiades Publishing remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

AI tools may have been used in the translation or editing of this article.