On an Inverse Problem for an Abstract Differential Equation of Fractional Order

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Abstract—In Banach space, we consider the problem of determining the solution and a summand of a differential equation of fractional order from the initial and redundant conditions containing fractional Riemann–Liouville integrals. It is shown that the solvability of the problem under consideration depends on the distribution of zeros of the Mittag–Leffler function.

Key words: *differential equation of fractional order, Riemann–Liouville integral, densely defined linear operator, Mittag–Leffler function, Cesàro mean, Banach space.*

Suppose that E is a Banach space and A is a linear, closed, densely defined operator in E with domain of definition $D(A)$ and with a nonempty resolvent set. Consider the problem of determining a function $u(t) \in C^1((0,1], E)$ belonging to $D(A)$ for $t \in (0,1]$ and an element $p \in E$ from the conditions

$$D^\alpha u(t) = Au(t) + t^{k-1}p, \tag{1}$$

$$\lim_{t\to 0} I^{1-\alpha} u(t) = u_0, \tag{2}$$

$$\lim_{t\to 1} I^{\beta} u(t) = u_1, \tag{3}$$

where $k > 0$,

$$I^\beta u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s)\, ds$$

is the left-sided fractional Riemann–Liouville integral of order $\beta \geq 0$ (I^β is the unit operator for $\beta = 0$), $\Gamma(\,\cdot\,)$ is the gamma function, and

$$D^\alpha u(t) = \frac{d}{dt} I^{1-\alpha} u(t)$$

is the left-sided fractional Riemann–Liouville derivative of order $\alpha \in (0,1)$.

Definition. By a *solution of problem* (1)–(3) we mean the pair $(u(t), p)$, where $u(t) \in D(A)$ is a continuous (for $t > 0$) function such that $I^{1-\alpha}u(t)$ is continuously differentiable (for $t > 0$) function, $p \in E$, and finally, $u(t)$ and p satisfy (1)–(3).

Following the existing terminology, we call problem (1)–(3) an *inverse problem* in contrast to a direct problem of Cauchy type (1), (2) with a known element $p \in E$. The problem under consideration can be regarded as the recovery of the nonstationary summand $t^{k-1}p$ in Eq. (1) with the help of the additional boundary condition (3).

For a survey of works on inverse problems of the form (1)–(3) for $\alpha = 1$, $\beta = 0$ and various constraints on the operator A, see the monograph [1] as well as the papers [2]–[5]. As to the inverse problem (1)–(3), it is considered here for the first time. In contrast to the inverse problems studied

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earlier, the additional condition (3) involves an integral of fractional order that can be regarded as the Cesàro mean over the interval $[0, 1]$. In what follows, it will shown that, as far as the solvability of the inverse problem is concerned, assigning the Cesàro mean over the interval $[0, 1]$ is more advantageous than assigning the final value $u(1)$.

It follows from [6], [7] that the assignment of relation (2) makes the initial-value problem for the equation

$$D^{\alpha}u(t) = Au(t) + f(t), \qquad t > 0, \tag{4}$$

with $\alpha \in (0, 1)$ and a sufficiently smooth function $f(t)$, well posed. Conditions on the operator A and the function $f(t)$ ensuring that problem (4) (2) is well-posed were given in [6]. These conditions in a Banach space E possessing the Radon-Nikodima property (see [8, p. 15]), are stated in terms of the estimates of the norm of the derivatives of the resolvent $R(\lambda^{\alpha}) = (\lambda^{\alpha}I - A)^{-1}$ (λ^{α} is the principal branch of the power function), which exists at the point λ^{α} for $\operatorname{Re}\lambda > \omega$. These estimates are of the form

$$\left\| \frac{d^n R(\lambda^{\alpha})}{d\lambda^n} \right\| \le \frac{M\Gamma(n+\alpha)}{(\operatorname{Re}\lambda - \omega)^{n+\alpha}}, \qquad n = 0, 1, 2, \dots, \tag{5}$$

and constitute a necessary and sufficient condition for the uniform well-posedness of problem (4), (2). Note that another approach to the derivation of conditions of uniform well-posedness involving a regularization method for a contour integral was described in [9].

In what follows, we shall need the following conditions.

Condition 1. The operator A is such that problem (4), (2) is uniformly well posed.

Condition 2. One of the following requirements is imposed:

(i) $f(t) \in C((0, \infty), E)$ is absolutely integrable at zero and assumes values in $D(A)$, while $Af(t) \in C((0, \infty), E)$ and is also absolutely integrable at zero;

(ii) $D^{\alpha}f(t) \in C((0, \infty), E)$ and is absolutely integrable at zero.

In particular, if Condition 2 holds and the operator A is bounded, then Condition 1 also holds and the solution of problem (4), (2) is of the form (see [7])

$$u(t) = t^{\alpha-1}E_{\alpha,\alpha}(t^{\alpha}A)u_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^{\alpha}A)f(s)\,ds, \qquad u_0 \in E, \tag{6}$$

where

$$E_{\alpha,\mu}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \mu)}$$

is the Mittag–Leffler function.

In the case of an unbounded operator A satisfying Condition 1, the solution of problem (4), (2) for $f(t) \equiv 0$ is of the form (see [6])

$$u(t) \equiv T_{\alpha}(t)u_0 = \frac{1}{2\pi i} D^{1-\alpha} \int_{\sigma-i\infty}^{\sigma+i\infty} \lambda^{\alpha-1}e^{\lambda t}R(\lambda^{\alpha})u_0\,d\lambda, \tag{7}$$

where $u_0 \in D(A)$, $\sigma > \max(0, \omega)$, while, in the general case (see [7]),

$$u(t) = T_{\alpha}(t)u_0 + \int_0^t T_{\alpha}(t-s)f(s)\,ds;$$

here the function $f(t)$ must satisfy Condition 2.

Theorem 1. *Suppose that A is a linear bounded operator in E and $u_0, u_1 \in E$. For problem (1)–(3) to have a unique solution, it is necessary and sufficient that the condition*

$$E_{\alpha,k+\alpha+\beta}(z) \neq 0, \qquad z \in \sigma(A) \tag{8}$$

hold on the spectrum $\sigma(A)$ of the operator A.

Proof. In view of (6), problem (1), (2) can be reduced to that of finding a function $u(t)$ and an element $p \in E$ such that the following relation holds:

$$u(t) = t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)s^{k-1}p\,ds. \tag{9}$$

Relation (9) and the boundary condition (3) imply the equation

$$\lim_{t\to 1} I^\beta \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)s^{k-1}p\,ds = u_1 - \lim_{t\to 1} I^\beta(t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha A)u_0),$$

which serves to determine the unknown element p, or, using the semigroup property of the operation of fractional integration and calculating the fractional integrals, we obtain the following equation in operator form:

$$B_0 p = q_0, \tag{10}$$

where

$$B_0 p = \lim_{t\to 1} I^{k+\beta}(t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha A)p) = E_{\alpha,k+\alpha+\beta}(A)p, \qquad q_0 = \frac{1}{\Gamma(k)}(u_1 - E_{\alpha,\alpha+\beta}(A)u_0). \tag{11}$$

Thus, a necessary and sufficient condition for the unique solvability of problem (1)–(3) with bounded operator A and arbitrary values $u_0, u_1 \in E$ is the solvability of Eq. (10) for any $q \in E$, i.e., the absence of points $\lambda = 0$ in the spectrum $\sigma(B_0)$ of the operator B_0.

It follows from (11) that the operator B_0 is an analytic function of the operator A. By the theorem on the mapping of the spectrum of a bounded operator, we have

$$\sigma(B_0) = \sigma(E_{\alpha,k+\alpha+\beta}(A)) = E_{\alpha,k+\alpha+\beta}(\sigma(A)).$$

Therefore, the value of $\lambda = 0$ is not a point of the spectrum of the operator B_0 only if the function $E_{\alpha,k+\alpha+\beta}(z)$. does not vanish on the spectrum of the operator A. The theorem is proved. □

Corollary. *Under the assumptions of Theorem 1, the solution $(u(t), p)$ is linear and, therefore, depends continuously on the given limiting values $u_0, u_1 \in E$.*

It follows from Theorem 1 that the location of zeros of the function $E_{\alpha,k+\alpha+\beta}(z)$ determines whether problem (1)–(3) with bounded operator A is uniquely solvable. As indicated in [2], for $\alpha = k = 1$, $\beta = 0$ and even for $\alpha = 1$, $k + \beta = 1$ (in these cases, the zeros of the Mittag–Leffler function $E_{1,2}(z)$ can be written out explicitly) Condition (8) for an unbounded operator A is no longer a sufficient condition for unique solvability, although the location of zeros also plays an important role. Therefore, we present the required results from [10] concerning their location. In Theorem 1 [10], it was established that, for $\alpha \in (0,1)$, $k + \alpha + \beta > 0$, and an appropriate numbering, all sufficiently large (in absolute value) zeros μ_n, $n \in \mathbb{Z} \setminus \{0\}$, of the function $E_{\alpha,k+\alpha+\beta}(z)$ are simple and the following asymptotics as $n \to \pm\infty$ holds:

$$\mu_n^{1/\alpha} = 2\pi n i + (k+\beta-1)\left(\ln 2\pi|n| + \frac{\pi i}{2}\operatorname{sign} n\right) + \ln\frac{\alpha}{\Gamma(k+\beta)} + o(1), \qquad n \to \pm\infty. \tag{12}$$

Further, we shall establish a necessary condition for the uniqueness of the solution of the inverse problem (1)–(3) with unbounded operator A.

Theorem 2. *Suppose that A is a linear closed operator in E. Suppose that the inverse problem (1)–(3) has a solution $(u(t), p)$. For this solution to be unique, it is necessary that no zero μ_n of the entire function $E_{\alpha,k+\alpha+\beta}(z)$ be an eigenvalue of the operator A.*

Proof. Assume the converse; suppose that some zero μ_n from a countable set of zeros of the function $E_{\alpha,k+\alpha+\beta}(z)$ is an eigenvalue of the operator A with eigenvector $h_n \neq 0$.

Consider the function $w(t) = \psi(t)h_n$ and choose a scalar function $\psi(t)$ so that the function $w(t)$ satisfies Eq. (1) for $p = h_n$ and the zero initial condition (2). It is easy to verify that the function $\psi(t)$ must be a solution of the following Cauchy problem:

$$D^\alpha \psi(t) = \mu_n \psi(t) + t^{k-1}, \tag{13}$$

$$\lim_{t\to 0} I^{1-\alpha}\psi(t) = 0. \tag{14}$$

Problem (13), (14) has a unique solution (see [11, p. 602], which can be expressed as

$$\psi(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) s^{k-1}\, ds.$$

Since μ_n is a zero of the function $E_{1,k+\alpha+\beta}(z)$, just as for (10), (11), we obtain

$$\lim_{t\to 1} I^\beta \psi(t) = \Gamma(k) E_{\alpha,k+\alpha+\beta}(\mu_n) = 0.$$

Thus, the function $w(t) = \psi(t)h_n$ satisfies Eq. (1) for $p = h_n$ and the zero conditions (2) and (3), which contradicts the assumption on the uniqueness of the solution, because the pair $(u(t) + w(t), p + h_n)$ is also a solution of problem (1)–(3). The theorem is proved. □

To establish the unique solvability of problem (1)–(3) with unbounded operator A satisfying Condition 1, we reduce this problem to the operator equation by taking (7) into account:

$$Bp = q, \tag{15}$$

where

$$Bp = \frac{1}{\Gamma(k)} \lim_{t\to 1} I^\beta \int_0^t s^{k-1} T_\alpha(t-s)p\, ds = \lim_{t\to 1} I^{k+\beta} T_\alpha(t)p, \qquad B\colon E \to E, \tag{16}$$

$$q = \frac{1}{\Gamma(k)}\Big(u_1 - \lim_{t\to 1} I^\beta T_\alpha(t)u_0\Big), \qquad q \in D(A); \tag{17}$$

here the function $f(t) = t^{k-1}p$ must satisfy Condition 2.

Thus, the unique solvability of problem (1)–(3) can be reduced to the problem of the existence of an operator given on a subset of the Banach space E and inverse to the bounded operator B given by relation (16). To amplify this point, let us obtain a more convenient representation for the operator B with the help of the resolvent

$$R(\lambda^\alpha) = (\lambda^\alpha I - A)^{-1},$$

simultaneously restricting the domain of definition of the operator B to a dense set $D(A)$ in E.

Theorem 3. *Suppose that the operator A satisfies Condition* 1 *and $k > \alpha$. Then, for any $p \in D(A)$, the following expression is valid*:

$$Bp = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} z^{\alpha-1} E_{1,k+\alpha+\beta}(z) R(z^\alpha)p\, dz, \tag{18}$$

where $\lambda \in \rho(A)$, $\rho(A)$ is the resolvent set of the operator A, $\operatorname{Re}\lambda > \sigma > \omega$.

Proof. First, suppose that $p \in D(A^2)$, and hence,

$$p = R^2(\lambda)p_0, \qquad p_0 \in E.$$

Then, using the semigroup property of fractional integration and Hilbert's identity, after integrating by parts, from (16), (7) we obtain

$$Bp = \frac{1}{\Gamma(k+\beta)}\int_0^1 (1-s)^{k+\beta-1}\, ds \cdot \frac{1}{2\pi i} D^{1-\alpha} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp(zs)}{z^{\alpha-1}} R(z^\alpha) R^2(\lambda) p_0\, dz$$

$$= \frac{1}{\Gamma(k+\alpha+\beta)} \lim_{t\to 1} \frac{d}{dt} \int_0^t (t-s)^{k+\alpha+\beta-1} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{\alpha-1} \exp(zs)$$
$$\times \left(\frac{R(z^\alpha)p_0}{(\lambda - z^\alpha)^2} - \frac{R^2(\lambda)p_0}{\lambda - z^\alpha} - \frac{R(\lambda)p_0}{(\lambda - z^\alpha)^2} \right) dz\, ds$$
$$= \frac{1}{\Gamma(k+\alpha+\beta)} \lim_{t\to 1} \frac{d}{dt} \int_0^t (t-s)^{k+\alpha+\beta-1}\, ds \cdot \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp(zs) R(z^\alpha) p_0}{z^{1-\alpha}(\lambda - z^\alpha)^2}\, dz; \qquad (19)$$

here the integrals along the line $\operatorname{Re} z = \sigma$ of functions of the form

$$\frac{z^{\alpha-1}\exp(zs)R^j(\lambda)p_0}{(\lambda - z^\alpha)^{3-j}}, \qquad j = 1, 2$$

vanish because of Jordan's lemma.

The last integral is absolutely convergent; therefore, by changing the order of integration and using relation 1.17 from [12]

$$z^\mu E_{1,\mu+1}(\lambda z) = \frac{1}{\Gamma(\mu)} \int_0^z e^{\lambda z}(z-s)^{\mu-1}\, ds, \qquad \mu > 0, \qquad (20)$$

from (19) we obtain the representation

$$Bp = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\alpha+\beta}(z) R(z^\alpha) p_0}{z^{1-\alpha}(\lambda - z^\alpha)^2}\, dz$$
$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\alpha+\beta}(z) R(z^\alpha)((\lambda - z^\alpha)I + (z^\alpha I - A))(\lambda I - A)p}{z^{1-\alpha}(\lambda - z^\alpha)^2}\, dz$$
$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\alpha+\beta}(z)}{z^{1-\alpha}(\lambda - z^\alpha)} R(z^\alpha)(\lambda I - A)p\, dz, \qquad p \in D(A^2). \qquad (21)$$

We let $p_1 = (\lambda I - A)p$; then $p_1 \in D(A)$ and $p = R(\lambda)p_1$. Therefore, relation (20) takes the form

$$BR(\lambda)p_1 = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\alpha+\beta}(z)}{z^{1-\alpha}(\lambda - z^\alpha)} R(z^\alpha)p_1\, dz, \qquad p_1 \in D(A). \qquad (22)$$

The left-hand and right-hand sides of relation (22) are bounded operators coinciding on $D(A)$. Since $D(A)$ is dense in E, relation (21) holds for all $p_1 \in E$. But, in that case, $p = R(\lambda)p_1 \in D(A)$ and, for such p, the following expression is valid:

$$Bp = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\alpha+\beta}(z)}{z^{1-\alpha}(\lambda - z^\alpha)} R(z^\alpha)((\lambda - z^\alpha)I + (z^\alpha I - A))p\, dz$$
$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{\alpha-1} E_{1,k+\alpha+\beta}(z) R(z^\alpha)p\, dz.$$

The theorem is proved. □

Remark 1. In Theorem 3, the constraint $k > \alpha$ is imposed in order that the function $f(t) = t^{k-1}p$ satisfy Condition 2, (ii). We can replace by this constraint the requirement of the smoothness of the element p, namely, $p \in D(A)$. Then the function $f(t) = t^{k-1}p$ will satisfy Condition 2, (i).

Let us now turn to establishing sufficient conditions for the unique solvability of problem (1)–(3). It follows from Theorem 2 that we must require that no zero μ_n of the function $E_{\alpha,k+\alpha+\beta}(z)$ be an eigenvalue of the operator A. Moreover, in order to establish solvability, we require that all zeros belong to the resolvent set $\rho(A)$. Taking their asymptotics (12) into account, we note that, for $k + \beta > 1$, the condition will be imposed only on a finite number of zeros μ_n, $n = 1, 2, \dots, n_0$, with $\operatorname{Re} \mu_n^{1/\alpha} < \sigma$, because the others automatically belong to $\rho(A)$. In the case of $k + \beta \le 1$ zeros with $\operatorname{Re} \mu_n^{1/\alpha} < \sigma$, the set of zeros is countable.

Theorem 4. *Suppose that the operator A satisfies Condition 1, $k > \alpha$, $k + \beta > 1$, $\sigma > \omega$, and $u_0, u_1 \in D(A^3)$. If each zero μ_n, $n = 1, 2, \ldots, n_0$, of the function $E_{\alpha,k+\alpha+\beta}(z)$ with $\operatorname{Re}\mu_n^{1/\alpha} < \sigma$ belongs to $\rho(A)$, then problem (1)–(3) has a unique solution.*

Proof. We have already noted that the existence of a unique solution of problem (1)–(3) (or of the operator equation (15)) can be reduced to the proof of the existence of an operator inverse to the bounded operator B defined by relation (16) (or (18)). For $u_0, u_1 \in D(A^3)$, in view of the invariance of $D(A)$ with respect to $T_\alpha(t)$, the right-hand side of Eq. (15) belongs to $D(A^3)$. Let us show that the operator B has an inverse operator $B^{-1}\colon D(A^3) \to E$.

Since each zero $\mu_n^{1/\alpha}$ of the function $E_{\alpha,k+\alpha+\beta}(z^\alpha)$ with $\operatorname{Re}\mu_n^{1/\alpha} < \sigma$ belongs to $\rho(A)$, then it belongs to $\rho(A)$ together with some disk neighborhood Ω_n. Suppose that Γ is the contour in the complex plane consisting of the line $\operatorname{Re} z = \sigma > \omega$ and the boundaries γ_n of the disk neighborhoods Ω_n, i.e.,

$$\Gamma = \{\operatorname{Re} z = \sigma\} \bigcup_{\operatorname{Re}\mu_n^{1/\alpha} < \sigma} \gamma_n.$$

We take $\lambda \in \rho(A)$, $\operatorname{Re}\lambda > \sigma > \omega$ and consider the bounded operator

$$\Upsilon q = \frac{\alpha}{2\pi i} \int_\Gamma \frac{z^{\alpha-1} R(z^\alpha) q \, dz}{E_{\alpha,k+\alpha+\beta}(z^\alpha)(z^\alpha - \lambda)^3}, \qquad \Upsilon\colon E \to E. \tag{23}$$

Note that the integral in (23) is absolutely convergent by the choice of the contour Γ, estimate (5), the asymptotics (12), and the well-known (see [12, p. 134]) asymptotic behavior of the Mittag–Leffler function for $0 < \alpha < 2$ and $|z| \to \infty$,

$$E_{\alpha,\mu}(z) = \frac{1}{\alpha} z^{(1-\mu)/\alpha} \exp(z^{1/\alpha}) - \sum_{j=1}^{n} \frac{1}{\Gamma(\mu - \alpha j) z^j} + O\left(\frac{1}{|z|^{n+1}}\right), \tag{24}$$

$$|\arg z| \le \nu\pi, \quad \nu \in \left(\frac{\alpha}{2}, \alpha\right),$$

$$E_{\alpha,\mu}(z) = -\sum_{j=1}^{n} \frac{1}{\Gamma(\mu - \alpha j) z^j} + O\left(\frac{1}{|z|^{n+1}}\right), \qquad \nu\pi \le |\arg z| \le \pi. \tag{25}$$

Suppose that $q \in D(A)$, $\sigma < \sigma_1 < \operatorname{Re}\lambda$. Then, substituting (18) into (23) and using Hilbert's identity, we obtain

$$\begin{aligned} \Upsilon B q &= \frac{\alpha}{2\pi i} \int_\Gamma \frac{z^{\alpha-1} R(z^\alpha) \, dz}{E_{\alpha,k+\alpha+\beta}(z^\alpha)(z^\alpha - \lambda)^3} \cdot \frac{1}{2\pi i} \int_{\sigma_1 - \infty}^{\sigma_1 + \infty} \xi^{\alpha-1} E_{1,k+\alpha+\beta}(\xi) R(\xi^\alpha) q \, d\xi \\ &= \frac{\alpha}{(2\pi i)^2} \int_\Gamma \int_{\sigma_1 - \infty}^{\sigma_1 + \infty} \frac{z^{\alpha-1} \xi^{\alpha-1} E_{1,k+\alpha+\beta}(\xi)}{E_{\alpha,k+\alpha+\beta}(z^\alpha)(z^\alpha - \lambda)^3} \cdot \frac{R(z^\alpha) q - R(\xi^\alpha) q}{\xi^\alpha - z^\alpha} \, d\xi \, dz. \end{aligned} \tag{26}$$

The integral in (26) is absolutely convergent; therefore, changing the order of integration, we find that

$$\begin{aligned} \Upsilon B q = {} & \frac{\alpha}{(2\pi i)^2} \int_\Gamma \frac{z^{\alpha-1} R(z^\alpha) q \, dz}{E_{\alpha,k+\alpha+\beta}(z^\alpha)(z^\alpha - \lambda)^3} \cdot \int_{\sigma_1 - \infty}^{\sigma_1 + \infty} \frac{\xi^{\alpha-1} E_{1,k+\alpha+\beta}(\xi) \, d\xi}{\xi^\alpha - z^\alpha} - \\ & - \frac{\alpha}{(2\pi i)^2} \int_{\sigma_1 - \infty}^{\sigma_1 + \infty} \xi^{\alpha-1} E_{1,k+\alpha+\beta}(\xi) R(\xi^\alpha) q \, d\xi \int_\Gamma \frac{z^{\alpha-1} \, dz}{E_{\alpha,k+\alpha+\beta}(z^\alpha)(z^\alpha - \lambda)^3 (\xi^\alpha - z^\alpha)}. \end{aligned} \tag{27}$$

The inner integral (after the replacement $\eta = z^\alpha$) in the second summand (27) is zero owing to the choice of the contour Γ and Jordan's lemma, while, to calculate of the integrals in the first summand, we use relation (20), the formula (see [11, p. 33])

$$\int_0^{+\infty} \exp(-t\xi) E_{\alpha,1}(t^\alpha z^\alpha) \, dt = \frac{\xi^{\alpha-1}}{\xi^\alpha - z^\alpha}, \qquad \left|\frac{z^\alpha}{\xi^\alpha}\right| < 1,$$

the equality

$$I^{\nu} E_{\alpha,1}(t^{\alpha} z^{\alpha}) = t^{\nu} E_{\alpha,\nu+1}(t^{\alpha} z^{\alpha}), \qquad \nu > 0,$$

and Jordan's lemma. Thus, for $q \in D(A)$ the following relation holds:

$$\begin{aligned}\Upsilon Bq &= \frac{\alpha}{(2\pi i)^2} \int_{\Gamma} \frac{z^{\alpha-1} R(z^{\alpha}) q\, dz}{E_{\alpha,k+\alpha+\beta}(z^{\alpha})(z^{\alpha}-\lambda)^3} \cdot \lim_{t\to 1} I^{k+\alpha+\beta-1} \int_{\sigma_2-\infty}^{\sigma_2+\infty} \frac{\xi^{\alpha-1} \exp(\xi t)\, d\xi}{\xi^{\alpha} - z^{\alpha}} \\ &= \frac{\alpha}{2\pi i} \int_{\Gamma} \frac{z^{\alpha-1} R(z^{\alpha}) q\, dz}{E_{\alpha,k+\alpha+\beta}(z^{\alpha})(z^{\alpha}-\lambda)^3} \cdot \lim_{t\to 1} I^{k+\alpha+\beta-1} E_{\alpha,1}(t^{\alpha} z^{\alpha}) \\ &= \frac{\alpha}{2\pi i} \int_{\Gamma} \frac{z^{\alpha-1} R(z^{\alpha}) q\, dz}{(z^{\alpha}-\lambda)^3} = \frac{1}{2\pi i} \int_{(\Gamma)_{\alpha}} \frac{R(\eta) q\, d\eta}{(\eta-\lambda)^3} = R^3(\lambda) q,\end{aligned}$$

where $(\Gamma)_{\alpha}$ is the contour obtained from the contour Γ after the replacement

$$\eta = z^{\alpha}\colon z \in \Gamma,\ \eta \in (\Gamma)_{\alpha}.$$

The commuting operators Υ, B, $R(\lambda)$ are bounded and the domain of definition $D(A)$ is dense in E; therefore, we also have the equality

$$\Upsilon Bq = R^3(\lambda) q \qquad \text{for} \quad q \in E, \quad \Upsilon B\colon E \to D(A^3).$$

This implies that the operator

$$B^{-1} q = (\lambda I - A)^3 \Upsilon q, \qquad \text{for} \quad q \in D(A^3)$$

is inverse to B. Indeed,

$$\begin{aligned} BB^{-1} q = B(\lambda I - A)^3 \Upsilon q = R^3(\lambda)(\lambda I - A)^3 q = q, \qquad q \in D(A^3), \\ B^{-1} Bq = (\lambda I - A)^3 \Upsilon Bq = q, \qquad q \in E.\end{aligned}$$

As to the solution of problem (1)–(3), the element p belonging to E is of the form

$$p = (\lambda I - A)^3 \Upsilon q,$$

where the element $q \in D(A^3)$ is defined by relation (17), the operator Υ is defined by relation (23),

$$\lambda \in \rho(A), \qquad \operatorname{Re} \lambda > \sigma > \omega,$$

while, for the function $u(t)$, the following expression is valid:

$$u(t) = T_{\alpha}(t) u_0 + \int_0^t T_{\alpha}(t-s) p\, ds.$$

The theorem is proved. □

In the case $k + \beta \le 1$, as already noted, the set of zeros μ_n with $\operatorname{Re} \mu_n^{1/\alpha} < \sigma$ of the function $E_{\alpha,k+\alpha+\beta}(z)$ is countable; therefore, we require that the following condition be satisfied.

Condition 3. Each zero μ_n, $n \in \mathbb{Z} \setminus \{0\}$, of the function $E_{\alpha,k+\alpha+\beta}(z)$ with $\operatorname{Re} \mu_n^{1/\alpha} < \sigma$ belongs to $\rho(A)$, and there exists a $d > 0$ such that

$$\sup_{\operatorname{Re} \mu_n^{1/\alpha} < \sigma} \left\| \frac{R(\mu_n)}{\mu_n^k} \right\| \le d.$$

Theorem 5. *Let Conditions* 1 *and* 3 *be satisfied,* $k > \alpha$, $k + \beta \le 1$, *and* $u_0, u_1 \in D(A^3)$. *Then problem* (1)–(3) *has a unique solution.*

Proof. Just as in Theorem 4, we consider the operator Υ defined by relation (23). In the case under consideration, the contour Γ will contain a countable set of circles γ_n and, in order to prove the absolute convergence of the integral, we consider the integral over the circles γ_n in relation (23). Suppose that $(\gamma_n)_\alpha$ is the contour obtained from γ_n by replacing

$$\xi = z^\alpha\colon z \in \gamma_n, \xi \in (\gamma_n)_\alpha.$$

Then

$$\frac{\alpha}{2\pi i}\int_{\bigcup \gamma_n} \frac{z^{\alpha-1}R(z^\alpha)q\,dz}{E_{\alpha,k+\alpha+\beta}(z^\alpha)(z^\alpha-\lambda)^3} = \frac{1}{2\pi i}\int_{\bigcup(\gamma_n)_\alpha} \frac{R(\xi)q\,dz}{E_{\alpha,k+\alpha+\beta}(\xi)(\xi-\lambda)^3}$$
$$= \sum_{n=-\infty,\,n\neq 0}^{+\infty} \frac{R(\mu_n)q}{E'_{\alpha,k+\alpha+\beta}(\mu_n)(\lambda-\mu_n)^3}\,, \tag{28}$$

Using the relation (see [12, formula (1.5), p. 118])

$$E'_{\alpha,k+\alpha+\beta}(\mu_n) = \frac{1}{\alpha\mu_n}(E_{\alpha,k+\alpha+\beta-1}(\mu_n) - (k+\alpha+\beta-1)E_{\alpha,k+\alpha+\beta}(\mu_n))$$

and taking the asymptotics (24) of the Mittag–Leffler function and the asymptotics (12) into account, we obtain

$$E'_{\alpha,k+\alpha+\beta}(\mu_n) = \frac{1}{\alpha\mu_n}\biggl(\frac{\mu_n^{(2-k-\beta)/\alpha-1}(2\pi|n|)^{k+\beta-1}\exp(i\operatorname{Im}\mu_n)}{\Gamma(k+\beta)} - \frac{1}{\Gamma(k+\beta-1)\mu_n}$$
$$- \frac{(k+\alpha+\beta-1)\mu_n^{(2-k-\beta)/\alpha-2}(2\pi|n|)^{k+\beta-1}\exp(i\operatorname{Im}\mu_n)}{\Gamma(k+\beta)}$$
$$- \frac{k+\alpha+\beta-1}{\Gamma(k+\beta)\mu_n} + O\biggl(\frac{1}{|\mu_n|^2}\biggr)\biggr).$$

Thus, we have

$$|E'_{\alpha,k+\alpha+\beta}(\mu_n)| = \frac{1}{|\mu_n|^{2-1/\alpha}}\biggl(\frac{1}{\alpha\Gamma(k+\beta)} + O\biggl(\frac{1}{|\mu_n|}\biggr)\biggr). \tag{29}$$

In view of relation (29), Condition 3, and the asymptotics (12), the series (28) and hence also the integral over $\bigcup\gamma_n$ are absolutely convergent.

The convergence of the integral along the line $\operatorname{Re} z = \sigma$ in relation (23) follows, obviously, from Condition 3 and the asymptotics (25).

The subsequent proof follows along the same lines as the proof of Theorem 4, and so it is omitted. The proof of Theorem 5 is thus complete. □

Remark 2. Taking Remark 1 into account in Theorems 4 and 5, we can replace the constraint $k > \alpha$ by the requirement of additional smoothness of the data of problem (1)–(3); namely, for $0 < k \le \alpha$, the elements u_0 and u_1 must belong to $D(A^4)$. In addition, we have $p \in D(A)$, and the solution $(u(t), p)$ of problem (1)–(3) is defined by the same formulas as in Theorems 4 and 5.

Thus, Theorems 4 and 5 imply that, for $k + \beta > 1$, the inverse problem (1)–(3) has a solution under less severe constraints on the operator A.

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