ON PERTURBATIONS OF ABSTRACT FRACTIONAL DIFFERENTIAL EQUATIONS BY NONLINEAR OPERATORS

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Abstract. We prove the unique solvability of a Cauchy-type problem for an abstract parabolic equation containing fractional derivatives and a nonlinear perturbation term. The result is applied to establish the solvability of the inverse coefficient problem for a fractional-order equation.

CONTENTS

Introduction. Setting the Problem

We consider the following Cauchy-type problem in a Banach space E:

$$D^\alpha u(t) = Au(t) + F(t, B(t)u(t)), \quad t > 0, \tag{1}$$

$$\lim_{t\to 0} D^{\alpha-1}u(t) = u_0, \tag{2}$$

where $0 < \alpha < 1$,

$$D^{\alpha-1}u(t) = I^{1-\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t (t-s)^{-\alpha}u(s)\,ds$$

is the left-side fractional Riemann–Liouville integral of order $1-\alpha$ (for $\alpha = 1$, we assume that $I^{1-\alpha}$ is the identity operator), $D^\alpha u(t) = \dfrac{d}{dt}I^{1-\alpha}u(t)$ is the left-side fractional Riemann–Liouville derivative of order α, $\Gamma(\cdot)$ is the gamma-function, A is a linear closed densely defined operator, $B(t)$ is a linear closed densely defined operator depending on t (no assumptions about the boundedness of $B(t)$ are imposed), and $F(t, w)$ is a nonlinear operator acting in E for any $t \geq 0$; the latter operator is treated as a perturbation of the operator A.

The results presented below are related to the perturbation theory of generators of semigroups (see [8, Chap. 9]). We add to problem (1), (2) a term with a nonlinear operator subordinated in a way to the operator A and investigate how this affects the solvability of the problem. Sufficient conditions providing the solvability of the problem with the perturbed operator A are presented.

In [5], solvability results for equations with fractional Riemann–Liouville derivatives perturbed by a linear closed operator $B(t)$ are obtained. The study of problems of that kind is motivated by numerous applications of fractional-order differential equations in physics and mathematical modelling (see, e.g., [18, Chap. 8], [13, Chap. 5], and [9, Chap. 8]).

Apart from problem (1), (2), consider the following problem without perturbations (assuming that $\beta \geq \alpha$):

$$D^{\beta}u(t) = Au(t), \quad t > 0, \tag{3}$$

$$\lim_{t\to 0} D^{\beta-1}u(t) = u_0. \tag{4}$$

Definition 1. A function $u(t)$ is called a *solution of problem* (3), (4) if it is continuous for $t > 0$, takes values in $D(A)$ (here $D(A)$ is the domain of the operator A), $I^{1-\beta}u(t)$ is continuously differentiable for $t > 0$, and $u(t)$ satisfies (3) and (4).

Definition 2. We say that problem (3), (4) is *uniformly well posed* if there exist an operator-valued function $T_\beta(t)$ defined on E and commuting with A, a positive M_1, and a real ω such that, for any $u_0 \in D(A)$, the function $T_\beta(t)u_0$ is the unique solution of problem (3), (4) and

$$\|T_\beta(t)\| \leq M_1 t^{\beta-1}e^{\omega t}. \tag{5}$$

According to Definition 2, problem (3), (4) is uniformly well posed if its solution exists, is unique, and continuously depends on the initial data uniformly with respect to t from any compact set of $(0,\infty)$. The latter property follows from (5). Apart from those standard requirements, Definition 2 includes additional information about the behavior of the solution as $t \to 0$ and $t \to \infty$ (see inequality (5)).

Condition 1. *There exists $\beta \in [\alpha, 1]$ such that problem* (3), (4) *is uniformly well posed and u_0 belongs to $D(A)$.*

In [4, 6, 10], the uniform well-posedness of problem (3), (4) is studied for $0 < \beta < 1$. If $\beta = 1$, then the uniform well-posedness of the Cauchy problem requires that the operator A be a generator of a C_0-semigroup.

Condition 2. (i) *The domain of the operator $B(t)$, denoted as D, does not depend on t and $D(A) \subset D$.*

(ii) *Let $x \in D$. Then either the function $w(t) = B(t)x$ belongs to $C((0,\infty), E)$, is absolutely integrable at the origin, and takes values in $D(A)$ and the function $Aw(t)$ belongs to $C((0,\infty), E)$ and is absolutely integrable at the origin or the function $I^{1-\alpha}w(t) = I^{1-\alpha}B(t)x$ is continuous for $t \geq 0$, continuously differentiable for $t > 0$, and $D^\alpha w(t)$ is absolutely integrable at the origin.*

(iii) *For any $x \in E$, there exist $M_2 > 0$, $\gamma \in (0, 1)$, and $\omega \in \mathbb{R}$ such that $T_\beta(\tau)x \in D$ (the smoothing effect) and*

$$\|B(t)T_\beta(\tau)x\| \leq M_2\tau^{-\gamma}e^{\omega\tau}\|x\|, \quad t, \tau \in (0,\infty). \tag{6}$$

Note that if the operator $-A$ is strongly positive in the sense of [11], i.e., if

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{M_3}{1+|\lambda|}, \quad \operatorname{Re}\lambda \geq 0,\ M_3 > 0,$$

then we can assign $\beta = 1$ in Condition 1. In this case, $\omega = 0$ and inequality (6) means that the operator $B(t)$ is subordinated to the fractional power $(-A)^\gamma$ (see [11, p. 298]).

If the operator $B(t)$ is bounded and the operator A satisfies Condition 1, then inequality (6) is valid for $\gamma = 1 - \beta$.

The operators A and $B(t)$ are not assumed to commute.

Condition 3. (i) *The function F acts from $(0,\infty) \times E$ to E; if a function $w(t) = B(t)x$, $x \in D$, satisfies item* (ii) *of Condition* 2*, so does $w_1(t) = F(t, w(t))$.*

(ii) *The following inequality is valid for $w = 0$: $\|F(t, 0)\| \leq C_0(1 + t^{\mu-1})$, mu > 0, $C_0 > 0$.*

(iii) *The operator $F(t,w)$ satisfies the following Lipschitz condition uniformly with respect to $t>0$:*

$$\|F(t,w_2)-F(t,w_1)\|\le L\|w_2-w_1\| \qquad \text{for all } w_1,w_2\in E.$$

Condition 4. *The Banach space E possesses the Radon–Nikodým property* (*see* [1, p.15]), *i.e., any absolutely continuous function $F:\mathbb{R}_+\longrightarrow E$ is differentiable almost everywhere.*

For example, reflexive Banach spaces possess that property (see [1, Corollary 1.2.7]), while the spaces $L_1(a,b)$, $C[a,b]$, and the space c_0 of sequences converging to zero do not (see [1, Example 1.2.8 and Propositions 1.2.9 and 1.2.10]).

We will show below that Conditions 1–4 guarantee the unique solvability of problem (1), (2).

The following function (see [7, p. 357]) is needed for the proof:

$$f_{\tau,\nu}(t)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\exp\left(tz-\tau z^{\nu}\right)dz, & t\ge 0,\\ 0, & t<0,\end{cases} \tag{7}$$

where $\sigma>0,\ \tau>0,\ 0<\nu<1$, and the branch of the function z^ν is chosen to satisfy the inequality $\operatorname{Re} z^\nu>0$ for $\operatorname{Re} z>0$. This branch is a one-valued function on the complex z-plane cut along the negative part of the real axis. The convergence of the integral in (7) is guaranteed by the factor $\exp(-\tau z^\nu)$.

Below, we present certain properties of the function $f_{\tau,\nu}(t)$ (see also [7, p. 358–361, Propositions 1–3]).

Consider the integral defining the function $f_{\tau,\nu}(t)$ and replace the line of integration $\operatorname{Re} z=\sigma>0$ by the contour consisting of the rays $z=r\exp(-i\theta)$ and $z=r\exp(i\theta)$, where $0<r<\infty$ and $\pi/2\le\theta\le\pi$. This yields the following representation of the function $f_{\tau,\nu}(t)$ for $t>0$:

$$f_{\tau,\nu}(t)=\frac{1}{\pi}\int\limits_0^\infty \exp\left(tr\cos\theta-\tau r^\nu\cos\nu\theta\right)\sin\left(tr\sin\theta-\tau r^\nu\sin\nu\theta+\theta\right)dr. \tag{8}$$

The function $f_{\tau,\nu}(t)$ is nonnegative, and the following relations are valid:

$$\int\limits_0^\infty f_{\tau,\nu}(t)\,dt\equiv 1 \tag{9}$$

$$\exp\left(-\tau\lambda^\nu\right)=\int\limits_0^\infty \exp(-\lambda t)\,f_{\tau,\nu}(t)\,dt, \qquad \tau>0,\ \lambda>0,\ 0<\nu<1. \tag{10}$$

We also note that the function $f_{\tau,\nu}(t)$ can be expressed via the Wright function (see [9, p. 54]) for $t>0$:

$$f_{\tau,\nu}(t)=t^{-1}\phi\left(-\nu,0;-\tau t^{-\nu}\right), \qquad \phi(a,b;z)=\sum_{k=0}^\infty\frac{z^k}{k!\,\Gamma(ak+b)}.$$

Another representation via the more general Wright-type function (see [16, Ch. 1]) is valid as well:

$$f_{\tau,\nu}(t)=t^{-1}e^{1,0}_{1,\nu}\left(-\tau t^{-\nu}\right), \qquad e^{\mu,\delta}_{\alpha,\beta}(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(\alpha k+\mu)\,\Gamma(\delta-\beta k)}, \tag{11}$$

where $\beta<1,\ \delta+\beta>0,\ \max\{0;\beta\}<\alpha<2,\ \alpha+\beta<2$, and $\mu,z\in C$.

1. Cauchy-Type Problems for Fractional-Order Equations: Inhomogeneous Equations

The following theorem establishes conditions under which the uniform well-posedness of problem (3), (4) implies the uniform well-posedness of the corresponding Cauchy-type problem for the equation of order α, where $0 < \alpha < \beta \le 1$.

Theorem 1.1. *Let $\alpha < \beta \le 1$, Conditions 1 and 4 be satisfied, and $\omega = 0$ in inequality (5). Then the problem*

$$D^{\alpha}u(t) = Au(t), \quad t > 0, \tag{1.1}$$

$$\lim_{t\to 0} D^{\alpha-1}u(t) = u_0 \tag{1.2}$$

is uniformly well posed and its resolving operator is of the form

$$T_{\alpha}(t)u_0 = \int_0^{\infty} f_{\tau,\nu}(t)T_{\beta}(\tau)u_0\, d\tau, \tag{1.3}$$

where $\nu = \alpha/\beta$ and the function $f_{\tau,\nu}(t)$ is defined by relation (7).

Proof. The following is proved in [4]. If problem (3), (4) is uniformly well posed and $\omega = 0$ in inequality (5), then λ^{β} belongs to the resolvent set $\rho(A)$ of the operator A for $\operatorname{Re}\lambda > 0$, the resolvent $R(\lambda^{\beta}) = (\lambda^{\beta}I - A)^{-1}$ is representable in the form

$$R(\lambda^{\beta})x = \int_0^{+\infty} \exp(-\lambda t)T_{\beta}(t)x\, dt \tag{1.4}$$

for any $x \in E$, and the following inequalities are valid for any nonnegative integer n:

$$\left\| \frac{d^n R\left(\lambda^{\beta}\right)}{d\lambda^n} \right\| \le \frac{M\Gamma\left(n+\beta\right)}{\left(\mathrm{Re}\ \lambda\right)^{n+\beta}}, \quad \mathrm{Re}\ \lambda > 0. \tag{1.5}$$

If the Banach space E possesses the Radon–Nikodým property, then the validity of inequalities (1.5) (even for real positive λ) is a sufficient condition for the uniform well-posedness of problem (3), (4). The resolving operator for this problem is of the form (see [4, formula (13)])

$$T_{\beta}(t)u_0 = D^{1-\beta}\frac{1}{2\pi i}\int_{\omega_0-i\infty}^{\omega_0+i\infty} \lambda^{\beta-1}\exp(\lambda t)R(\lambda^{\beta})u_0\, d\lambda, \qquad \omega_0 > 0. \tag{1.6}$$

Taking into account (1.4), (10), and (5), for $\nu = \alpha/\beta$, we have

$$R\left(\mu^{\alpha}\right)x = \int_0^{\infty} \exp\left(-\mu^{\nu}t\right)T_{\beta}(t)x\, dt = \int_0^{\infty} T_{\beta}(t)x\, dt \int_0^{\infty} \exp\left(-\tau\mu\right) f_{t,\nu}\left(\tau\right) d\tau.$$

In (8), we take $\theta \in [\pi/2, \pi]$ such that $\cos\theta < 0$ and $cos\nu\theta > 0$. To achieve that, we actually take it from the interval $(\pi/2, \min\{\pi/(2\nu); \pi\})$.

Hence, by virtue of (8), (9), and the theorem on the differentiability of integrals with respect to a parameter, the following inequalities are valid:

$$\left\| \frac{d^n R\left(\mu^{\alpha}\right)x}{d\mu^n} \right\| \le M_1\left\|x\right\| \int_0^{\infty} t^{\beta-1}\, dt \int_0^{\infty} \tau^n \exp\left(-\tau \mathrm{Re}\ \mu\right) d\tau \int_0^{\infty} \exp\left(\tau s\cos\theta - ts^{\nu}\cos\nu\theta\right) ds$$

$$= M_4\left\|x\right\| \int_0^{\infty} \tau^n \exp\left(-\tau \mathrm{Re}\ \mu\right) d\tau \int_0^{\infty} s^{-\alpha}\exp\left(\tau s\cos\theta\right) ds$$

$$= M_5 \|x\| \int_0^\infty \tau^{n-1+\alpha} \exp(-\tau \mathrm{Re}\ \mu)\, d\tau = \frac{M_6 \Gamma(n+\alpha) \|x\|}{(\mathrm{Re}\ \mu)^{n+\alpha}}.$$

This proves the uniform well-posedness of problem (1.1), (1.2).

Due to (1.6), (1.4), (7), and (11), the resolving operator for this problem is of the form

$$T_\alpha(t)\, u_0 = D^{1-\alpha} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \lambda^{\alpha-1} \exp(\lambda t)\, R(\lambda^\alpha)\, u_0\, d\lambda$$
$$= D^{1-\alpha} \int_0^\infty T_\beta(\tau) u_0\, d\tau \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \lambda^{\alpha-1} \exp(\lambda t - \lambda^\nu \tau)\, d\lambda$$
$$= D^{1-\alpha} \int_0^\infty t^{-\alpha} e^{1,1-\alpha}_{1,\nu}(-\tau t^{-\nu}) T_\beta(\tau) u_0\, d\tau. \quad (1.7)$$

The following relation (implied by [16, formula (1.1.13)]) for the Laplace transformation was used:

$$L\left[t^{-\alpha} e^{1,1-\alpha}_{1,\nu}\left(-\tau t^{-\nu}\right); \lambda\right] = \lambda^{\alpha-1} \exp\left(-\tau \lambda^\nu\right).$$

Now we use the following relation for the fractional derivatives of Wright-type functions (see [16, formula (1.2.12)]):

$$D^{1-\alpha}\left(t^{-\alpha} e^{1,1-\alpha}_{1,\nu}(-\tau t^{-\nu})\right) = t^{-1} e^{1,0}_{1,\nu}(-\tau t^{-\nu}) = f_{\tau,\nu}(t).$$

Combining it with the limit relation

$$\lim_{x\to+\infty} x e^{1,0}_{1,\nu}(-x) = -\lim_{x\to+\infty} e^{0,\nu}_{1,\nu}(-x) = 0$$

(see [16, formulas (1.2.3) and (1.2.6)] and note that this relation and the estimate in (5) guarantee the convergence of the integral in (1.3) for $\omega = 0$) and using (1.7), we obtain (1.3). This is the required representation. □

Remark 1.1. Consider the particular case where $\nu = \alpha/\beta = 1/2$. Then (see [7, p. 369, formula (32)])

$$f_{\tau,1/2}(t) = \frac{\tau}{2t\sqrt{\pi t}} \exp\left(-\frac{\tau^2}{4t}\right).$$

Thus, the relation in (1.3) takes the form

$$T_{\beta/2}(t)\, u_0 = \frac{1}{2t\sqrt{\pi t}} \int_0^\infty \tau \exp\left(-\frac{\tau^2}{4t}\right) T_\beta(\tau) u_0\, d\tau. \quad (1.8)$$

The representation in (1.8) can provide the smoothing effect (see item (iii) in Condition 2) for the resolving operator $T_{\beta/2}(t)$ in the case where this effect for the operator $T_\beta(t)$ is absent. For example, this takes place if A and B are differential operators.

The following assertion is the solvability theorem for the Cauchy problem for the inhomogeneous equation.

Theorem 1.2. *Let $\beta < 1$, and let Condition 1 be satisfied. Let one of the following two conditions hold:*

(a) *a function $h(t)$ belongs to $C((0,\infty), E)$, is absolutely integrable at the origin, and takes values in $D(A)$ and the function $Ah(t)$ belongs to $C((0,\infty), E)$ and is absolutely integrable at the origin;*

(b) *a function $h(t)$ is such that the function $I^{1-\beta}h(t)$ is continuous for $t \geq 0$ and continuously differentiable for $t > 0$ and $D^{\beta}h(t)$ is absolutely integrable at the origin.*

Then the problem

$$D^{\beta}u(t) = Au(t) + h(t), \quad t > 0, \tag{1.9}$$

$$\lim_{t\to 0} D^{\beta-1}u(t) = u_0 \tag{1.10}$$

has a unique solution, which is defined by the relation

$$u(t) = T_{\beta}(t)u_0 + \int_0^t T_{\beta}\,(t-\xi)\,h(\xi)\,d\xi. \tag{1.11}$$

Proof. It suffices to check that the function

$$v\,(t) = \int_0^t T_{\beta}\,(t-\xi)\,h\,(\xi)\;d\xi$$

satisfies Eq. (1.9) and condition (1.10), which is the zero initial condition.

Let condition (a) be satisfied. Then, for $t > 0$, we have

$$D^{\beta}v\,(t) = \frac{1}{\Gamma\,(1-\beta)}\frac{d}{dt}\int_0^t (t-\tau)^{-\beta}\,d\tau \int_0^{\tau} T_{\beta}\,(\tau-\xi)\,h\,(\xi)\;d\xi$$

$$= \frac{1}{\Gamma\,(1-\beta)}\frac{d}{dt}\int_0^t d\xi \int_{\xi}^t (t-\tau)^{-\beta}T_{\beta}\,(\tau-\xi)\,h\,(\xi)\;d\tau$$

$$= \frac{1}{\Gamma\,(1-\beta)}\frac{d}{dt}\int_0^t d\xi \int_0^{t-\xi} (t-\xi-x)^{-\beta}T_{\beta}(x)h(\xi)\,dx.$$

Since the integrand (with respect to ξ) is a continuous function of the variable $t-\xi$, it follows that

$$D^{\beta}v(t) = \frac{1}{\Gamma\,(1-\beta)}\lim_{\xi\to t}\int_0^{t-\xi} (t-\xi-x)^{-\beta}T_{\beta}\,(x)\,h\,(\xi)\;dx$$

$$+\frac{1}{\Gamma\,(1-\beta)}\int_0^t d\xi\frac{d}{dt}\int_0^{t-\xi} (t-\xi-x)^{-\beta}T_{\beta}\,(x)\,h\,(\xi)\;dx$$

$$= \lim_{t-\xi\to+0} D^{\beta-1}T_{\beta}\,(t-\xi)\,h\,(\xi) + \int_0^t D^{\beta}T_{\beta}\,(t-\xi)\,h\,(\xi)\;d\xi$$

$$= h\,(t) + \int_0^t T_{\beta}\,(t-\xi)\,Ah(\xi)\,d\xi = h\,(t) + Av(t).$$

Hence, the function $v(t)$ satisfies Eq. (1.9).

Further, we check that the function $v(t)$ satisfies condition (1.10). We have

$$\lim_{t\to+0} D^{\beta-1}v\,(t) = \frac{1}{\Gamma\,(1-\beta)}\lim_{t\to+0}\int_0^t (t-\tau)^{-\beta}\,d\tau\int_0^{\tau} T_{\beta}\,(\tau-\xi)\,h(\xi)\,d\xi.$$

Since $T_\beta(t)$ satisfies the estimate in (5), it follows that

$$\left\| \int_0^t (t-\tau)^{-\beta}\, d\tau \int_0^\tau T_\beta(\tau-\xi)\, h(\xi)\, d\xi \right\|$$
$$\leq M \int_0^t (t-\tau)^{-\beta}\, d\tau \int_0^\tau (\tau-\xi)^{\beta-1} \|h(\xi)\|\, d\xi = M\ B(\beta,1-\beta) \int_0^t \|h(\xi)\|\, d\xi,$$

for $t\in[0,1]$, where $B(\cdot,\cdot)$ is the beta-function. Hence, the function $v(t)$ satisfies condition (1.10).

Now, let condition (b) hold. Then

$$D^\beta v(t) = D^\beta \int_0^t T_\beta(\tau)h(t-\tau)\, d\tau = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\xi)^{-\beta}\, d\xi \int_0^\xi T_\beta(\tau) h(\xi-\tau)\, d\tau$$
$$= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t T_\beta(\tau)\, d\tau \int_0^{t-\tau} (t-\tau-x)^{-\beta} h(x)\, dx$$
$$= T_\beta(t) \lim_{\tau\to t} \frac{1}{\Gamma(1-\beta)} \int_0^{t-\tau} (t-\tau-x)^{-\beta} h(x)\, dx + \frac{1}{\Gamma(1-\beta)} \int_0^t T_\beta(\tau)\, d\tau \frac{d}{dt} \int_0^{t-\tau} (t-\tau-x)^{-\beta} h(x)\, dx$$
$$= T_\beta(t)\, D^{\beta-1} h(0) + \int_0^t T_\beta(\tau)\, D^\beta h(t-\tau)\, d\tau = T_\beta(t) D^{\beta-1} h(0) + \int_0^t T_\beta(t-\xi) D^\beta h(\xi)\, d\xi. \quad (1.12)$$

On the other hand, it follows from the relation

$$I^\beta D^\beta h(x) = h(x) - \frac{I^{1-\beta}h(x)}{\Gamma(\beta)} x^{\beta-1}, \qquad 0<\beta<1 \quad (1.13)$$

(see [18, formula (2.61)]), that

$$v(t) = \int_0^t T_\beta(t-\xi) \left(\frac{1}{\Gamma(\beta)} D^{\beta-1} h(0) \xi^{\beta-1} + I^\beta D^\beta h(\xi) \right) d\xi = \frac{1}{\Gamma(\beta)} \int_0^t \xi^{\beta-1} T_\beta(t-\xi) D^{\beta-1} h(0)\, d\xi$$
$$+ \frac{1}{\Gamma(\beta)} \int_0^t T_\beta(t-\xi)\, d\xi \int_0^\xi (\xi-\tau)^{\beta-1} D^\beta h(\tau)\, d\tau = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} T_\beta(\tau) D^{\beta-1} h(0)\, d\tau$$
$$+ \frac{1}{\Gamma(\beta)} \int_0^t T_\beta(\tau)\, d\tau \int_0^{t-\tau} (t-\tau-\xi)^{\beta-1} D^\beta h(\xi)\, d\xi. \quad (1.14)$$

Using (1.13) and the closedness of the operator A again, we obtain

$$\frac{1}{\Gamma(\beta)} A \int_0^t (t-\tau)^{\beta-1} T_\beta(\tau) v_0\, d\tau = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} D^\beta T_\beta(\tau) v_0\, d\tau$$
$$= I^\beta D^\beta T_\beta(\tau) v_0 = T_\beta(t) v_0 - \frac{t^{\beta-1}}{\Gamma(\beta)} D^{\beta-1} T_\beta(0) v_0 = T_\beta(t) v_0 - \frac{t^{\beta-1}}{\Gamma(\beta)} v_0. \quad (1.15)$$

It follows from (1.13)–(1.15) that

$$Av(t) = T_\beta(t)D^{\beta-1}h(0) - \frac{t^{\beta-1}}{\Gamma(\beta)}D^{\beta-1}h(0) + \int\limits_0^t \left(T_\beta(t-\xi)D^\beta h(\xi) - \frac{(t-\xi)^{\beta-1}}{\Gamma(\beta)}D^\beta h(\xi) \right) d\xi$$

$$= T_\beta\,(t)\,D^{\beta-1}h(0) - \frac{t^{\beta-1}}{\Gamma(\beta)}D^{\beta-1}h(0) + \int\limits_0^t T_\beta(t-\xi)D^\beta h(\xi)\;d\xi - h(t) + \frac{t^{\beta-1}}{\Gamma(\beta)}D^{\beta-1}h(0) =$$

$$D^\beta v(t) - h(t).$$

Hence, the function $v(t)$ satisfies Eq. (1.9).

To verify that the function $v(t)$ satisfies condition (1.10) if condition (b) is satisfied, one should represent $D^\beta v(t)$ as follows:

$$D^{\beta-1}v(t) = \int\limits_0^t T_\beta(s)I^{1-\beta}h(t-s)\;ds.$$

□

2. Cauchy-Type Problems for Perturbed Fractional-Order Equations

We pass to the investigation of the perturbed problem (1), (2). In the sequel, we use the following function of the Mittag-Leffler type (see [2, Chap. III-IV]):

$$E_{\mu,\rho}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\mu k+\rho)}.$$

Theorem 2.1. *Let $\alpha < \beta \le 1$, Conditions 1 and 2 be satisfied, and $\omega = 0$ in inequalities (5) and (6). Let Conditions 3 and 4 be satisfied. Then problem (1), (2) has a unique solution satisfying the estimate*

$$\|u(t)\| \le \frac{M_1\Gamma(\beta)}{\Gamma(\alpha)}t^{\alpha-1}\,\|u_0\| + \frac{C_0M_1\Gamma(\beta)}{\Gamma(\alpha+1)}t^\alpha + \frac{C_0M_1\Gamma(\beta)\Gamma(\mu)}{\Gamma(\alpha+\mu)}t^{\alpha+\mu-1}$$
$$+ LM_1M_2\Gamma(\beta)\Gamma(\delta/\nu)\left(t^{\alpha+\delta-1}E_{\delta,\alpha+\delta}\left(LM_2\Gamma(\delta/\nu)t^\delta\right)\|u_0\|\right.$$
$$\left.+C_0t^{\alpha+\delta}E_{\delta,\alpha+\delta+1}\left(LM_2\Gamma(\delta/\nu)t^\delta\right) + C_0\Gamma(\mu)t^{\alpha+\delta+\mu-1}E_{\delta,\alpha+\delta+\mu}\left(LM_2\Gamma(\delta/\nu)t^\delta\right)\right), \quad (2.1)$$

where $\delta = \nu(1-\gamma)$.

Proof. Taking into account Theorems 1.1 and 1.2, we reduce problem (1), (2) to an integral equation. By virtue of (1.3) and (1.11), this integral equation can be written as follows:

$$u(t) = \int\limits_0^\infty f_{\tau,\nu}(t)T_\beta(\tau)u_0\,d\tau + \int\limits_0^t\int\limits_0^\infty f_{\tau,\nu}(t-s)T_\beta(\tau)F(s,B(s)u(s))\,d\tau ds, \quad (2.2)$$

where u_0, $T_\beta(\tau)u_0 \in D(A) \subset D$ and $\nu = \alpha/\beta$. Denoting $B(t)u(t)$ by $w(t)$, we obtain

$$w(t) = \int\limits_0^\infty f_{\tau,\nu}(t)B(t)T_\beta(\tau)u_0\,d\tau + \int\limits_0^t\int\limits_0^\infty f_{\tau,\nu}(t-s)B(t)T_\beta(\tau)F(s,w(s))\,d\tau ds. \quad (2.3)$$

To solve Eq. (2.3), we use the iteration method, assigning

$$w_0(t) = 0, \qquad w_1(t) = \int\limits_0^\infty f_{\tau,\nu}(t)B(t)T_\beta(\tau)u_0\,d\tau + \int\limits_0^t\int\limits_0^\infty f_{\tau,\nu}(t-s)B(t)T_\beta(\tau)F(s,0)\,d\tau ds,$$

$$w_{n+1}(t) = \int_0^\infty f_{\tau,\nu}(t)B(t)T_\beta(\tau)u_0\,d\tau + \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)B(t)T_\beta(\tau)F(s,w_n(s))\,d\tau ds, \quad n \in N.$$

Using inequality (6) and item (ii) of Condition 3, we estimate the norm

$$\|w_1(t)\| \le M_2\,\|u_0\| \int_0^\infty f_{\tau,\nu}(t)\tau^{-\gamma}\,d\tau + M_2C_0\int_0^t\int_0^\infty f_{\tau,\nu}(t-s)\tau^{-\gamma}(1+s^{\mu-1})\,d\tau ds. \tag{2.4}$$

Taking into account that the function $f_{\tau,\nu}(t)$ is defined by relation (7) and using [15, integrals 2.3.4.1 and 2.3.3.4], we obtain

$$\int_0^\infty f_{\tau,\nu}(t)\tau^{-\gamma}\,d\tau = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zt}\,dz \int_0^\infty \tau^{-\gamma}\exp\left(-\tau z^\nu\right)\,d\tau$$

$$= \frac{\Gamma(1-\gamma)}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zt}z^{-\nu(1-\gamma)}\,dz = \frac{\Gamma(1-\gamma)}{\Gamma(\nu(1-\gamma))}\,t^{\nu(1-\gamma)-1}, \quad t>0. \tag{2.5}$$

Applying relation (2.5) to (2.4) twice and computing the obtained integral, we have

$$\|w_1(t)\| \le M_2\,\|u_0\|\,\frac{\Gamma(1-\gamma)}{\Gamma(\nu(1-\gamma))}t^{\nu(1-\gamma)-1} + M_2C_0\frac{\Gamma(1-\gamma)t^{\nu(1-\gamma)}}{\Gamma(\nu(1-\gamma)+1)}$$

$$+ M_2C_0\frac{\Gamma(1-\gamma)\Gamma(\mu)t^{\nu(1-\gamma)+\mu-1}}{\Gamma(\nu(1-\gamma)+\mu)} \le \frac{M_2\Gamma(\delta/\nu)}{\Gamma(\delta)}\left(t^{\delta-1}\,\|u_0\| + \frac{C_0}{\delta}t^\delta + \frac{C_0\Gamma(\delta)\Gamma(\mu)}{\Gamma(\delta+\mu)}t^{\delta+\mu-1}\right).$$

Using item (iii) in Condition 3, we estimate (in the same way) the norm of the difference

$$\|w_2(t)-w_1(t)\| \le \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)\,\|B(t)T_\beta(\tau)\left(F(s,w_1)-F(s,0)\right)\|\,d\tau ds$$

$$\le \frac{LM_2^2\Gamma(\delta/\nu)}{\Gamma(\delta)}\int_0^t\int_0^\infty f_{\tau,\nu}(t-s)\tau^{-\gamma}\left(s^{\delta-1}\,\|u_0\| + \frac{C_0}{\delta}s^\delta + \frac{C_0\Gamma(\delta)\Gamma(\mu)}{\Gamma(\delta+\mu)}s^{\delta+\mu-1}\right)\,d\tau ds$$

$$\le \frac{LM_2^2\Gamma^2(\delta/\nu)}{\Gamma(2\delta)}\left(t^{2\delta-1}\,\|u_0\| + \frac{C_0}{2\delta}t^{2\delta} + \frac{C_0\Gamma(2\delta)\Gamma(\mu)}{\Gamma(2\delta+\mu)}t^{2\delta+\mu-1}\right). \tag{2.6}$$

Taking into account (2.6), for $n \in N$, by induction, we obtain the inequality

$$\|w_n(t)-w_{n-1}(t)\| \le \frac{L^{n-1}M_2^n\,\Gamma^n(\delta/\nu)}{\Gamma(n\delta)}\left(t^{n\delta-1}\,\|u_0\| + \frac{C_0}{n\delta}t^{n\delta} + \frac{C_0\Gamma(n\delta)\Gamma(\mu)}{\Gamma(n\delta+\mu)}t^{n\delta+\mu-1}\right). \tag{2.7}$$

Hence, the series

$$\sum_{n=1}^\infty \left(w_n(t)-w_{n-1}(t)\right)$$

uniformly converges on any segment $[t_0,t_1]$, $0<t_0<t_1$. Therefore, $w_n(t)$ uniformly converges to a function $w(t)$ on the same segment, where $w(t)$ is continuous on $[t_0,t_1]$ and satisfies Eq. (2.3). By virtue of (2.7), the following estimate holds for that function:

$$\|w(t)\| \le \sum_{n=1}^\infty \|w_n(t)-w_{n-1}(t)\|$$

$$\le \sum_{k=0}^{\infty} \frac{L^k M_2^{k+1}\Gamma^{k+1}(\delta/\nu)}{\Gamma((k+1)\delta)} \left(t^{(k+1)\delta-1} \|u_0\| + \frac{C_0}{(k+1)\delta} t^{(k+1)\delta} + \frac{C_0\Gamma((k+1)\delta)\Gamma(\mu)}{\Gamma((k+1)\delta+\mu)} t^{(k+1)\delta+\mu-1} \right)$$

$$\le M_2\Gamma(\delta/\nu) \left(t^{\delta-1}\|u_0\| \sum_{k=0}^{\infty} \frac{L^k M_2^k \Gamma^k(\delta/\nu) t^{k\delta}}{\Gamma((k+1)\delta)} + C_0 t^{\delta} \sum_{k=0}^{\infty} \frac{L^k M_2^k \Gamma^k(\delta/\nu) t^{k\delta}}{\Gamma((k+1)\delta+1)} \right.$$

$$\left. + C_0\Gamma(\mu)\, t^{\delta+\mu-1} \sum_{k=0}^{\infty} \frac{L^k M_2^k \Gamma^k(\delta/\nu) t^{k\delta}}{\Gamma((k+1)\delta+\mu)} \right)$$

$$= M_2\Gamma(\delta/\nu) \left(t^{\delta-1} E_{\delta,\delta}\left(LM_2\Gamma(\delta/\nu)t^{\delta}\right)\|u_0\| + C_0 t^{\delta} E_{\delta,\delta+1}\left(LM_2\Gamma(\delta/\nu)t^{\delta}\right) \right.$$

$$\left. + C_0\Gamma(\mu) t^{\delta+\mu-1} E_{\delta,\delta+\mu}\left(LM_2\Gamma(\delta/\nu)t^{\delta}\right)\right), \quad (2.8)$$

where $E_{\sigma,\rho}(\cdot)$ is a function of the Mittag-Leffler type, $t \in [t_0, t_1]$, $0 < t_0 < t_1$.

Since the segment $[t_0, t_1]$ is chosen arbitrarily, it follows that the function $w(t)$ is a solution of Eq. (2.3) continuous on $(0, \infty)$ and satisfying inequality (2.8) on $(0, \infty)$, i.e., $w(t)$ is absolutely integrable at the origin. Moreover, from relation (2.3), we conclude that the function $w(t)$ satisfies item (ii) in Condition 2.

Finally, using relation (2.2) and Theorem 1.2, we obtain the following representation of the solution $u(t)$ of problem (1), (2):

$$u(t) = \int_0^{\infty} f_{\tau,\nu}(t) T_\beta(\tau) u_0 \, d\tau + \int_0^t \int_0^{\infty} f_{\tau,\nu}(t-s) T_\beta(\tau) F(s, w(s)) \, d\tau ds.$$

By virtue of (5), (2.8), (2.5), and item (ii) in Condition 3, it satisfies the inequality

$$\|u(t)\| \le \int_0^{\infty} f_{\tau,\nu}(t) \|T_\beta(\tau)u_0\| \, d\tau$$

$$+ \int_0^t \int_0^{\infty} f_{\tau,\nu}(t-s) \|T_\beta(\tau)F(s,0)\| \, d\tau ds + \int_0^t \int_0^{\infty} f_{\tau,\nu}(t-s) \|T_\beta(\tau)(F(s,w(s)) - F(s,0)\| \, d\tau ds$$

$$\le \frac{M_1\Gamma(\beta)\; t^{\alpha-1}}{\Gamma(\alpha)} \|u_0\| + \frac{C_0 M_1 \Gamma(\beta) t^{\alpha}}{\Gamma(\alpha+1)} + \frac{C_0 M_1 \Gamma(\beta)\Gamma(\mu) t^{\alpha+\mu-1}}{\Gamma(\alpha+\mu)}$$

$$+ \frac{LM_1M_2\Gamma(\beta)\Gamma(1-\gamma)}{\Gamma(\alpha)} \|u_0\| \int_0^t (t-s)^{\alpha-1} s^{\delta-1} E_{\delta,\delta}\left(LM_2\Gamma(\delta/\nu)s^{\delta}\right) ds$$

$$+ \frac{C_0 L M_1 M_2 \Gamma(\beta)\Gamma(\delta/\nu)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\delta} E_{\delta,\delta+1}\left(LM_2\Gamma(\delta/\nu)s^{\delta}\right) ds$$

$$+ \frac{C_0 L M_1 M_2 \Gamma(\beta)\Gamma(\delta/\nu)\Gamma(\mu)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\delta+\mu-1} E_{\delta,\delta+\mu}\left(LM_2\Gamma(\delta/\nu)s^{\delta}\right) ds.$$

Therefore, the solution satisfies the estimate

$$\|u(t)\| \le \frac{M_1\Gamma(\beta) t^{\alpha-1}}{\Gamma(\alpha)} \|u_0\| + \frac{C_0 M_1 \Gamma(\beta) t^{\alpha}}{\Gamma(\alpha+1)} + \frac{C_0 M_1 \Gamma(\beta)\Gamma(\mu) t^{\alpha+\mu-1}}{\Gamma(\alpha+\mu)}$$

$$+ LM_1M_2\Gamma(\beta)\Gamma(\delta/\nu)\left(t^{\alpha+\delta-1} E_{\delta,\alpha+\delta}\left(LM_2\Gamma(\delta/\nu)t^{\delta}\right)\|u_0\| \right.$$

$$+C_0t^{\alpha+\delta}E_{\delta,\alpha+\delta+1}\left(LM_2\Gamma(\delta/\nu)t^{\delta}\right)+C_0\Gamma(\mu)t^{\alpha+\delta+\mu-1}E_{\delta,\alpha+\delta+\mu}\left(LM_2\Gamma(\delta/\nu)t^{\delta}\right)\Big).$$

The following relation was used:

$$I^{\alpha}(t^{\rho-1}E_{\sigma,\rho}\left(ct^{\sigma}\right))=t^{\alpha+\rho-1}E_{\sigma,\alpha+\rho}\left(ct^{\sigma}\right),\quad \alpha,\sigma,\rho>0$$

(see [18, p. 141, formula (23)]).

To establish the uniqueness of the solution of problem (1), (2), we assume, to the contrary, that there exists another solution. We denote it by $U(t)$. Then, by virtue of Theorems 1.1 and 1.2, we have

$$U(t)=\int_0^{\infty}f_{\tau,\nu}(t)T_{\beta}(\tau)u_0\,d\tau+\int_0^t\int_0^{\infty}f_{\tau,\nu}(t-s)T_{\beta}(\tau)F(s,W(s))\,d\tau ds,$$

where $W(t)$ satisfies Eq. (2.3).

Let us prove the uniqueness of the solution of Eq. (2.3) in the class of functions continuous on $(0,\infty)$ and satisfying the estimate

$$\|W(t)\|\le Mt^{\delta-1}e^{\omega t},\quad M>0,\ \omega\ge 0, \tag{2.9}$$

where $\delta=\nu(1-\gamma)<1$. Note that the functions satisfying estimate (2.8) belong to the specified class due to the following asymptotic behavior of the Mittag-Leffler function for $0<\mu<2$ (see [2, p. 134]):

$$E_{\mu,\rho}(z)=\frac{1}{\mu}z^{(1-\rho)/\mu}\exp\left(z^{1/\mu}\right)-\sum_{j=1}^{n}\frac{1}{\Gamma(\rho-\mu j)\ z^{j}}+O\left(\frac{1}{|z|^{n+1}}\right),\quad z\in\mathbb{R},\ z\to+\infty. \tag{2.10}$$

Let $b>0$ and $t\in(0,b]$. Set

$$m=\sup_{t\in[0,b]}(t^{1-\delta}e^{-\omega t}\|W(t)-w(t)\|).$$

The supremum is finite because we consider the class of functions satisfying inequality (2.9).

The difference $W(t)-w(t)$ satisfies Eq. (2.3) for $u_0=0$. Therefore, taking into account relation (2.5), we have

$$\|W(t)-w(t)\|\le\frac{LM_2\Gamma(1-\gamma)}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}\|W(s)-w(s)\|\,ds$$
$$=LM_2\Gamma(1-\gamma)I^{\delta}(\|W(t)-w(t)\|). \tag{2.11}$$

Hence, the following inequality holds:

$$\|W(t)-w(t)\|\le\frac{LM_2\Gamma(1-\gamma)m}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}s^{\delta-1}e^{\omega s}\,ds=LM_2\Gamma(1-\gamma)m\ I^{\delta}(t^{\delta-1}e^{\omega t}). \tag{2.12}$$

Substituting (2.12) into (2.11), we obtain the inequality

$$\|W(t)-w(t)\|\le L^2M_2^2\Gamma^2(1-\gamma)mI^{2\delta}(t^{\delta-1}e^{\omega t}).$$

Continuing this procedure, we arrive at the inequality

$$\|W(t)-w(t)\|\le L^kM_2^k\Gamma^k(1-\gamma)mI^{k\delta}(t^{\delta-1}e^{\omega t})=\frac{L^kM_2^k\Gamma^k(1-\gamma)m}{\Gamma(k\delta)}\int_0^t(t-s)^{k\delta-1}s^{\delta-1}e^{\omega s}\,ds$$
$$\le\frac{L^kM_2^k\Gamma^k(1-\gamma)\Gamma(\delta)}{\Gamma((k+1)\delta)}t^{(k+1)\delta-1}e^{\omega t}m\quad\text{for all }k\in N. \tag{2.13}$$

Taking the supremum, we obtain the inequality

$$m \leq \frac{L^k M_2^k \Gamma^k(1-\gamma)\Gamma(\delta)}{\Gamma((k+1)\delta)} b^{k\delta} m.$$

The factor

$$\frac{L^k M_2^k \Gamma^k(1-\gamma)\Gamma(\delta)}{\Gamma((k+1)\delta)} b^{k\delta}$$

is the common term of the series defining the Mittag-Leffler function (cf. (2.8)). Therefore, it vanishes as $k \to \infty$. Thus,

$$m = \sup_{t\in[0,b]} (t^{1-\delta} e^{-\omega t} \|W(t) - w(t)\|) = 0.$$

Since the positive number b was chosen arbitrarily, it follows that $W(t) \equiv w(t)$ for $t > 0$. This completes the proof of the uniqueness. □

Note that estimate (2.1) contains a detailed dependence of the solution on the data of the problem. This dependence can be used in further research. If only the behavior of solution of problem (1), (2) as $t \to 0$ and $t \to \infty$ is investigated, then, taking into account (2.10), one can represent estimate (2.1) as follows:

$$\|u(t)\| \leq M t^{\alpha-1} e^{\omega_1 t} \|u_0\|, \quad M > 0,\ \omega_1 \geq 0. \tag{2.14}$$

Theorem 2.1 establishes the solvability of problem (1), (2) for any α provided that $0 < \alpha < \beta \leq 1$, Conditions 1–4 are satisfied, and $\omega = 0$ in inequalities (5) and (6). Let us prove that if $0 < \alpha = \beta < 1$, then similar results can be obtained without the requirement $\omega = 0$ in inequalities (5) and (6) and without Condition 4.

Theorem 2.2. *Let Conditions* 1–3 *be satisfied, and let* $\alpha = \beta < 1$. *Then problem* (1), (2) *has a unique solution satisfying estimate* (2.14).

Proof. Taking into account Theorem 1.2, we reduce problem (1), (2) to the integral equation

$$u(t) = T_\alpha(t)u_0 + \int_0^t T_\alpha(t-s) F\left(s, B(s)u(s)\right) ds. \tag{2.15}$$

Introducing $w(t) = B(t)u(t)$, we obtain the equation

$$w(t) = T_\alpha(t)u_0 + \int_0^t B(t)T_\alpha(t-s)F(s, w(s))\, ds. \tag{2.16}$$

To solve it by the iteration method, we set

$$w_0(t) = 0, \quad w_1(t) = T_\alpha(t)u_0, \quad w_{n+1}(t) = T_\alpha(t)u_0 + \int_0^t B(t)T_\alpha(t-s)F(s, w_n(s))\, ds, \quad n \in N.$$

Using inequalities (5) and (6) and item (iii) in Condition 3, we estimate the norm of the following difference:

$$\|w_2(t) - w_1(t)\| \leq LM_2 \int_0^t (t-s)^{-\gamma} e^{\omega(t-s)} \|w_1(s)\|\, ds \leq LM_1M_2\Gamma(1-\gamma)e^{\omega t} I^{1-\gamma}(t^{\alpha-1}) \|u_0\|. \tag{2.17}$$

Taking into account (2.17), by induction, we obtain the relation

$$\|w_n(t) - w_{n-1}(t)\| \leq M_1 L^{n-1} M_2^{n-1} \Gamma^{n-1}(1-\gamma) e^{\omega t} I^{(n-1)(1-\gamma)} \left(t^{\alpha-1}\right) \|u_0\|$$

$$= \frac{M_1 L^{n-1} M_2^{n-1} \Gamma(\alpha) \Gamma^{n-1}(1-\gamma)}{\Gamma(\alpha+(n-1)(1-\gamma))} t^{\alpha-1+(n-1)(1-\gamma)} e^{\omega t} \|u_0\|, \quad n \in N.$$

Further reasoning regarding the existence of a unique solution is similar to the proof of Theorem 2.1. The following estimate holds for the solution $w(t)$ of Eq. (2.16):

$$\|w(t)\| \leq M_1 t^{\alpha-1} e^{\omega t} \|u_0\| + \sum_{k=1}^{\infty} \frac{M_1 L^k M_2^k \Gamma(\alpha) \Gamma^k(1-\gamma) t^{\alpha-1+k(1-\gamma)} e^{\omega t} \|u_0\|}{\Gamma(\alpha+k(1-\gamma))} \leq M_0 t^{\alpha-1} e^{\omega_0 t} \|u_0\|, \tag{2.18}$$

where $M_0 > 0$ and $\omega_0 \geq \omega$.

Using (2.18), we deduce estimate (2.14) of the solution $u(t)$ of problem (1), (2) from relation (2.15). $\square$

Remark 2.1. An assertion similar to Theorem 2.2 is also valid for $\alpha = \beta = 1$, but item (ii) in Condition 2 should be replaced by the following assumption: for any $x \in D$, either the functions $B(t)x$ and $AB(t)x$ belong to $C([0,\infty), E)$ and the function $B(t)x$ takes values in $D(A)$ or the function $B(t)x$ belongs to $C^1([0,\infty), E)$.

The following assertion is a theorem on the continuous dependence of the solution of problem (1), (2) on the initial data.

Theorem 2.3. *Suppose that the conditions of Theorem* 2.1 *are satisfied and $u_n(t)$ is the sequence of solutions of the problem*

$$D^\alpha u_n(t) = A u_n(t) + F\left(t, B(t) u_n(t)\right), \quad t > 0, \tag{2.19}$$

$$\lim_{t \to 0} D^{\alpha-1} u_n(t) = g_n \in D(A). \tag{2.20}$$

If $g_n \to u_0 \in D(A)$, $Ag_n \to Au_0$, and $B(t)g_n \to B(t)u_0$ uniformly with respect to $t \in (0,b]$ for any positive b, then the sequence $u_n(t)$ of the solutions of problem (2.19), (2.20) *converges to a solution $u(t)$ of problem* (1), (2) *uniformly with respect to $t \in [t_0, b]$ for any $t_0 \in (0,b)$.*

Proof. Consider the sequence $U_n(t) = u_n(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} g_n$, which satisfies the problem

$$D^\alpha U_n(t) = A U_n(t) + F\left(t, B(t) U_n(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} B(t) g_n\right) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} A g_n, \tag{2.21}$$

$$\lim_{t \to 0} D^{\alpha-1} U_n(t) = 0. \tag{2.22}$$

By Theorems 1.1 and 1.2, the function $U_n(t)$ satisfies the integral equation

$$U_n(t) = \int_0^t \int_0^\infty f_{\tau,\nu}(t-s) T_\beta(\tau) \left(F\left(s, B(s) U_n(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} B(s) g_n\right) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} A g_n\right) d\tau ds.$$

Setting $W_n(t) = B(t) U_n(t)$, we obtain (as in the proof of Theorem 2.1) that

$$U_n(t) = \int_0^t \int_0^\infty f_{\tau,\nu}(t-s) T_\beta(\tau) \left(F\left(s, W_n(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} B(s) g_n\right) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} A g_n\right) d\tau ds, \tag{2.23}$$

where $W_n(t)$ satisfies the integral equation

$$W_n(t) = \int_0^t \int_0^\infty f_{\tau,\nu}(t-s) B(t) T_\beta(\tau) \left(F\left(s, W_n(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} B(s) g_n\right) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} A g_n\right) d\tau ds. \tag{2.24}$$

Let n and k be sufficiently large positive integers and $\varepsilon > 0$. Taking (2.24) into account, we obtain (as in the proof of Theorem (2.13)) that

$$\|W_n(t) - W_k(t)\| \le \frac{L\, M_2\, \Gamma(\delta/\nu)}{\Gamma(\delta)} \int\limits_0^t (t-s)^{\delta-1} \|W_n(s) - W_k(s)\|\, ds$$

$$+ \frac{M_2\, \Gamma(\delta/\nu)}{\Gamma(\delta)\Gamma(\alpha)} \int\limits_0^t (t-s)^{\delta-1}\, s^{\alpha-1} \left(\|Ag_n - Ag_k\| + L\, \|B(s)g_n - B(s)g_k\|\right) ds$$

and

$$m = \sup_{t\in[0,b]} \left(t^{1-\delta} e^{-\omega t} \|W_n(t) - W_k(t)\|\right) \le M_0 m + \varepsilon, \quad M_0 < 1.$$

Hence, $m \le \dfrac{\varepsilon}{1 - M_0}$. Then, by virtue of the completeness of the space E, the sequence $t^{1-\delta}e^{-\omega t}W_n(t)$ converges to a function $t^{1-\delta}e^{-\omega t}W(t)$ continuous on $[0, b]$ uniformly with respect to $t \in [0, b]$. Thus, $W_n(t)$ converges to a function $W(t)$ uniformly with respect to $t \in [t_0, b]$, $0 < t_0 < b$, where $W(t)$ satisfies inequality (2.9) and item (ii) in Condition 2.

Relation (2.23) implies the uniform (with respect to $t \in [t_0, b]$) convergence of $U_n(t)$ to the function

$$U(t) = \int\limits_0^t \int\limits_0^\infty f_{\tau,\nu}(t-s) T_\beta(\tau) \left(F\left(s, W(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} B(s)u_0\right) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} Au_0 \right) d\tau ds,$$

which satisfies problem (2.21), (2.22). Finally, $u_n(t)$ converges to the function $u(t) = U(t) + \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} u_0$ uniformly with respect to $t \in [t_0, b]$, while $u(t)$ satisfies problem (1), (2). □

Remark 2.2. An assertion similar to Theorem 2.3 on the continuous dependence of the solution of problem (1), (2) on the initial data can also be formulated and proved for $\alpha = \beta \le 1$.

In the particular case where the operator B does not depend on t and is bounded and Condition 4 is satisfied, the part of Theorem 2.2 regarding the unique solvability contains [6, Theorem 8]. In [6], it is proved that (in the specified particular case) for $\alpha = \beta < 1$, the resolving operator $T_\alpha(t, A + B)$ for problem (1), (2) is of the form

$$T_\alpha(t, A + B) = \sum_{n=0}^\infty S_n(t),$$

where $S_0(t) = T_\alpha(t, A)$ is the resolving operator for problem (3), (4) for $\beta = \alpha$ and

$$S_n(t) = \int\limits_0^t T_\alpha(t-s, A) B S_{n-1}(s)\, ds, \qquad n = 1, 2, \dots .$$

In [3], the perturbation theorem is proved for an equation which, unlike Eq. (1), contains the Caputo fractional derivative, provided that the operator A is a generator of an analytic semigroup and $\beta = 1$. The following example is given in [3].

Example 2.1. Let $E = L_2(\mathbb{R}^n)$. Then Condition 4 is satisfied (see [1, p. 20]). We define the operator A on the set $D(A) = W_2^{2m}(\mathbb{R}^n)$ as follows:

$$Au(t, x) = \sum_{|p|=2m} a_p(x) \frac{\partial^{p_1+\dots+p_n} u(t, x)}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}},$$

where

$$\sum_{|p|=2m} a_p(x)\xi^p \geq (-1)^{m+1} M_0 |\xi|^{2m}$$

for all $x,\ \xi \in \mathbb{R}^n$ and the coefficients $a_p(x)$, $|p| = 2m$, satisfy the Hölder condition uniformly in $\mathbb{R}^n$. It is known from [3] that the operator A satisfies Condition 1 if $\beta = 1$ and $\omega = 0$.

We define the operator $B(t)$ on $D = W_2^{2m-1}(\mathbb{R}^n) \supset D(A)$ as follows:

$$B(t)u(t,x) = \sum_{|p|\leq 2m-1} a_p(t,x) \frac{\partial^{p_1+\cdots+p_n} u(t,x)}{\partial x_1^{p_1}\cdots \partial x_n^{p_n}} + \int\limits_{\Omega} \sum_{|p|\leq 2m-1} b_p(t,x,\xi) \frac{\partial^{p_1+\cdots+p_n} u(t,\xi)}{\partial \xi_1^{p_1}\cdots \partial \xi_n^{p_n}}\, d\xi,$$

where $\Omega \subset \mathbb{R}^n$, the coefficients $a_p(t,x)$ are continuous and bounded with respect to $x \in \mathbb{R}^n$ for any $|p| \leq 2m-1$ and any $t \geq 0$ and satisfy the Hölder condition with respect to t with power $\mu > \alpha$ uniformly with respect to $x \in \mathbb{R}^n$, the coefficients $b_p(t,x,\xi)$ are continuous,

$$\int\limits_{\mathbb{R}^n}\int\limits_{\Omega} |b_p(t,x,\xi)|^2\, d\xi dx < +\infty,$$

and

$$\int\limits_{\mathbb{R}^n}\int\limits_{\Omega} |b_p(t_2,x,\xi) - b_p(t_1,x,\xi)|^2\, d\xi dx \leq C\,|t_2 - t_1|^{\mu}, \quad \mu > \alpha,\ C > 0.$$

It is known from [3] that there exists $\gamma \in (0,1)$ such that the operator $B(t)$ satisfies Condition 2 for $\omega = 0$.

Let the operator $F(t,w)$ satisfy Condition 3. Then, by virtue of Theorems 2.1 and 2.3, problem (1), (2) (the Cauchy-type problem for an integrodifferential equation) is well posed and uniquely solvable for $u_0(x) \in W_2^{2m}(\mathbb{R}^n)$ and $\alpha < 1$.

3. Loaded Fractional-Order Differential Equations

Consider the following Cauchy-type problem in a Banach space E:

$$D^{\alpha}u(t) = Au(t) + g(u(t))p, \quad t > 0, \tag{3.1}$$

$$\lim_{t\to 0} D^{\alpha-1}u(t) = u_0, \tag{3.2}$$

where $0 < \alpha < 1$, g is a nonlinear continuous functional defined on E, A is a linear closed densely defined operator, and p is a fixed element of the space E.

Problem (3.1), (3.2) is a particular case of problem (1), (2) for $F(t, B(t)u(t)) = g(u(t))p$. Equation (3.1) contains the functional g depending on the sought solution $u(t)$. Hence, it is natural to call it a *loaded differential equation* (see the definition of a loaded differential equation, e.g., in [12, Chap. 2]).

Condition 3.1. (i) *If the function $I^{1-\alpha}u(t)$ is continuous for $t \geq 0$ and continuously differentiable for $t > 0$, then the function $D^{\alpha}g(u(t))$ belongs to $C((0,\infty),E)$ and is absolutely integrable at the origin.*

(ii) *For any $u, v \in E$, there exists a positive L such that*

$$|g(u) - g(v)| \leq L\|u - v\|. \tag{3.3}$$

Theorems 2.1 and 2.2 imply the validity of the following assertions.

Theorem 3.1. *Let $\alpha < \beta \leq 1$, Condition* 1 *be satisfied, and $\omega = 0$ in inequality* (5). *Let Conditions* 4 *and* 3.1 *be satisfied. Then problem* (3.1), (3.2) *has a unique solution satisfying the estimate*

$$\|u(t)\| \leq Mt^{\alpha-1}e^{\omega_1 t}\|u_0\|, \quad M > 0, \quad \omega_1 > \omega. \tag{3.4}$$

Theorem 3.2. *Let $\alpha = \beta \leq 1$ and Conditions* 1 *and* 3.1 *be satisfied. Then problem* (3.1), (3.2) *has a unique solution satisfying estimate* (3.4).

The above solvability theorems for the Cauchy-type problem for loaded abstract equations can be used for the investigation of inverse coefficient problems for fractional-order equations.

4. Inverse Problems for Fractional-Order Differential Equations

Consider the problem of finding a pair $(w(t), \varphi(t))$ satisfying the following conditions:

$$D^\beta w(t) = Aw(t) + \varphi(t)p, \quad t > 0, \tag{4.1}$$

$$\lim_{t\to 0} D^{\beta-1}w(t) = u_0, \tag{4.2}$$

$$f(w(t)) = \psi(t), \tag{4.3}$$

where $0 < \beta < 1$, p and u_0 are fixed elements of $D(A)$, f is a linear continuous functional over E (i.e., f belongs to the adjoint space E^*), and $\psi(t)$ is a given scalar function.

For example, a problem to reconstruct the dependence of the perturbation on the time, using an additional observation at a space point, is a particular interpretation of the considered inverse problem.

Definition 4.1. A pair $(w(t), \varphi(t))$ is called a *solution of problem* (4.1)–(4.3) if $w(t)$ is an abstract function and $\varphi(t)$ is an absolutely integrable function such that $w(t)$ satisfies Eq. (4.1) and Conditions (4.2) and (4.3).

In [14], one can find a review of publications devoted to inverse problems for abstract integer-order differential equations. In [17], their particular implementations can be found. The inverse problem (4.1)–(4.3) for fractional-order equations was not considered before.

Condition 4.1. (i) *$0 < \beta < 1$ and $p \in D(A)$, where $D(A)$ is the domain of the operator A.*

(ii) *$f \in E^*$ and $f(p) \neq 0$ (the nondegeneracy condition).*

(iii) *The scalar function $I^{1-\beta}\psi(t)$ is continuous for $t \geq 0$ and continuously differentiable for $t > 0$, the fractional derivative $D^\beta\psi(t)$ is absolutely integrable at the origin, and the conjugation condition*

$$f(u_0) = \lim_{t\to 0} D^{\beta-1}\psi(t)$$

is satisfied.

In particular implementations, the nondegeneracy condition $f(p) \neq 0$ means that a reconstructable source acts at the observation point (see [17]).

Theorem 4.1. *Let Conditions* 1 *and* 4.1 *be satisfied. Then problem* (4.1)–(4.3) *has a unique solution.*

Proof. A solution of problem (4.1)–(4.3) is sought in the form

$$w(t) = \theta(t)p + u(t), \tag{4.4}$$

where

$$\theta(t) = I^\beta\varphi(t). \tag{4.5}$$

It is easy to verify that the function $u(t)$ satisfies the equation

$$D^\beta u(t) = Au(t) + \theta(t)Ap, \quad t > 0,$$

and the initial condition

$$\lim_{t\to 0} D^{\beta-1}u(t) = u_0. \tag{4.6}$$

Taking into account condition (4.3), we obtain the following linear equation for the function $\theta(t)$:

$$\psi(t) = \theta(t)f(p) + f(u(t)). \tag{4.7}$$

Thus, to solve inverse problem (4.1)–(4.3), it suffices to find a solution of the loaded equation

$$D^\beta u(t) = Au(t) + g(u(t))q, \quad t > 0, \tag{4.8}$$

satisfying condition (4.6), where $q = Ap \in E$ and

$$g(u(t)) = \frac{\psi(t) - f(u(t))}{f(p)}$$

is a continuous functional (its linearity is not assumed).

By assumption, the operator A satisfies Condition 1. Obviously, the functional $g(u(t))$ satisfies Condition 3.1. By virtue of Theorem 3.2, the Cauchy-type problem (4.8), (4.6) has a unique solution $u(t)$.

The function $\varphi(t)$ can be uniquely found from relations (4.5) and (4.7). It is of the form

$$\varphi(t) = D^{\beta}\theta(t) = \frac{1}{f(p)}\left(D^{\beta}\psi(t) - f\left(D^{\beta}u(t)\right)\right).$$

Finally, the function $w(t)$ is defined by relation (4.4). $\square$

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